The Micro Effects of Aggregate Shocks in Endogenous Trade Networks *

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Abstract

This paper examines how aggregate shocks affect production networks through a model where firms balance cost and risk in supplier choices. Global uncertainty reorganizes the network, changing shock transmission and GDP growth. In the model, the impact of reorganization on value added depends on a firm's position in the network: during crises, producers shift away from central country-industry pairs as these become riskier. In addition, the model attributes 16-20% of GDP growth during the financial crisis to network reorganization. A policy counterfactual shows lower trade costs boost expected consumption by reallocating the network to more productive suppliers but raise aggregate risk for many countries.

Keywords: Aggregate shocks, production networks, international trade, uncertainty, global financial crisis.

JEL classification: F44, F14, E32, D85, C68.

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1 Introduction

Cross-border trade links shape the degree of economic exposure to global risk through the production network. Empirical evidence suggests that (i) aggregate risk varies over time, such as during the global financial crisis or COVID-19, and (ii) firms adjust their production structure in response to risk changes Kopytov, Mishra, Nimark, and Taschereau-Dumouchel (2024). Although extensive research shows how micro-level heterogeneity enables micro shocks to generate aggregate fluctuations (Gabaix, 2011; Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012), less attention has been paid to how aggregate shocks alter micro-level heterogeneity. I develop a multi country quantitative model of endogenous formation of production networks to examine how aggregate risk shapes the global production network and affects aggregate variables.

I study this interaction using the 2009 global financial crisis. During the crisis, aggregate volatility increased, serving as an experiment to study network reorganization. In the crisis, the Japanese financial sector increased their global market share by almost half, while manufacturers of vehicles and metals in Indonesia lost almost one third of their global share. What explains these patterns?

In the model, firms face a trade-off when choosing suppliers. Before TFP shocks are realized, firms must balance selecting suppliers with lower expected prices against choosing safer suppliers whose prices are less correlated with aggregate conditions. This trade-off depends on technology, transportation costs and, crucially, the endogenous exposure of each producer to global shocks through the network. In my model, highly productive sectors or sectors with lower trade distortions gain large sales shares when there are no aggregate shocks. But as these sectors become central in the production network, they turn into poor hedges during high uncertainty because their prices co-move strongly with global conditions. As a result, during crises, firms move away from central suppliers for insurance reasons.

Empirical Motivation. The empirical analysis uncovers two patterns in World Input-Output Data (WIOD) covering 43 countries and 56 sectors: sectoral value-added growth is well approximated by normal distributions, while aggregate GDP growth exhibits heavy tails. These heavy tails, concentrated during the 2009 financial crisis, reveal that synchronized sectoral declines are crucial for explaining aggregate tail risk. These patterns extend the work of Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2017 to an international setting and provide empirical motivation for studying the impact of heteroskedastic aggregate shocks in global production networks

Environment. I develop a quantitative model in which firms endogenously reorganize their supplier networks in response to changes in global uncertainty. The framework extends the one in Kopytov et al., 2024 to a multi-country setting with heterogeneous trade costs, distinguishing between intermediate-good and final-good producers, incorporating endogenous labor supply, and persistent productivity shocks that feature a global component with time-changing variance. Firms choose their supplier portfolios before productivity shocks are realized, balancing a trade-off between expected efficiency and exposure to risk.

The model has three types of agents: households, intermediate-good producers, and final-good producers. Households provide labor, consume the final domestic product, and own the firms in their country. Intermediate-good producers combine labor with inputs sourced from all sectors and countries. Final-good producers aggregate inputs from all sectors of the world into a single good that is sold to the representative household in each country.

Firms face a two-stage decision process. In the first stage, before the realization of productivity shocks, firms select a portfolio of suppliers that minimizes expected costs given beliefs about the distribution of shocks. In the first stage, firms can modify their production technology by changing the Cobb-Douglas coefficients with which they use the inputs from each supplier. As in Acemoglu and Azar, 2020, departures from ideal technology imply a loss of efficiency. In the second stage, after the shocks are realized, they choose input quantities and output levels taking prices as given.

Productivity shocks are persistent and with time-changing variance. The key feature of the model is that it provides a tractable production function with two layers of substitution. In the first stage, the underlying elasticities of substitution are non-constant (unlike CES) and depend on second-moment terms that reflect risk, which resembles translog production functions. Two key differences from a standard translog are: (i) substitution responds to expected, not realized, prices; and (ii) the degree of substitutability is endogenous. Although all goods are substitutes, the extent to which firms shift away from a supplier when its expected price rises depends on transportation costs and the covariance structure of prices, which itself changes with the equilibrium network. Hence, substitution elasticities evolve with both the network configuration and the level of uncertainty. In the second stage, conditional on the chosen network, firms substitute away from goods with higher relative prices with unit elasticity (Cobb-Douglas).

The optimal network is characterized by risk-adjusted prices, which capture the expected cost to producers of increasing the input share of a given supplier. The risk-adjusted price of each supplier has two components: the expected price adjusted for transportation costs, and a covariance term reflecting how the supplier's price comoves with consumption. I use risk-adjusted prices to characterize how changes in aggregate uncertainty reshape the production network, showing that central sectors lose market share when aggregate volatility increases because they become poor hedges.

Existence and uniqueness. I show that a competitive equilibrium with endogenous production networks exists and provide sufficient conditions for uniqueness. I recast the competitive equilibrium as an static game between producers. This structure allows me to study strategic complementarities among producers' network choices and to derive parameter restrictions under which these complementarities do not generate multiple equilibria. Unlike Kopytov et al., 2024, which analyzes a planner's problem to establish existence and uniqueness, my model includes heterogeneous households and incomplete markets, making the decentralized equilibrium inefficient. Therefore, I directly analyze the existence and uniqueness of competitive equilibrium.

Quantification of the crisis. My next contribution is quantitative, using WIOD and ESCAP-World Bank Trade Cost Database data. I propose a method to estimate changes in aggregate shock variance in short panels, and develop a numerical algorithm to solve competitive equilibrium with endogenous networks. I conduct a growth decomposition to assess network reorganization's impact on GDP growth. Lastly, I evaluate the welfare effects of changes in risk and transportation costs through two counterfactuals.

Based on the work of Huo, Levchenko, and Pandalai-Nayar, 2024; Caselli, Koren, Lisicky, and Tenreyro, 2020, I recover the cross-sectional distribution of TFP shocks that explains the observed distribution of VA and decomposes it into global, country, sector and product-specific components. The key innovation involves using this decomposition to estimate *time-changing* variances for each component in a short panel (14 years). I extend the *tail-movement* statistic of Acemoglu et al., 2017 to accommodate multiple factors and multiple quantiles. Next, assuming normality, I demonstrate that these probabilities depend only on the ratio of variances between different factors. This approach allows me to estimate period-by-period variances via GMM leveraging cross-sectional data on conditional quantile distributions. The key advantage of this method, compared to traditional methods such as GARCH, is that it uses cross-sectional variation to estimate changes in variances. For the case of aggregate risk, my estimate relies on 2.408 observations for each period (the total number of products in my sample) instead of just 14 observations.

Using these estimates, I present a quantitative analysis of the substitution patterns between goods during the 2009 crisis and illustrate how the mechanisms of the model contribute to explaining the observed reallocation of global market shares between countries and sectors during the crisis. Furthermore, I do a growth decomposition that divides the sources of growth into two: (i) growth generated by the transmission of productivity shocks throughout the network and (ii) growth generated by the novel mechanism of the model, the reorganization of the network. Changes in the network impact growth by changing the distribution of market shares across sectors. Sectors that increase their market share increase the demand for labor to meet the higher demand and experience higher growth. Network reorganization also impacts GDP growth as it impact real wages throughout their interaction with transportation costs. This decomposition reveals that network adjustment during the 2009 crisis accounts for 16-20% of observed GDP growth dynamics.

Transportation Costs and Risk. The preceding analysis shows that network reorganization during the 2009 crisis affected both GDP growth and welfare. However, firms' ability to adjust their networks is constrained by transportation costs. This raises two policy questions: would reducing trade barriers improve welfare and reduce the variance of GDP? The reorganization of the network changes the welfare of each country for two reasons. It redistributes consumption across countries. Second, it induces changes in aggregate volatility. Critically, with incomplete markets, changes in country specific risk due to network reorganization are themselves welfare-relevant.

On variance of GDP, I revisit Caselli et al., 2020 who argued that reduced trade distortions enable diversification of risk through the global production network, potentially reducing aggregate risk. In their model, inputs from different countries are perfect substitutes. This allows complete reallocation away from countries with adverse shocks, leading to significant diversification benefits. Conversely, my model uses a Cobb-Douglas production function which allows me to match the empirical production network. Under Cobb-Douglas producers maintain fixed proportions of suppliers ex-post, limiting diversification benefits. The second and crucial difference is that in my model, the diversification benefit is endogenous. Lower trade costs allow firms to either enhance insurance by adjusting their supplier network (reducing risk) or focus on more productive suppliers (increasing risk).

My results indicate that while lower trade distortions increase aggregate volatility, they also boost expected consumption due to increased average productivity, enhancing welfare. These welfare gains are smaller during periods of higher productivity shock variance, such as 2009.

1.1 Related Literature

The main precursors of this paper are Kopytov et al., 2024 and Huo et al., 2024. I extend Kopytov et al., 2024 analysis by explicitly modeling aggregate shocks and establish how aggregate uncertainty has heterogeneous effects across sectors and industries according to their position in the network. My model extends the one in Kopytov et al., 2024, considering an open economy with heterogeneous countries, incomplete markets, endogenous labor supply, and distortions in trade. I develop a solution algorithm to solve the competitive equilibrium beyond the planner's problem considered by Kopytov et al., 2024, and I provide characterizations of how network adjustments affect GDP growth.

Also, I build on Huo et al., 2024, who develop a model of international propagation of shocks with exogenous input-output linkages. I use their environment as a benchmark and then endogenize the production network, asking how network's optimal reorganization of change the transmission of shocks.

This paper belongs to the emerging trade literature that studies how changes in risk shape trade and the global production network. Closest to my approach are Fan and Luo, 2025 and Kleinman, Liu, and Redding, 2025, who also model supplier choice under uncertainty in a global environment. Fan and Luo, 2025 main focus is on how uncertainty on tariff reshape international sourcing and trade flows. In contrast, I study how changes in global macro conditions, captured by shifts in the variance of an aggregate shock, lead firms to reorganize linkages, and how this reorganization feeds into GDP growth and volatility.

While Kleinman et al., 2025 focuses on portfolio choice in final goods production, my model also includes portfolio choice in intermediate goods production. In their model, intermediate goods are produced using only labor so the exposure of each country to risk is summarized by the portfolio of suppliers used in the production of final goods. In my model, however, input-output links in intermediate goods generate an additional mechanism through which aggregate risk impacts consumption and welfare. Aggregate shocks affect more central sectors in the production network, so changes in its variance also changes comparative advantage, shifting production from central to peripheral sectors. Thus, the reorganization of the intermediate-input network adds a channel for aggregate shocks to impact consumption and welfare.

I also contribute to the literature on endogenous production networks, such as Arkolakis, Huneeus, and Miyauchi, 2025, Acemoglu and Azar, 2020, and Elliott, Golub, and Leduc, 2022. These papers model how firms optimally choose suppliers and how the production network reorganizes in response to shocks or policy. Relative to this work, my model maintains Cobb-Douglas tractability but allows for endogenous reorganization of both domestic and cross-country networks in response to time-varying aggregate shocks, and it links these network adjustments directly GDP growth. Other closely related papers are Oberfield, 2018, Lim, 2017, Taschereau-Dumouchel, 2025, and Huneeus, 2018, who also study optimal network formation and its implications for shock propagation.

In addition, my paper is related to Acemoglu et al., 2017, who introduce a tail comovement statistic to study how sectoral shocks generate macro tail risk in a U.S. input-output network. I extend their approach to a global production network, documenting heavy tails and tail comovement in sectoral value-added growth using WIOD data for 43 countries and 56 sectors, and using the cross-sectional distribution of sectoral growth to infer time-varying aggregate and sectoral volatilities. This evidence complements recent work on tail risk and centrality in production networks, such as Auer, Levchenko, and Sauré, 2019 and Dew-Becker, 2023, by linking tail comovement to time-varying aggregate risk and endogenous network reorganization in a global setting. Another important reference in this literature is Dew-Becker, Tahbaz-Salehi, and Vedolin, 2021, who show that even with homoscedastic sectoral shocks, strong complementarities in a CES production network can generate skewness and time-varying higher-order moments of GDP

The paper is also related to work on aggregation and distortions in production networks, such as Jones, 2011, Baqaee and Farhi, 2019, and Bigio and La'O, 2020, which show how network structure and wedges shape aggregate productivity and the mapping from micro shocks to macro outcomes. I share their focus on distorted networks, but emphasize how time-varying risk and endogenous supplier choice jointly determine both volatility and cross-country comovement in a global production network.

Another important reference in the trade literature is important example is Baqaee and Farhi, 2024, who extend closed-economy production network results on CES production network to open economies and show how changes in expenditure shares can amplify the effects of transportation costs. In contrast, I work with an endogenous Cobb-Douglas production network in which expenditure shares adjust because firms reoptimize their supplier portfolios when the distribution of shocks changes, providing a different way for substitution across goods.

I also build on Caselli et al., 2020, who study how changes in trade costs affect aggregate volatility through risk diversification in the global production network. In their model, inputs from different countries are perfect substitutes, so firms can fully reallocate expenditure away from countries hit by adverse shocks, generating large diversification benefits and potentially reducing aggregate risk. By contrast, my framework uses a Cobb-Douglas production function calibrated to the empirical production network, which implies fixed expenditure shares across suppliers ex post and thus sharply limits ex-post diversification. In addition, whereas diversification in Caselli et al., 2020 is driven by lower trade costs, in my setting the diversification margin is endogenous: lower trade costs induce firms to reorganize their supplier portfolios, trading off insurance against higher average productivity.

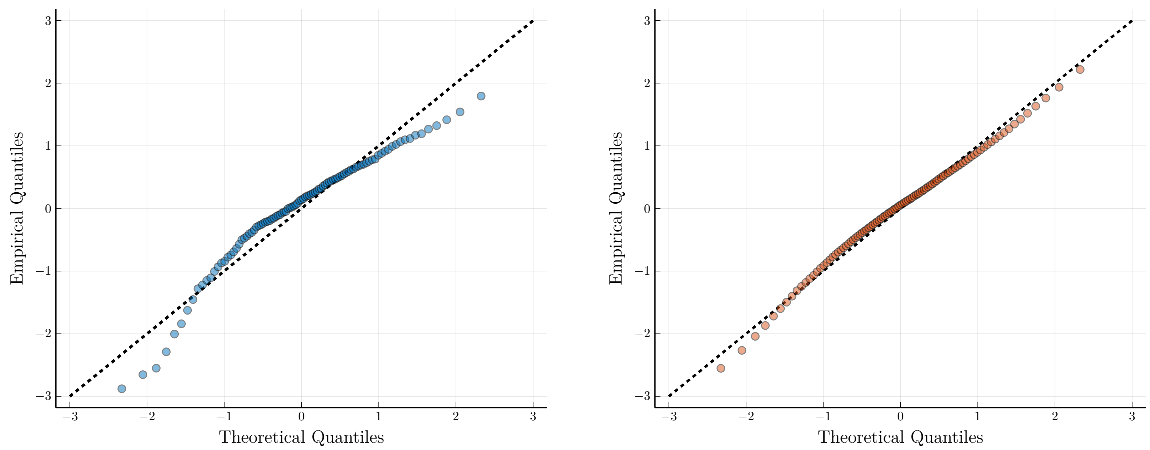
Outline This paper is organized as follows. In Section 2, I present the empirical motivation. In Section 3, I present the model. In Section 4, I study the optimal choice of the network. In Section 5 the effect of aggregate shocks. In addition, in Section 6 I describe the calibration strategy. And in Sections 7, and 8 I describe the quantitative results. Finally, Section 9 presents some conclusions.

2 Empirical Motivation

This section documents an empirical pattern using WIOD data (43 countries, 56 sectors, 2000-2014): sectoral value-added growth is well approximated by a normal distribution, yet aggregate GDP growth exhibits heavy-tailed distribution.

2.1 Distributions.

Figure 1 presents a comparison of the empirical quantiles against normal distributions (Q-Q plots). Panel 1a shows GDP that growth deviates in the tails. Extreme negative events occur far more frequently than predicted by a normal distribution. In contrast, Panel 1b reveals sectoral VA growth aligns closely with normality across all quantiles.



(a) Country GDP growth

(b) Sectoral VA growth

Figure 1: Q-Q plot

2.2 The Global Financial Crisis

Table 1 reveals that the heavy-tailed distribution of GDP growth is mainly a phenomenon of the crisis in 2009.

Table 1: Heavy Tails in GDP Growth: The 2009 Concentration

Metric	**Full Sample** (2001-2013)	**2009 Only** (43 obs)	**Excluding 2009** (516 obs)
Excess kurtosis	1.04	1.085	0.21
Beyond 2σ (%)	5.10	65.12	0.97
Beyond 3σ (%)	0.36	4.65	0.00

Notes: GDP growth standardized using full-sample mean and variance. The 2009 column shows the percentage of 2009 observations exceeding each threshold.

Although the whole sample exhibits substantial excess kurtosis (1.04), removing 2009 observations reduces kurtosis to near-normal levels (0.21). Furthermore, 65% of the countries experienced growth shocks beyond 2σ in 2009, compared to less than 1% in all other combined years.

2.3 Tail Comovement

Following Acemoglu et al., 2017, tail comovement measures the probability that a sector i experiences an extreme negative outcome when global GDP is also in its lower tail:

$$\text{Tail Comovement} = \Pr\left(g_{i,t} < q_i(\tau) \,\middle|\, g_t^{\text{world}} < q_{\text{world}}(\tau)\right), \tag{1}$$

Table 2 reveals massive synchronization during 2009, when global GDP hit its lowest percentile. The key finding: 60% of countries fell into their bottom 5% simultaneously. This synchronization spike explains how normal sectoral distributions generate aggregate fat tails in country GDP.

Table 2: Tail Comovement (2009 Crisis)

Metric	**VA Growth** (2,408 sectors)	**GDP Growth** (43 countries)
Mean percentile	29.0	6.7
In bottom 5%	27.3%	60.5%
In bottom 10%	40.0%	88.4%

Notes: Tail comovement is the share of units in extreme negative outcomes when global GDP is also in its lower tail.

Discussion and Implications This section highlights that fluctuations in GDP growth deviate significantly from what would be expected under a constant-variance Gaussian model. It also shows that heavy-tailed episodes tend to be synchronized global episodes, suggesting time-varying intensity of global risk factors. This observation led me to develop a model that incorporates time-varying variance in aggregate shocks.

3 Environment

There is a set of C countries. In each country, there are J producers that belong to a global sector. In total, there are $N = C \times J$ representative firms that produce differentiated intermediate goods. Each country has a representative firm that combines intermediate goods from all firms (N) into a final good that is sold to a representative household in each country. Finally, each household provides labor L_c, to the J industries within its country c, consumes, and owns the firms.

The timing is as follows. At the beginning of the period, firms update their beliefs about productivity shocks and choose the production technique facing uncertainty about productivity and prices. Next, after shocks are realized, the markets are clear. All firms and households choose the demand and supply of all intermediate and final goods. There are no dynamic choices connecting the equilibrium between periods.

3.1 Households

The representative household in the country c derives its utility from consumption and leisure. Its preferences have the following form:

$$W_c = \sum_t \frac{\left(Z_{c,t} - \sum_i^J \left(L_{ci,t}\right)^{1+\frac{1}{\psi}}\right)^{1-\gamma}}{1-\gamma} \tag{2}$$

I follow Greenwood, Hercowitz, and Huffman, 1988 in defining quasi-linear preferences over consumption and leisure to eliminate any income effect on labor supply. Households choose consumption after uncertainty is realized. Following Heathcote and Perri, 2002, among others, who show that models with financial autarky perform well in accounting for business-cycle comovement, I assume financial autarky. Thus, in each period and for each state of the world, the budget constraint is:

$$P_{c,t} Z_{c,t} \leq \sum_i W_{ci,t} L_{ci,t} \tag{3}$$

$P_{c,t}$ is the price index of the final goods and W_{ci} is the nominal wage. The optimality condition for labor in each sector is given by the following:

$$L_{ci,t} = \psi \frac{W_{ci,t}}{P_{c,t}} \tag{4}$$

Equation (4) gives the standard optimal condition that relates labor supply to the real wage. Finally, I define the price kernel of the household as follows:

$$\Lambda_{c,t} = \frac{u'\left(Z_{c,t} - \sum_i^J \left(L_{ci,t}\right)^{1+\frac{1}{\psi}}\right)}{P_{c,t}}$$

which measures the marginal utility of consumption of households in each state of the world.

3.2 Intermediate Goods Producers

Let $\{c, i\}$ identify the representative firm located in a country $c \in C$ and belonging to the sector $i \in J$. When it does not lead to confusion, I will omit the subscript time t to avoid the cumbersome notation.

Technology The set $\mathcal{A}$ consists of feasible production *techniques*. Following Acemoglu and Azar, 2020, a technique $\eta_{ci} \in \mathcal{A}$ specifies how inputs are to be combined using sector-specific coefficients $\eta_{ci,\hat{c}k}$ in a Cobb-Douglas production function. It also determines a productivity shifter $A_{ci}(\eta_{ci})$. In particular, I assume that the production function has the following form:

$$F_{ci}(\eta_{ci}, Z_c) = e^{A_{ci}(\eta_{ci})\ \epsilon_{ci}} L_{ci}^{\mu_{ci}} \left(\prod_{\hat{c}} \prod_{k} X_{ci,\hat{c}k}^{\eta_{ci,\hat{c}k}}\right)^{(1-\mu_{ci})} \tag{5}$$

where $X_{ci,\hat{c}k}$ is the intermediate input of sector $\hat{c}k$ used in sector ci. L_{ci} is the amount of labor (augmented), and ϵ_{ci} is an exogenous productivity shifter. I impose the following assumptions:

Assumption 1. *The set of production techniques $\mathcal{A}$ lies in the unit simplex:*

$$\mathcal{A} \;=\; \left\{ \eta \in \mathbb{R}_{+}^{NxN} : \sum_{\hat{c}} \sum_{k} \eta_{ci,\hat{c}k} = 1 \right\}.$$

And $A_{ci}(\eta_{ci})$ has the following form:

$$A_{ci}(\eta_{ci}) = -\sum_{\hat{c}} \sum_{k} \kappa \left(\eta_{ci,\hat{c}k} - \eta^{0}_{ci,\hat{c}k}\right)^2$$

These assumptions are both technical and substantial. The first part of the assumption ensures that all coefficients remain positive. The second part guarantees that the composite good has a constant return to scale. In addition, $\eta^{0}_{ci,\hat{c}k}$ is a constant parameter that identifies the technology that maximizes productivity.

Inputs and production In the second stage, given the choice of technology, the firms use a Cobb-Douglas production function. Using the first order conditions of each input, the cost of production of the good ci can be expressed as:

$$K_{ci}\left(\eta_{ci}, P\right) = e^{-\epsilon_{ci} A_{ci}(\eta_{ci})} W_{ci}^{\mu_{ci}} \left(\prod_{\hat{c}} \prod_{k} P_{ci,\hat{c}k}^{\eta_{ci,\hat{c}k}} \right)^{(1-\mu_{ci})} \tag{6}$$

where P is the entire price vector of all firms in the global economy. Equation (6) is the is the standard cost derived from the Cobb-Douglas production function and shows that the cost function is given by the geometric average price of the inputs used in the production multiplied by the productivity of the firm.

Technique choice The optimization problem of the firms in the first stage is to maximize profits using as weights for profits the pricing kernel of the households in different states of the world as follows:

$$\eta^{\star}_{ci} \in arg \max_{\eta_{ci} \in \mathcal{A}} \mathbf{E}\left[\Lambda_c Y_{ci}(P_{ci} - K_{ci}\left(\eta_{ci}, P\right))\right] \tag{7}$$

where Y_{ci} is the equilibrium demand and P_{ci} is the price of production at the factory gate. Because markets are competitive, the representative firm takes Y_{ci}, P_{ci}, and Λ_c as given, so the only term in the optimization problem is the unit cost $K_{ci}\left(\eta_{ci}, P\right)$. As profits are discounted by Λ_c, firms effectively inherit the risk attitude of domestic households.

3.3 Final Goods Producers

Each country has a representative firm owned by the domestic household that produces a country-specific final good bundle that combines intermediate goods. The production function has the following form:

$$F_c(\alpha_c, Z_c) = e^{A_c(\alpha_c)} \prod_{\hat{c}} \prod_k Z_{c,\hat{c}k}^{\alpha_{c,\hat{c}k}} \tag{8}$$

where $\alpha_c \in \mathcal{A}$ and $A_c(\alpha_c)$ have the same form as the endogenous productivity shifter described in the Assumption 2. In addition, $Z_{c,\hat{c}k}$ is the demand for final goods in the country c from goods produced in the country $\hat{c}$ and the sector k. Given a technology choice; the cost function of one unit of the final good is:

$$K_c(\alpha_c, P) = e^{-A_c(\alpha_c)} \prod_{\hat{c}} \prod_k p_{c,\hat{c}k}^{\alpha_{c,\hat{c}k}} \tag{9}$$

Technique choice The optimization problem of the final good firms in the first stage is as follows:

$$\alpha_c^\star \in arg \max_{\alpha_c \in \mathcal{A}} \mathbf{E} \left[\Lambda_c Z_c \left(P_c - K_c(\alpha_c, P) \right) \right] \tag{10}$$

where Z_c is the equilibrium demand for consumption and P_c is the price of the final good. Note that this problem is completely symmetric with respect to the intermediate good producers.

3.4 Iceberg Trade Cost

I assume that there is an iceberg-trade cost and that there is no arbitrage in shipping. It implies that the following formula relates to the price at the factory gate with the price at the final destination of the good:

$$P_{c,\hat{c}i} = P_{\hat{c},\hat{c}i} \tau_{c,\hat{c}i} \tag{11}$$

With $\tau_{c,\hat{c}i} \geq 1$. That is, the price of the good c, i in the country c', is the price of the good in the country where it is produced (c) times the cost of transport. Transportation costs are allowed to be heterogeneous between destination country, sector, and country of origin.

3.5 Stochastic Structure

The source of uncertainty in the model is the TFP supply shocks. The stochastic state of the economy is the collection of productivity shifters for all intermediate good producers, which I summarize as $\epsilon_t = \{\epsilon_{11,t}, ..\epsilon_{CJ,t}\}$.

Assumption 2. *The element ci of the productivity vector ϵ_t is:*

$$\epsilon_{ci,t} = g_t + \chi_{c,t} + \zeta_{i,t} + u_{ci,t}$$

where g_t, χ_{ct} and $\zeta_{i,t}$ are independent $AR(1)$ processes characterized by a mean $\theta = \{\theta_g, \theta_c, \theta_i, \theta_{ci}\}$ and variances $\Sigma_\epsilon = \{\sigma_{gt}, \sigma_{ct}, \sigma_{it}, \sigma_{ci}\}$.

Hence g_t is a global component that affects all sectors in the world. $\chi_{c,t}$ is a country-specific component that affects all sectors in the same country, and $\zeta_{i,t}$ is a sector component that affects all products in the same sector. This decomposition follows Caselli et al., 2020, the only change is the inclusion of a global component.

Each shock component k follows a persistent process with the parameter ρ_k. As a result, both the conditional mean and the variance of the shocks are time-varying. In particular, the expected value θ_k in period t depends on the productivity in the previous period.

3.6 Equilibrium

We are ready to define a competitive equilibrium. All firms are competitive, so the price is be equal to the cost. The equilibrium of the economy with the endogenous network is defined as

Definition 1. (Competitive Equilibrium) An equilibrium is a sequence of allocations for intermediate firms $\{Y_{ci}, Z_{ci,\hat{c}k}, \eta_{ci}\}_{i,c}$; a sequence of allocations for final good producers $\{Y_c, Z_{c,\hat{c}k}, \alpha_c\}$; a sequence of allocations for households $\{Z_c, L_{ci}\}_c$; and a sequence of prices $\{P_{ci}, W_{ci}, P_c\}_{ci}$ such that all agents in the economy solve their problem.

3.7 Definitions and Notation

Before describing the solution of the model, I define the statistics of interest and introduce a helpful notation.

Output The nominal output or Gross Domestic Product (GDP) for country c is the sum of the value of the goods produced in the country minus the value of intermediate inputs. This coincides with the total value added earned by producers located in the country:

$$GDP_c = \sum_i \left(P_{ci} y_{ci} - \sum_k \sum_{\hat{c}} p_{ci,\hat{c}k} X_{ci,\hat{c}k} \right) = \sum_{i \in N_c} W_{ci} L_{ci} \tag{12}$$

Following the convention on national accounts, the real GDP of country c is defined as follows:

$$G_c = \sum_i^J \left(\overline{P}_{ci} Y_{ci} - \sum_{\hat{c}} \sum_k \overline{P}_{\hat{c}k} Z_{ci,\hat{c}k} \right) \tag{13}$$

Hence $\overline{P}_{ci}$ is the price of the good ci in a reference year.

Input-Output Matrix The Input-Output matrix (IO) in this economy is the $N \times N$ matrix $\boldsymbol{\eta}$ whose $ci, \hat{c}k$th element is equal to the coefficient of the Cobb-Douglas production function of sector ci that controls the intensity at which it uses input from sector $\hat{c}k$.

Final Goods Matrix The Final Goods Matrix (FG) in this economy is the $N \times N$ matrix $\boldsymbol{\alpha}$ that stores the Cobb-Douglas coefficients in the final goods production function. All columns for sectors that belong to the same country have the same values.

Labor Shares I also define $\boldsymbol{\mu}$ as a diagonal $N \times N$ matrix that stores the labor production shares μ_{ci}.

Domar Weights Let S be a vector $1 \times N$ whose ci th element is equal to the share measured in sales of each sector (the Domar weight) of sector ci in the GDP of the world, in particular:

$$\omega_{ci} = \frac{P_{ci} Y_{ci}}{GDP}. \tag{14}$$

The vector S_c, whose element ci is the local Domar weight, represented by:

$$\hat{\omega}_{c,ci} = \frac{P_{ci} Y_{ci}}{GDP_c}. \tag{15}$$

It will also be useful to define:

$$\Omega_c = \sum_i \omega_{c,ci} \tag{16}$$

$$\hat{\Omega}_i = \sum_c \omega_{c,ci} \tag{17}$$

Where Ω_c is the Domar weight of the country and $\hat{\Omega}_c$ is the Domar weight of the industry. As in Baqaee and Farhi, 2024, the local Domar weight is related to the global Domar weight by equation $\hat{\omega}_{c,ci} = \omega_{ci} \frac{GDP_c}{GDP}$.

4 Optimal Network

In this section, I characterize the solution of the model. I will analyze the model backwards, so I first characterize the equilibrium and covariance structure of the economy given a network. Next, I use those results to characterize the optimal choice of the network. It is useful to define $s_t = \{\eta_t^\star, \alpha_t^\star\}$, which is the aggregate network.

4.1 Equilibrium Given a Network

The analysis in this section is closely related to the extensive literature on trade and comovement; see, for example, Huo et al., 2024 and de Soyres and Gaillard, 2022. From now on I will denote x_{ci} as the log of the variable $X_{ci,t}$. In addition, I will denote $\boldsymbol{x}$ as the vector that contains all the values of x_{ci}.

Domar weighs Let us begin by characterizing the distribution of sales shares (Domar weights) as functions of the equilibrium network as follows.

Lemma 1. *Let $\omega(s)$ be the Domar weights in the global economy. Then $\omega(s)$ is the eigenvector associated with eigenvalue 1, which solves this system:*

$$\omega = \left[\boldsymbol{\mu}\boldsymbol{\alpha}^{\star} + (1 - \boldsymbol{\mu})\boldsymbol{\eta}^{\star}\right]\omega$$

Proof. See Appendix A.1 □

While $\boldsymbol{\mu}\boldsymbol{\alpha}^{\star}$ accounts for revenue coming from sales of final goods, $(1 - \boldsymbol{\mu})\boldsymbol{\eta}^{\star}$ accounts for revenues coming from sales of intermediate good uses. Intuitively, the revenue in one sector increases because other intermediate good producers increase their consumption of its good. Or because households increase their demand for final goods, which in turn will make final good producers increase their expenditures in that sector. Given the Cobb-Douglas structure, the Domar weights do not change with the stochastic state ϵ_t.

Finally, I highlight that in this model the relative size of trade in final and intermediate goods depends on the vector of labor shares μ. When labor weights are high, a large fraction of production is consumed by households, so the relative size of final goods in the economy is high.

Prices I normalize prices so that world GDP equals one and serves as the numeraire. Given the normalization, prices are characterized as functions of the production structure and the vector of productivity shocks as follows.

Lemma 2. *Let $\boldsymbol{p}$ be the logarithmic vector of prices in the economy and let ϵ be the vector of TFP productivity shocks. Then $\boldsymbol{p}$ is characterized by:*

$$\boldsymbol{p}(s,\epsilon) = \mathcal{L}(s)\ (B(s) - \epsilon)$$

Where $B(s)$ depends on the network and $\mathcal{L}(s)$ is the extended Leontief-inverse defined as:

$$\mathcal{L}(s) = \left(I - \boldsymbol{\mu}\frac{\psi}{1+\psi}\boldsymbol{\alpha}^{\star} - (1-\boldsymbol{\mu})\boldsymbol{\eta}^{\star}\right)^{-1}$$

$$B(s) = -A(\eta^{\star}_{ci}) - \frac{\psi}{1+\psi}\,\mu\, A(\alpha^{\star}_{c}) + \text{diag}\Big(\tfrac{\psi}{1+\psi}\left[\mu\alpha^{\star} + (I-\mu)\eta\right]\Big)\ \log\tau + \frac{1}{1+\psi}\,(\log\mu + \log\nu)$$

Proof. See Appendix A.2 □

Lemma 2 characterizes the price of each industry-country pair of the world economy as a function of the network and the realization of the N productivity shocks (ϵ). Whereas the FG matrix ($\boldsymbol{\alpha}$) and the IO matrix ($\boldsymbol{\eta}$) record the *direct* link of one producer to another, the Leontief inverse matrix $\mathcal{L}$ records the *direct and indirect* exposures through the production network. To see this clearly, we can write the Leontief inverse as.

$$\mathcal{L}(s) = I - \left(\boldsymbol{\mu}\frac{\psi}{1+\psi}\boldsymbol{\alpha}^{\star} - (1-\boldsymbol{\mu})\boldsymbol{\eta}^{\star}\right) - \left(\boldsymbol{\mu}\frac{\psi}{1+\psi}\boldsymbol{\alpha}^{\star} - (1-\boldsymbol{\mu})\boldsymbol{\eta}^{\star}\right)^{2} \dots. \tag{18}$$

This matrix is studied by Huo et al., 2024) and extends the classical Leontief inverse, with an additional term $\mu\,\frac{\psi}{1+\psi}\,\boldsymbol{\alpha}$, which comes from the optimality condition of the labor supply (4). It captures how a TFP shock in industry-country pair ci affects wages and therefore costs in all industry-country pairs.

In the Leontief inverse, the expenditure shares in final and intermediate goods are weighted by the elasticity of labor supply ψ, while in the Domar weights, both types of sales enter symmetrically. Intuitively, to measure Domar weights we sum sales of final and intermediate goods, but to measure *transmission* , we must account for how price changes feed into input costs and wages. When labor supply is inelastic, wage responses are muted, and prices of intermediates is the main transmission channel. When labor supply is elastic, changes in final good prices have a stronger effect on wages, so trade in final goods becomes the key transmission channel.

The vector $B(s)$ summarizes how the equilibrium network s tilts prices even in the absence of shocks. The terms $A(\alpha_c^\star)$ and $A(\eta_{ci}^\star)$ capture how costly reallocation of suppliers lowers effective productivity and therefore raises prices in industry-country pairs that deviate more from the ideal technology. The trade cost component, built from $\mu\alpha^\star+(1-\mu)\eta^\star$ and $\log\tau$, makes prices higher in sectors that rely more intensively on expensive foreign inputs. Finally, the $\log\mu$ and $\log\nu$ terms reflect how labor cost shares and market shares shape the pass through of wages and revenues into sectoral prices, bigger sectors and labor intensive sectors relatively more expensive. Finally, it is useful to define:

Corollary 1 (Final-good price index)**.** *For each country c, the (log) price index of the final good is given by*

$$\boldsymbol{p}_c(s,\epsilon) = A(\alpha_c^\star) + \boldsymbol{\alpha}^\star\left(\boldsymbol{p}(s,\epsilon) + \log\tau\right),$$

$p_c(s,\epsilon)$ is the price index of final good, which is the sum of prices adjusted by transportation cost and using as weight the optimal network of each final good producer (α_c). It capture the exposure of each representative households to shocks and changes in the network in other sectors of the global economy.

Real Value Added (VA) Using the Leontief-inverse and the matrix with Domar weights, it is possible to characterize the real VA as follows:

Lemma 3. *The log of the real VA, a_{ci}, is given by:*

$$a_{ci}(s,\epsilon) = \epsilon_{ci} + A(\eta_{ci}^\star) + \mu_{ci}\, l_{ci}(s,\epsilon)$$

Where $l_{ci}(s,\epsilon)$ is the logarithmic of the labor supply of the sector ci, characterized by:

$$l_{ci}(s,\epsilon) = \frac{\psi}{1+\psi}\Big[-p_c(s,\epsilon) + \log\mu_{ci} + \log(\nu_{ci})\Big]$$

Proof. See Appendix A.3 □

Lemma 3 decomposes the value added into productivity and labor supply. In turn, endogenous labor supply has a stochastic component, which depends on the vector of productivity in all sectors, and a deterministic component. The stochastic part is driven by the transmission of TFP shocks through wages: a productivity shock in any sector (c, i) changes the price index, the real wage and thus affects the labor supply and VA in all other sectors. This channel, first emphasized by Backus, Kehoe, and Kydland, 1992, is what generates international comovement.[1] The strength of the response of domestic price and real wage to TFP depends on the exposure of the final good firm, summarized by $\alpha_c^\star$ to each good: shocks to goods that are heavily used in domestic final production have a higher impact on domestic real wages.

The deterministic part depends on: (i) sales shares (Domar weights), larger sectors employ more labor in equilibrium; (ii) the deterministic component of final good prices, which reflects the expenditures of final good producers on trade costs; and (iii) labor shares, since sectors with higher labor intensity employ more labor in equilibrium. Changes in the production network impact the deterministic component of labor across sectors. Analyze this effects is the main objective of the paper.

Centrality An important concept for the analysis is the centrality of each producer in the network.

Definition 2 (Centrality)**.** The sectoral propagation centrality of (c, i) is

$$\ell_{ci} = \sum_{\hat{c}k} \mathcal{L}_{ci,\hat{c}k},$$

and the exposure of country c is

$$\ell_c = \sum_{\hat{c}k} \alpha^\star_{c,\hat{c}k}\, \ell_{\hat{c}k}.$$

where ℓ_{ci} is the row sum of the Leontief inverse, measuring the Bonacich-type centrality of sector (c, i) in the production network. In addition, $\ell_c = \sum_{\hat{c}k} \alpha^\star_{c,\hat{c}k}\, \ell_{\hat{c}k}$ is a measure of how final good producers in country c are exposed to central suppliers.

With endogenous labor supply, the Domar weight is no longer the relevant measure of centrality, in contrast to Acemoglu et al., 2012 and related work. This result is connected to Baqaee and Farhi, 2024, who show that revenue based transmission (Domar weights) generally differs from cost based transmission in economies with wage determination.

[1] See also Huo et al., 2024, who characterize deviations from steady-state value added (VA) using the Leontief inverse in a model with CES production.

Here, the divergence arises even in efficient economies once labor supply is endogenous.

I now characterize the pricing kernel of the households as a function of the network and shocks.

Lemma 4. *Let Λ_c be the pricing kernel of the representative household in country c. Then:*

$$\log \Lambda_c(s, \epsilon) = (\gamma - 1)\, p_c(s, \epsilon) + C_c(s),$$

where

$$C_c(s) = -\gamma\Big(1 + \frac{1}{\psi}\Big)\log\Big(\sum_i (\mu_{ci}\omega_{ci}(s))^{\frac{\psi}{1+\psi}}\Big) + \gamma \log \psi$$

Proof. See Appendix A.6 □

Lemma 4 shows that, as in Kopytov et al. (2024), the pricing kernel depends on the final good price $p_c(s, \epsilon)$ through its effect on consumption. However, with endogenous labor supply under GHH preferences, we obtain an additional term $C_c(s)$ that depends on the network structure through the interaction of equilibrium Domar weights $\omega_{ci}(s)$ and labor shares μ_{ci}. This term depends on the network but is independent of productivity shocks ϵ.

With endogenous labor supply, network reorganization generates two distinct effects. First, it changes the level of the stochastic discount factor through its impact on C_c. Second, it changes the covariance between the discount factor and prices by altering how p_c is exposed to shocks. Importantly, the additional effect of endogenous labor supply on the pricing kernel operates only through the level of the kernel and is independent of shocks. This separation will be important for understanding the optimal choice of the network.

Expectations Given the characterization of the equilibrium in all states ϵ, we are ready to study the expectations. The key object of interest is the distribution of the prices, as they will be the main input to solve the optimal network.

Lemma 5. *Let $\epsilon_t \sim \mathcal{N}(\theta_t, \Sigma_{\epsilon,t})$. Then $p_{ci}(s, \epsilon)$ is normal distributed with mean and variances given by:*

$$\mathbf{E}\left[\mathbf{p}(s,\epsilon)\right] = \mathcal{L}(s)\,\left(B(s)-\theta\right)$$

$$\mathbf{V}\left[\mathbf{p}(s,\epsilon)\right] = \mathcal{L}(s)^T\,\Sigma_\epsilon\,\mathcal{L}(s)$$

Proof. See Appendix A.4 □

The lemma 5 relates the first and second moments in the price distribution to the shock distribution and the network. One critical property of the prices is that they are log-normally distributed. It requires the Cobb-Douglas production function and normal shocks.

Now I provide a decomposition of the price distribution into shocks and the network. To develop intuition, I focus on the case in which productivity shocks have only two components, global and product specific component:

Corollary 2. *Let $\epsilon_{ci} = g_t + u_{ci,t}$. Then the distribution of prices is characterized by:*

$$Cov(p_{ci}, p_{\hat{c}k}) = \underbrace{\sigma_g^2 \ell_{ci}\ell_{\hat{c}k}}_{Aggregate} + \underbrace{\sum_{c'=1}^{C}\sum_{j=1}^{J} \sigma_{c'j}^2 \mathcal{L}_{ci,c'j}\mathcal{L}_{\hat{c}k,c'j}}_{Product\text{-}specific}$$

$$\mathbf{E}\left[p_{ci}(s,\epsilon)\right] = \mathcal{L}(s)_{[ci]} B(s)_{[ci]}^T - \underbrace{\theta_g\,\ell_{ci}}_{Aggregate} - \underbrace{\sum_{c'=1}^{C}\sum_{j=1}^{J} \theta_{c'j}\mathcal{L}_{ci,c'j}}_{Product\text{-}specific}$$

Changes in aggregate risk disproportionately increase the covariance of prices in central sectors, as measured by $\ell_{\hat{c}k}$. This is reminiscent of Acemoglu et al., 2012, where central sectors play a special role in shock transmission. The key difference here is that, because the price distribution of central sectors is more exposed to aggregate shocks, changes in the mean θ_g or variance σ_g of the aggregate component have a stronger impact on these sectors in the first stage as I will describe in the next.

4.2 Optimal Production Networks

Firms in both both final and intermediate sectors solve analogous optimization problems. Given the aggregate state s, they choose a supplier vector $s \in \mathcal{A}$ to minimize the expected discounted production costs. As shown by Kopytov et al., 2024, using the domestic pricing kernel of the households, their problem can be written as

Proposition 1 (Risk-Adjusted Prices). *Let $m = \alpha_c$ for the final good producers and $m = \eta_{ci}$ for the intermediate-good producers. In equilibrium, their choice can be written as*

$$m^\star \in \arg\max_{s\in\mathcal{A}} \; - \sum_{\hat{c},k} \kappa_{\hat{c}k} (m_{\hat{c}k} - m^0_{\hat{c}k})^2 \; - \; \sum_{\hat{c},k} m_{\hat{c}k} \, R_{\hat{c}k}(s),$$

where the risk-adjusted adjusted price of supplier $(\hat{c}, k)$ is

$$R_{\hat{c}k}(s) = E[p_{\hat{c}k}(s,\epsilon)] + \log \tau_{\hat{c}k} + (\gamma - 1) Cov(p_c(s,\epsilon), p_{\hat{c}k}(s,\epsilon)),$$

and p_c is the domestic final-good price index.

Proof. See Appendix A.6. □

The risk-adjusted cost of supplier $(\hat{c}, k)$ combines its expected log price $E[p_{\hat{c}k}]$, trade costs $\log \tau_{\hat{c}k}$, and a risk term $(1 - \gamma)\,\mathrm{Cov}(p_c, p_{\hat{c}k})$ that measures its insurance value for domestic producers. As shown in the proof of Proposition 1, consumption in country c is a function of p_c, so firms prefer suppliers whose prices covary negatively with the domestic price index.

Proposition 1 nests the risk-adjusted cost index in Kopytov et al., 2024. With a single country and no trade costs ($\tau_{\hat{c}k} = 1$), $R_{\hat{c}k}(s)$ reduces to their risk-adjusted price[2]. The first contribution here is the presence of $\log \tau_{\hat{c}k}$, which introduces a standard gravity channel: firms prefer domestic sectors or geographically close trading partners. The second contribution lies in the distribution of prices: in a global multi country network, the covariance term $\mathrm{Cov}(p_c, p_{\hat{c}k})$ reflects cross country heterogeneity, since it depends on each country's equilibrium final good network $\alpha_c^\star$.

I highlight that the structure of the risk-adjusted prices closely follows Kopytov et al.

[2]Under $\tau = 1$, Corollary 1 and Lemma 4 imply $R_{\hat{c}k}(s) = E[p_{\hat{c}k}(s,\epsilon)] + (\gamma - 1)\mathcal{L}(s)^{\mathrm{T}}\Sigma_\epsilon \mathcal{L}(s)\alpha_c^\star$. With a single country and fixed input shares in consumption (fixed $\alpha^\star$), this reduces to Lemma 2 in Kopytov et al., 2024.

(2024) despite having endogenous labor supply. The intuition of the result lies in 4, were I showed that the effect of changes in labor supply on the pricing kernel of households is deterministic in my model. As a result the insurance motives in the choice of the network remain unchanged compared to the case with exogenous labor supply (Kopytov et al. (2024)).

Lemma 6 (Optimal Network)**.** *Let $m \in \{I, F\}$. Given quadratic adjustment costs, the optimal network has the closed form*

$$m_{\hat{c}k} = \max\left\{m^0_{ck} - \frac{1}{2\kappa^m}\left(\lambda^m - R_{\hat{c}k}(s)\right),\ 0\right\},$$

where the multiplier λ^m is

$$\lambda^m = \frac{\sum_{(\hat{c},k)\in N^s} R_{\hat{c}k}(s)}{N^s}$$

ensures $\sum_{\hat{c},k} s_{\hat{c}k} = 1$ and $N^s = \{(\hat{c}, k)\ :\ s_{\hat{c}k} > 0\}$ determines the set of active suppliers.

Proof. See Appendix A.7. □

Lemma 6 shows that equilibrium networks solve a constrained linear problem. The max operator captures the non-negativity constraint, while λ^m enforces constant returns to scale. The resulting network is a fixed point: risk adjustments $R_{\hat{c}k}$ depend on the optimal networks, and the optimal networks depend on these same risk adjustments. In the quantitative section I describe how I use Lemma 6 to solve the competitive equilibrium.

In the first stage, all inputs are substitutes due to a constant-returns-to-scale constraint ($\sum_{\hat{c},k} m_{\hat{c}k} = 1$) acting as a portfolio restriction, meaning an increase in one supplier's share decreases others. However, the degree of substitutability is endogenous, depending on whether suppliers are actively used, which depends on trade costs and the covariance structure of prices which in turn depends on the optimal network (s). As a result, substitution patterns in the first stage are state-dependent. This differs from CES preferences where substitutability remains constant regardless of the economic environment.

4.3 Existence and Uniqueness

Next, we study the existence and uniqueness. I prove that the competitive equilibrium exists and establish conditions on the set of parameters for uniqueness. Kopytov et al., 2024 provide a proof of uniqueness and existence for an efficient equilibrium using the planing problem. This approach is not feasible as I have incomplete markets and there is not planner representation for the equilibrium. Instead in this section I write the competitive equilibrium as a static game and establish existence and analysis conditions for uniqueness.

Proposition 2 (Existence and Uniqueness)**.** *Let $\kappa^F > 0$ and $\kappa^I > 0$, and for any vector x, let $\bar{x} := \max x$ and $\underline{x} := \min x$. Then the equilibrium $s^*, p(s^*, \epsilon)$ exists. Moreover, $s^*, p(s^*, \epsilon)$ is unique if*

$$\underline{\kappa} \;>\; \frac{(1+\psi)^2}{\underline{\mu}^3}\sqrt{(1-\underline{\mu})^2 + \left(\frac{\psi\,\bar{\mu}}{1+\psi}\right)^2}\left[2(\gamma-1)(1+\psi)\,\bar{\sigma}^2 \;+\; (\bar{\theta}+\bar{B})\,\underline{\mu}\right]$$

where

$$\bar{B} := \frac{\psi}{1+\psi}\,\bar{\mu}\,\bar{A} + \frac{\psi}{1+\psi}\,\log\bar{\tau} + \frac{1}{1+\psi}\big(\log\bar{\mu} + \log\bar{\nu}\big),$$

Proof. In Appendix B □

In Appendix B I characterize existence and provide sufficient conditions for uniqueness of equilibrium. The argument is cast the equilibrium as a fix pint between the aggregate network and the best response of all firms. Existence follows from a standard fixed-point argument: the strategy space is a product of simplexes ($\mathcal{A}$), also prices $p(s, \epsilon)$ is uniquely determined by the Leontief inverse, and convexity from quadratic adjustment costs makes each firm's best response continuous. Brouwer's theorem then guarantees the existence of at least one fixed point s^*.

The more involved part of the proof is uniqueness. Changes in η and α shift the covariance structure affecting the risk-adjusted prices, which in turn induces other producers to adjust their networks. These general equilibrium (GE) feedbacks can generate multiple equilibria. In contrast, quadratic adjustment costs κ push firms back toward their ideal technologies (α^0, η^0), weaken complementarities across producers, and tilt the economy toward uniqueness. The condition in Proposition 2 places an upper bound on GE feedbacks by measuring how much risk-adjusted costs can move when the aggregate network changes. If the adjustment cost parameters exceed this bound, the best response map from the aggregate network to firms' optimal choices is a contraction, and the equilibrium is unique[3].

4.4 Macroeconomic Outcomes

Having characterized the equilibrium network and prices, I now define the two macroeconomic outcomes central to my analysis: aggregate GDP growth, and welfare. These objects measure how shocks propagate through the global production network and affect country level outcomes.

Real GDP growth First, the real GDP growth ($\Delta \log(GDP)$) for each country.

Lemma 7. *Let $\hat{\omega}_{ci}(s)$ be the local Domar weight and $g_{ci,t} = a_{ci,t} - a_{ci,t-1}$ be the growth in VA of sector ci. Then GDP growth is given by:*

$$g_{c,t}(s, \epsilon) = \sum_i \omega_{ci,t-1}(s) \left(\Delta \epsilon_{ci,t} + \Delta A(\eta^{\star}_{ci,t}) + \mu_{ci} \, \Delta l_{ci,t} \right)$$

Proof. See Appendix A.5 □

Lemma 7 aggregates the growth of VA in each sector into GDP growth. It uses the base year Domar weights $\omega_{ci,t-1}(s)$ to aggregate sectoral VA into GDP. The first two terms inside the brackets provide a Hulten aggregation of changes in exogenous and endogenous TFP, while the last term captures how the equilibrium network and the transmission of shocks to other sectors in the network generate growth through the endogenous labor supply into aggregate growth.

Welfare Next, I define the welfare measure for the representative household in country c in terms of the optimal network.

Lemma 8. *At each period t welfare in country c is:*

[3] See, Morris and Shin, 2002 for another example of games with linear quadratic payments where it possible to establish boundary conditions on complementarities across players so that the equilibrium is unique.

$$W_c = b + \left(1 + \frac{1}{\psi}\right) \log \left(\sum_i (\omega_{ci}(s)\mu_{ci})^{\frac{\psi}{1+\psi}} \right) \; - \; \mathbb{E}[p_c(s,\epsilon)] \; + \; \frac{1}{2}(1-\gamma) \; \mathrm{Var}\big(p_c(s,\epsilon)\big)$$

where b is a constant.

Proof. See Appendix A.8. □

Welfare in country c has an expected-consumption component and a risk component. By Lemma 8, equilibrium consumption can be written as a function of the price of the final good p_c and the interaction between Domar weights and labor shares. By Lemma 3, changes in the network affect labor supply by shaping the structure of the price index of the final good and the relative size of the domestic sectors in the world economy. As a result those two variables determine how the network affect expected consumption in each country. In addition, the welfare cost of uncertainty is fully summarized by $\mathrm{Var}(p_c)$ as it pin down the variance of consumption and labor. In particular higher price volatility due to network adjustment reduces welfare when $\gamma > 1$.

5 Macroeconomic Impact of Uncertainty

Now we are ready to study the main question of this paper. How does the endogenous respond of the network shape the effect of aggregate shocks? In this section I will study the impact on real VA of two shocks. Changes to the aggregate productivity g_t and changes in the variance of aggregate shocks σ_{gt}.

5.1 Aggregate TFP

I start by analyzing the effect of changes on the global which can be summarized as follows:

Proposition 3 (Total Effect on Value-Added: Current and Future)**.** *An aggregate TFP innovation θ_g at time t affects sector (ci)'s value-added as follows:*

1. ***Contemporaneous effect (period*** t***):***

$$\frac{\partial a_{ci,t}}{\partial g_t} = 1 + \mu_{ci}\frac{\psi}{1+\psi}\ell_c$$

2. ***Lagged effect (period*** $t+n$***):***

$$\frac{da_{ci,t+1}}{dg_t} = \rho_g^n \left[\underbrace{1 + \mu_{ci}\frac{\psi}{1+\psi}\ell_c}_{\textit{Direct Effect}} + \underbrace{\sum_{\textit{all } (c'i',c\hat{k})} \frac{\partial a_{ci,t+1}}{\partial s_{c'i',c\hat{k}}} \frac{ds^*_{c'i',c\hat{k}}}{dg_t}}_{\textit{Network Adjustment}} \right]$$

Proof. See Appendix C □

Proposition 3 reveals the differential impact of aggregate TFP shocks on current versus future VA. When aggregate productivity increases at time t, the contemporaneous effect consists of the direct productivity impact plus standard transmission through input-output linkages. As in Huo et al., 2024, the transmission operates through changes in the price of final goods. The effect of an aggregate shocks on the price index of each country depends by the exposure of the final good producer to central sectors measured by ℓ_c.

The effect on future VA at $t+n$ contains two new components. First, the direct persistence effect mirrors the contemporaneous response, scaled by persistence of the shock (ρ_g^n). Second, and crucially, future VA incorporates an additional margin of adjustment: the endogenous network reorganization, which I characterize in the next proposition:

Proposition 4 (Network Adjustment)**.** *An aggregate TFP innovation g at time t induces the following first-order adjustments in the network chosen for period $t+1$:*

$$\frac{ds^*_{ci,c\hat{k},t+1}}{dg_t} = \frac{\rho_g}{2\kappa_I}(\ell_{c\hat{k}} - \bar{\ell}_{ci})\left(1 - \ell_c\frac{\psi}{1+\psi}\right)$$

where $\bar{\ell} = \frac{1}{N^m}\sum_{\hat{c}k' \in L^m} \ell_{\hat{c}k'}$ is the average centrality among active suppliers.

Proof. See Appendix C □

Proposition 4 establishes how aggregate TFP shocks induce network reorganization based on centrality. The key result is that central sectors, those with $\ell_{c\hat{k}} > \bar{\ell}$ in the initial equilibrium, lose market share following negative aggregate productivity shocks, while peripheral sectors gain share. This occurs because central sectors' prices are more sensitive to aggregate shocks: their expected prices increase more when aggregate TFP decreases, making them relatively less attractive.

The network adjustment represents a different margin of substitution from the contemporaneous effect. While the immediate response operates through ex-post Cobb-Douglas substitution with unit elasticity, the network reorganization reflects ex-ante portfolio optimization under uncertainty. The elasticity of this substitution in the first stage is governed by the adjustment cost parameters κ_I and κ_F: lower adjustment costs enable stronger reallocation toward central sectors.

The adjustment magnitude depends critically on the buyer's position through $(1-\ell_c \frac{\psi}{1+\psi})$, which captures how aggregate shocks affect expected consumption and risk sensitivity. Peripheral buyers (small ℓ_c) substitute away from central suppliers whose prices increase most. Buyers highly exposed to central producers (large ℓ_c) experience lower expected consumption from negative shocks, making their marginal utility less sensitive to volatility. When $(1 - \ell_c \frac{\psi}{1+\psi}) < 0$, they exhibit substitution toward central suppliers despite higher expected prices, accepting greater risk exposure to maintain productivity. At $\ell_c = \frac{1+\psi}{\psi}$, productivity costs exactly offset risk benefits, yielding no adjustment.

5.2 Aggregate Risk

I now analyze the aggregate risk σ_g^2. In the next proposition I effects of changes in aggregate risk on VA:

Proposition 5 (Effect of Aggregate Risk on Value-Added)**.** *A change in aggregate risk σ_g^2 affects sector (ci)'s value-added as follows:*

$$\frac{da_{ci,t}}{d\sigma_g^2} = \sum_{(c'i',c\hat{k})} \frac{\partial a_{ci}}{\partial s_{c'i',c\hat{k}}} \cdot \frac{ds^*_{c'i',c\hat{k}}}{d\sigma_g^2} \tag{19}$$

Proof. See Appendix C □

Proposition 5 shows that changes in aggregate risk affect value-added exclusively through network reorganization, with no direct productivity effect. The effect of an increase in the volatility of the aggregate shock has a contemporaneous effect as I assumed producers observe the new risk level when choosing their networks at the beginning of period t,

Proposition 6 (Effect of Aggregate Risk)**.** *The first order effect of a change in aggregate risk σ_g^2 on expenditure shares is:*

$$\frac{\partial m_{ci,\hat{c}k}}{\partial\, \sigma_{gt}} = -\frac{(\gamma-1)}{2\kappa_I}(\ell_{c\hat{k}} - \bar{\ell}_c)\ell_c \frac{\psi}{1+\psi}$$

where $\bar{\ell}_c = \frac{1}{N^m}\sum_{\hat{c}k' \in L^m} \ell_{\hat{c}k'}$ is the average centrality among active suppliers.

Proof. See Appendix C □

Proposition 6 reveals a "flight to safety" mechanism in production networks. When aggregate risk increases, sectors with above-average centrality ($\ell_{\hat{c}k} > \bar{\ell}_c$) lose market share. This occurs because central sectors' prices exhibit stronger covariance with aggregate shocks, making them riskier inputs. The adjustment intensity depends on the buyer's exposure ℓ_c: central buyers (large ℓ_c) show strong flight to safety.

This result relates to Kopytov et al., 2024, who show that sectors with higher variance in their country-industry pair specific productivity lose market share. Instead, I show that central sectors endogenously become riskier under higher aggregate uncertainty: their higher exposure to aggregate risk through the production network makes them unattractive when aggregate volatility rises. This mechanism combines the network centrality insights of Acemoglu et al., 2012 with ex-ante portfolio optimization under uncertainty. This insight provides a new testable implication of the model, during periods were aggregate risk increases the market shares of central sectors of the network should go down. Next, I will calibrate the model and test this new mechanism.

6 Quantitative Methodology

In this section, I describe the calibration strategy. I first discuss the calibration strategy. There are three sets of parameters.

6.1 External Parameters

The first group is the parameters that I take directly from the data. Those are the trade cost and the sector-specific labor shares. In the last part of the calibration, estimate the risk aversion of households and the adjustment cost. For the analysis, I used WIOD data for 43 countries and 56 industries from 2000 to 2014.

Trade costs (τ) are obtained from the ESCAP-World Bank Trade Cost Database, which provides bilateral, sector-specific estimates derived from gravity equations based on a CES framework. While my model implies alternative gravity relationships in the first stage, I use these estimates as a approximation that allows me to isolate the effects of changes in the network due to risk.

6.2 Stochastic Structure

In this subsection, I describe the estimation of the productivity shocks. The main innovation is an estimation strategy that uses the percentile distribution of shocks, building on the tail-comovement concept in Acemoglu et al., 2017, to identify changes in aggregate risk.

Productivity Shocks The next set of parameters is the distribution of supply shock. I align the network with average trade flows and use both the model to estimate shock distribution. I start by estimating productivity shocks that explain observed VA, following Huo et al., 2024. In particular I invert Lemma 3, which characterizes VA in terms of productivity shocks given at the empirical network, to estimate productivity shock every period.

Common Factors The next step is to perform the decomposition of the shocks. In this step I follow Caselli et al., 2020. In particular, I estimate the following factor structure.

$$\epsilon^D_{ci,t} = g^D_t + \xi^D_{c,t} + \omega^D_{i,t} + \iota^D_{ci} \tag{20}$$

Next, I use OLS to estimate to find the persistence coefficient in the $AR(1)$ process for each factor. Armed with the factors estimated with this decomposition, we now proceed to estimate the time-changing variance of each factor.

Estimating Time-Varying Aggregate Risk For expositional clarity,I describe the estimation procedure in the simplest setting with a single global factor. This one-factor case conveys the main intuition behind how changes in the variance of the aggregate shock are inferred from the data. The general procedure, which allows for multiple factors (global, country, and sectoral) is presented in the Appendix D.

In this case $\hat{\epsilon}_{i,t} = \hat{g}_t + \hat{u}_{ci,t}$ denotes innovation in the AR process of the country-industry pair ci at time t. The total variance of $\hat{\epsilon}_{i,t}$ is $\sigma^2_{i,t} = \sigma_g(t) + \sigma_i$. Define the conditional probabilities as:

$$R^2_{ci,t} \equiv \frac{\sigma_g(t)}{\sigma^2_{ci}}, \qquad z_{g,t} \equiv \frac{g_t}{\sqrt{\sigma_g(t)}}.$$

Where $R^2_{ci,t}$ relative importance of the aggregate variance $\sigma_g(t)$ compared to the total variance $\sigma^2_{i,t}$. In addition, $z_{g,t}$ is the standardized global factor. Then, conditional probability implied by the normal distribution that $\epsilon_{i,t}/\sigma_{i,t}$ falls below z_τ is

$$B_t^{\text{mod}}(q;\sigma_g(t)) = \frac{1}{N}\sum_{i=1}^{N} \Phi\left(\frac{z_\tau - \sqrt{R^2_{g,i,t}}\, z_{g,t}}{\sqrt{1 - R^2_{g,i,t}}}\right). \tag{21}$$

Note that these theoretical probabilities depend only on the ratio $R^2_{ci,t}$ and the realization of the global factor g_t. If the common component becomes more volatile, more country-industry pairs fall into the lower quantiles simultaneously, raising $B_t^{\text{mod}}(\tau)$ for small τ.

Empirical quantile probabilities. Using estimated shocks, and for a grid of quantile levels $\mathcal{T} = \{\tau_k\}$, I compute the empirical quantile probability as follows:

$$B_t^{\text{emp}}(q) \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\frac{\epsilon_{i,t} - \rho\, \epsilon_{i,t-1}}{\sigma_{i,t}} \leq z_q\right\}. \tag{22}$$

This statistic captures the share of standardized shocks below each quantile level in the period t.

Identification of time-varying variance. Because (21) depends solely on variance ratios, we can estimate $\sigma_g(t)$ by matching the empirical and average model-implied moments across quantiles and periods. Specifically, $\sigma_g(t)$ minimizes the squared distance:

$$\min_{\sigma_g(t)>0} \sum_{q\in\mathcal{T}} \left[B_t^{\text{emp}}(q) - \overline{B}_t^{\text{mod}}(q;\sigma_g(t))\right]^2 \tag{23}$$

where $\overline{B}_t$ are the average conditional probability implied by the normal distribution. Intuitively, the optimal $\sigma_g(t)$ is the value of the global variance that makes the implicit distribution of the standardized shocks of the model best match the empirical distribution in that period.

This procedure provides a direct distribution based measure of the aggregate risk that varies over time. When the estimated $\sigma_g(t)$ increases, the probability mass in the lower quantiles increases, reflecting periods of high global uncertainty. This procedure relates to Chen, Dolado, and Gonzalo, 2021 and Bonhomme and Robin, 2009 who use the entire distribution to estimate the key moments of the distribution. Compared to standard methods such as GARCH, this methodology leverages cross-sectional information for inference, which is particularly useful for this data set. The dimension of the global factor's T consists of only observations $T = 14$, which complicates the estimation of a variance that changes over time. In contrast, the proposed method uses $N = 2408$ observations, providing a more robust estimate. They key intuition is that comparing the distribution across sectors is informative about the variance of an aggregate factor as in an economy where the aggregate factor is the driver of volatility all sectors will get productivity in their lower quantile when aggregate risk is also in a lower quantile.

The same logic of this derivation extends naturally to the multi-factor setting described in Appendix D, where country and sectoral components are introduced alongside the global variance. The additional step is that we need to guess the variance of all country and sector factors, estimate the global variance variance and update until we find the fixed point.

6.3 Calibration of risk aversion and adjustment costs

The key parameters governing network formation are the household risk aversion γ and the adjustment costs κ^I and κ^F for intermediate and final producers. I discipline these parameters by fitting the model-implied relationship between input shares and the distribution of prices. Appendix F.1 provides details the derivation.

Using Lemma 6, the optimality condition for input choice implies that, for a given buyer ci (a country–industry) and any supplier $c'j$ in network $\tau \in \{I, F\}$, the excess share of $\hat{c}k$ relative to a reference supplier $c'j$ can be written as

$$s_{ci,\hat{c}k} - s_{ci,c'j} = a_n^\tau - \frac{1}{2\kappa^\tau}\Big(\mathbb{E}[p_{\hat{c}k}] - \mathbb{E}[p_{c'i}]\Big) - \frac{1-\gamma}{2\kappa^\tau}\Big(\text{Cov}(p_c, p_{\hat{c}k}) - \text{Cov}(p_c, p_{c'j})\Big), \quad (24)$$

Equation (24) shows that excess shares depend on relative risk-adjusted prices, with two components: differences in expected prices and differences in covariances with the buyer's state.

In the data, I compute for each buyer-supplier pair the *excess expected price* and *excess covariance* relative to that buyer's reference supplier, and regress the excess input share $(s_{ci,\hat{c}k} - s_{ci,c'j})$ on these two moments. I pool intermediate and final networks, allowing the coefficient on excess expected prices to differ between them (via dummies for intermediate and final producers), while imposing a common coefficient on excess covariances and including buyer fixed effects. Comparing the estimated coefficients to the model implied link between excess shares and relative risk-adjusted prices yields closed form expressions for κ^I, κ^F, and γ.

The baseline calibration implies $\gamma = 5.8$, $\kappa^I = 365$, and $\kappa^F = 189$. This strategy parallels the elasticity estimation in Huo et al. (2024) and related work, which identifies CES elasticities by regressing log changes in expenditure shares on log changes in relative prices. Unlike CES, my model's expenditure shares depend on the price distribution rather than realized prices. As a result, κ^I and κ^F capture the curvature of network adjustment around the ideal technology, while γ governs how strongly buyers shift away from suppliers whose prices covary with their own state.

7 Baseline Results

Parameters In table 3, I present some statistics for the baseline parameters. I report the 10th and 90th percentiles of the distribution of the shocks, transportation cost and labor shares. The other parameter that is externally calibrated is ψ which I set equal to 1 which is a standard value in the literature.

Appendix F describes the numerical algorithm to solve the model. Following the iterative approach common in quantitative trade models, we guess the equilibrium networks $\alpha^\star, \eta^\star$, compute the implied prices and risk-adjusted costs $R(s)$, then update the networks until convergence. The key computational challenge compared to standard CES models is that $R(s)$ depends on both expected prices and price covariances. Importantly, the existence and uniqueness conditions derived in Proposition 2 ensure that this iterative procedure converges to the unique competitive equilibrium.

Table 3: Parameters

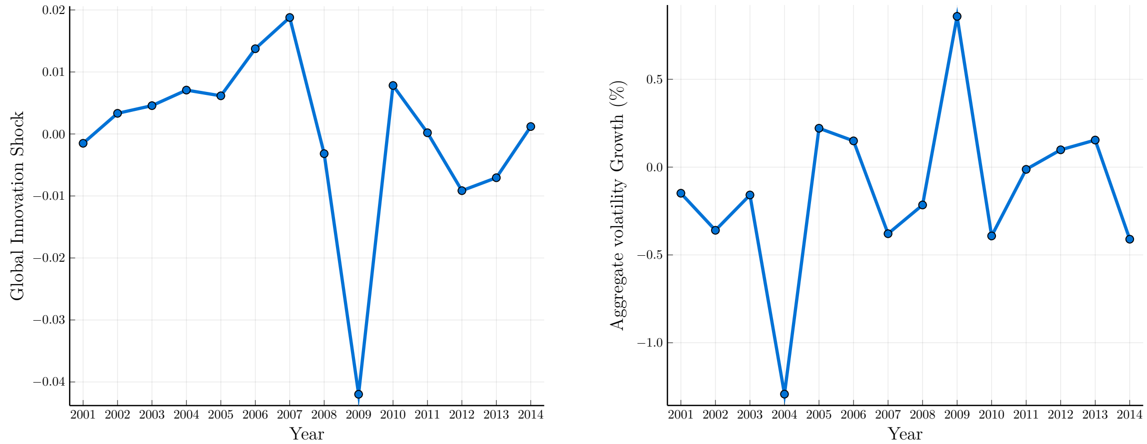
Parameter	Value	Source	Related to
External			
ψ	1.0	Standard	Frisch elasticity
μ_j	[0.2, 0.68]	WIOT	Labor shares
τ	[1.0, 3.2]	ESCAP/WB	Trade cost
Calibrated			
κ^I	365	–	Adjustment cost - Intermediate
κ^F	189	–	Adjustment cost - Final
γ	5.8	–	Risk aversion

Notes: The table reports the 10th and 90th percentiles of the range.

7.1 Aggregate Risk

I now present the result of the estimation of the aggregate shock. Figure 2 reports the estimated realization and growth in variance of the global factor's innovations. The level exhibits a pronounced trough in 2009 with a swift recovery thereafter, while the variance spikes during 2009 and subsequently mean-reverts towards pre-crisis values. This joint pattern indicates that the crisis can be understood as a temporary surge in aggregate uncertainty and a temporary drop in aggregate productivity.

These estimates extend Kopytov et al., 2024, who document increased volatility during the financial crisis for the United States, to a global setting. I find that aggregate risk increased across all countries in the sample, with the variance of the global component rising by 0.8% in 2009. The decomposition shows that there are differences in risk changes across different countries and sectors. Some countries experience additional increases in the variance of their productivity.



(a) Level (b) Variance

Figure 2: Global Factor

Figure 3 describes how the estimation works using conditional percentile probabilities. In the data, the conditional probability that the productivity of a given sector is in the left tail ($q = 1$) increases in 2009. In order to rationalize the spike in the conditional probability under normal distribution, the estimation identifies a high variance of the global component in 2009. This captures the idea that changes in tail comovement measured by the conditional quantile probabilities help to identify changes in the variance of the aggregate component.

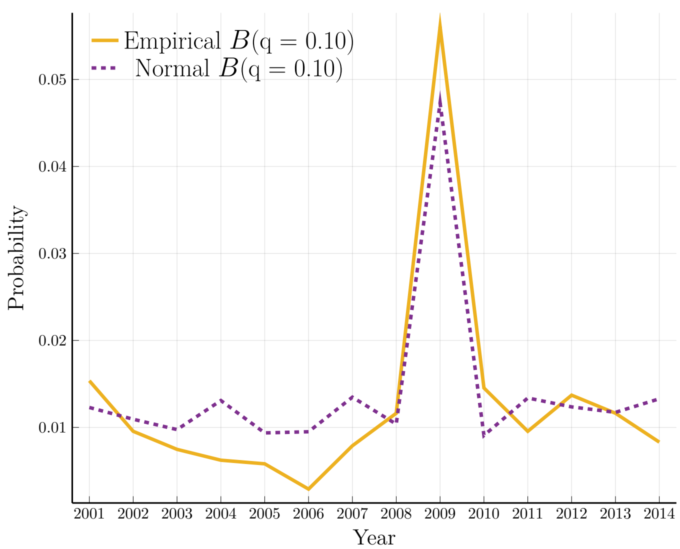
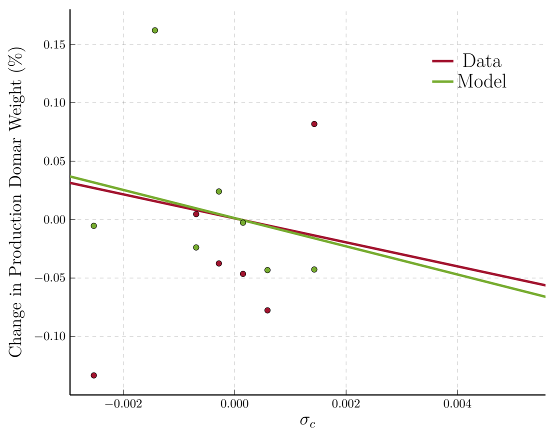


Figure 3: Conditional probability (q= 1)

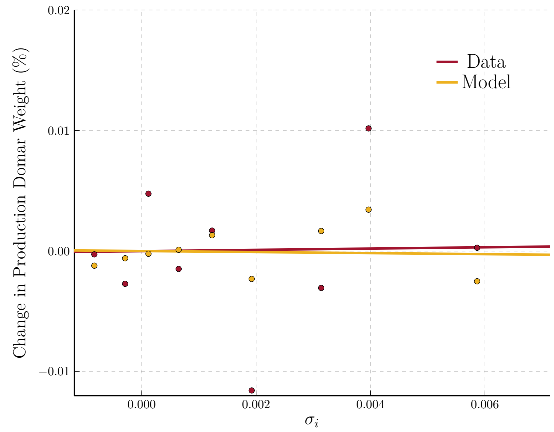
7.2 Expenditures shares

I begin by validating the model's key predictions about network reorganization.

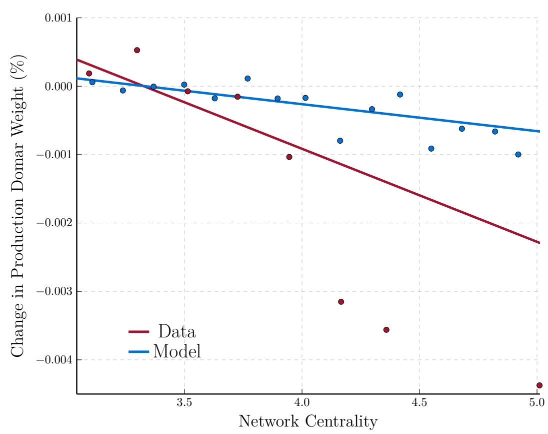
Testable Implications of the Model As noted by Kopytov et al., 2024, when the variance of a group of sectors increases, their relative size should decline in equilibrium. Figure 4 tests this implication against the data by relating changes in expenditure weights to changes in risk at the country and sector level.



(a) Country



(b) Sector



(c) Full Sample

Figure 4: Change in Expenditure shares in the Crisis

Panel (a) plots the change in each country's global sales share (the sum of the sectoral shares within the country) against the change in the estimated variance of that country's

TFP component. There is a clear negative relationship: countries whose productivity became more volatile lost market share during the crisis. Panel (b) repeats this exercise at the sector level. In both the data and the model, there is no strong relationship between a sector's own variance and its size.

This quantitative evidence extends the results in Kopytov et al., 2024, who show for the U.S. that sectors with higher estimated variance tend to have lower sales shares. Instead of focusing on country-industry pai specific variance, I examine country-specific shocks in a global setting and show that countries with larger increases in the variance of their productivity experienced larger declines in their global market share around 2009. In this sense, I provide an empirical test of the Kopytov et al., 2024 mechanism at the country level.

In Panel (c), the mechanism of the paper (Sub-section 5.1) is empirically validated. The prediction of the theory is that the central sector will experience a decrease in their expenditure shares when aggregate risk increases, as during the crisis in 2009. For the data, I used the observed network to compute the empirical measure of centrality using the Leontief-inverse. Panel (c) shows that the higher centrality of the network correlates with the reduction of sales shares in both the data and the model, suggesting a move away from central sectors during crises.

In Table 4, I formally compare reallocation patterns in the data and model across three aggregations.

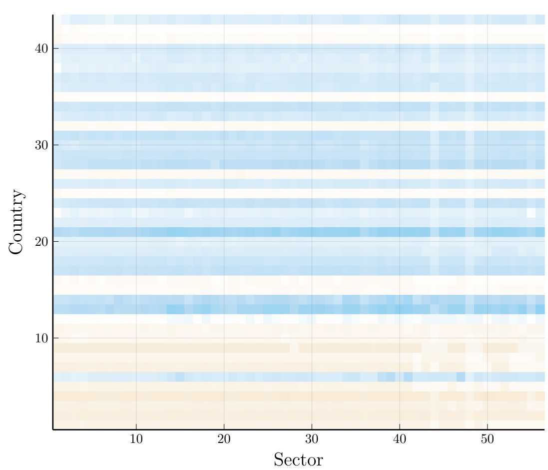
Table 4: Empirical Validation

	Data	Model
Full Sample		
Correlation	−0.122	−0.107
OLS coefficient	−0.002	−0.001
Country		
Correlation	−0.087	−0.221
OLS coefficient	−17.647	−8.767
Sector		
Correlation	0.001	−0.002
OLS coefficient	0.002	−0.001

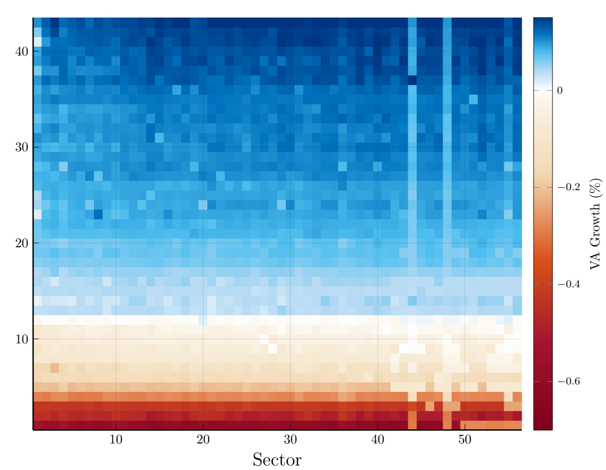
I computed the correlation and the OLS estimation that related both changes in sales shares with changes in risk and with centrality. The general pattern is that the model captures part of the negative correlation, but in the data expenditure shares change more in 2009.

Contribution to VA growth Now I turn to the implications for growth. Figure 5 plots two heat maps of the change in value-added (VA) growth of each country-industry pai induced by changes in expenditure shares, for 2009 in panel (a) and 2010 in panel (b). Countries are on the y-axis and sectors on the x-axis; each cell shows the contribution of changes in that sector's expenditure share to its VA growth. Countries and sectors are ordered by their contribution to growth in 2010. Blue cells correspond to country-industry pais that gain comparative advantage in that year: their world sales share rises and this reallocation contributes positively to their VA growth.

In 2009, the key change in expectations is a rise in aggregate volatility of about 1 percentage point (see Figure 2). As a result, most of the reallocation in expenditure shares that year is driven by the increase in aggregate risk. The contributions to VA growth in 2009 range from about -0.08 to 0.05 percentage points: the sector that loses the most market share experiences about 0.08 p.p. lower VA growth, while the one that gains the most market share enjoys about 0.05 p.p. higher VA growth.



(a) 2009



(b) 2010

Figure 5: Contribution of Expenditures Shares to VA

In 2010, the magnitudes are larger. Because aggregate productivity was low in 2009 (see

Figure 2) and the aggregate component is persistent ($\rho_g = 0.92$ in my estimates), expected aggregate productivity in 2010 rises sharply. The associated reallocation in expenditure shares generates contributions to VA growth between -0.6 and 0.15 percentage points, indicating a stronger impact of the global productivity shock in that year.

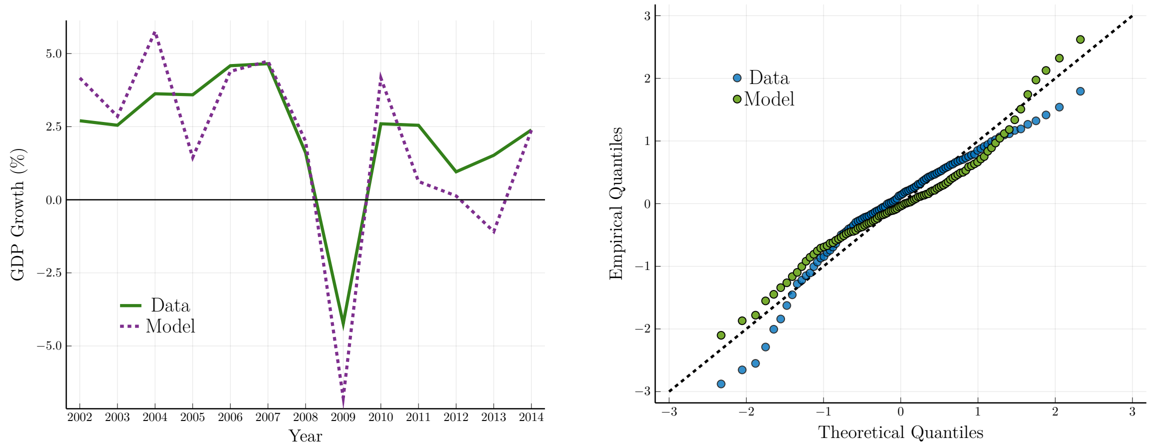
Interestingly, the positive contributions (indicated by blues) are concentrated in the same countries. I will focus on the two countries that experience the highest increase in market shares in 2010 Japan and Twain. Country-specific productivity increased during the crisis in these two countries. This leads other producers to increase their dependence on goods from Japan and Twain in 2010. As a result, these countries have also experienced an increase in their share of expenditure after crisis, which has fueled growth. However, Taiwan's variance increased in 2009, contributing to a reduction in maker shares of Taiwan in 2009.

7.3 GDP Growth

In this section I focus on how network adjustment affects GDP growth. Figure 6 compares the data and the model for mean GDP growth (panel (a)). The simulation uses the endogenous network implied by the model together with the shock process estimated under the observed network, as described in the previous subsection. If the model-implied network coincided exactly with the empirical network, the predicted GDP growth series in the model would match the data by construction. Hence, the deviations between model-predicted and observed GDP growth provide an error measure for how far the endogenous network is from the empirical one.

Figure 6 compares the data and model for the mean growth of GDP (panel a). The model used the endogenous network implied by the model and the estimation of shocks using the observed network, as described in the previous subsection. If the network in the model and in the data coincide, the predicted GDP growth in the model and in the data would also coincide (by construction the model matches GDP growth under the observed network). As a result the deviations of the predicted GDP in the model and the data provide an error measures of deviations of the network to the empirical one.

The model reproduces the cycle, capturing the pre-crisis expansion, the 2009 contraction, and the rebound, with small overshoots around the downturn. Because these moments were not targeted in the estimation, this alignment serves as a validation of the framework. In the next section, we use growth decompositions to identify the sources of growth.



(a) GDP (b) Q-Q Plot in the Model

Figure 6: Mean Growth

Panel (b) compares the empirical quantiles of GDP growth to those of a normal distribution. I compare the empirical distribution (using the empirical average and variance of GDP) with the model-implied distribution (using the model's predicted mean and variance). The key distinction is that the model features a time-varying mean and variance.

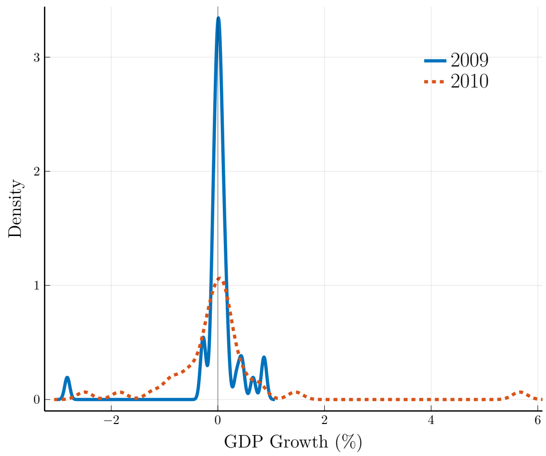
Around the median of the distribution, both show similar performance. However, in the tails, the empirical distribution deviates more strongly, with points bending away from the 45° line, indicating heavier tails than a normal distribution would predict. In contrast, the implied distribution of the model remains close to the 45° reference line in both tails.

This comparison shows that incorporating time-varying mean and variance in the model helps explain the extreme episodes observed in the data, where some countries' GDP deviated substantially from their averages.

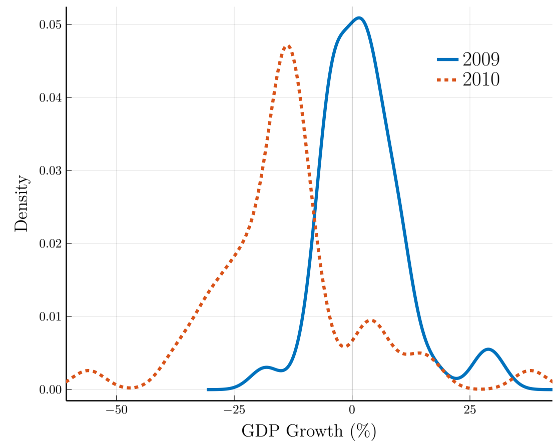
7.4 Growth decomposition

This subsection analyzes GDP growth contributions for the crisis year 2009 and the following year, 2010, highlighting the impact of two economic shocks. In 2009, increased volatility induced network adjustments, while a productivity drop affected expected 2010 productivity, inducing network adjustments and influencing GDP growth post-crisis.

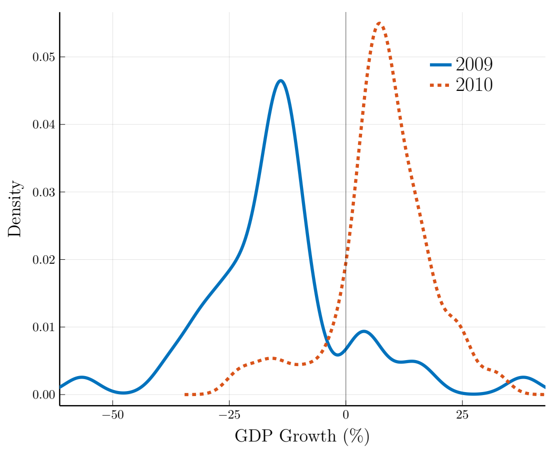
Figure 7 presents kernel densities for contributions to growth in 2009 and 2010 of four factors: network reorganization ($\Delta\mathcal{L}_t$), contemporaneous shock propagation ($\Delta\epsilon_t$) and lagged propagation ($\Delta\theta_t$), and deterministic supply change ($\Delta l_{ci,t}$).



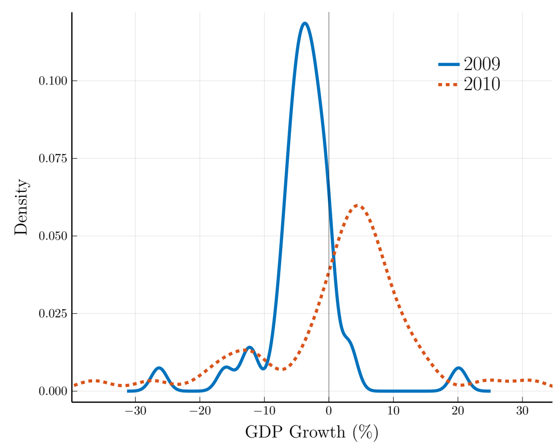
(a) Network Adjustment ($\Delta\mathcal{L}$)



(b) Lagged Propagation ($\Delta\theta$)



(c) Contemporaneous Propagation ($\Delta\epsilon$)



(d) Deterministic Supply (Δl_{ci})

Figure 7: Decomposition of GDP Growth During Crisis

Panel 7a shows the network adjustment channel: in 2009, the distribution skewed negative as firms shifted toward safer but less productive suppliers; in 2010, they reoptimized toward more productive sectors to offset the crisis-induced productivity decline. Panels 7b and 7c show shock transmission. Negative productivity shocks $\Delta\epsilon$ reduced 2009 growth, with persistent effects $\Delta\theta$ continuing to depress 2010 growth. Critically, Panel 7d reveals that changes in deterministic labor supply amplified these effects: centered below zero in 2009, this component exacerbated the crisis; its positive shift in 2010 accelerated the recovery.

Absolute Contribution Table 5 presents the contribution shares to GDP growth. Shock transmission ($\Delta\epsilon$ and $\Delta\theta$) accounts for most of the observed growth in both periods. However, network reorganization played a substantial role, contributing -3.7 percentage points (16%) in 2009 and +2.7 percentage points (20%) in 2010.

Table 5: GDP Growth Decomposition

	2009		2010	
Component	**Mean (pp)**	**Avg. Contribution (%)**	**Mean (pp)**	**Avg. Contribution (%)**
$\Delta\varepsilon$	−16.79	73.2	21.92	44.6
$\Delta\theta$	2.43	10.6	−14.10	34.9
Δl	−3.72	16.2	2.66	19.6
$\Delta\mathcal{L}$	0.01	0.04	−0.01	0.99

Notes: Mean shows the average growth in percentage points. Avg.

The 2009 collapse was mainly due to shock transmission through a fixed network, consistent with Acemoglu et al., 2012, accounting for 73% of the effect ($\Delta\epsilon = -16.8$ percentage points). The rest, 27%, was due to new components: lagged shocks ($\Delta\theta = +2.4$ percentage points, 11%) and endogenous network reorganization ($\Delta l = -3.7$ percentage points, 16%). In 2010, recovery showed inverse effects. Shock transmission rose by 21.9 percentage points (45%), offset by reduced lagged shocks ($\Delta\theta = -14.1$ percentage points, 35%). Support came from $\Delta l = +2.7$ percentage points (20%), while network reorganization changes ($\Delta\mathcal{L}$) were minimal.

Counterfactual with fixed expectations. In Appendix E, I compare the baseline model to a counterfactual in which the network does not reorganize in response to risk (expectations remain fixed in their pre-crisis value (2007). The counterfactual delivers a deeper downturn on average, approximately 2 percentage points larger at the peak of the crisis, with substantial cross country heterogeneity. Some economies experience markedly larger losses while others are only mildly affected. Welfare effects are also heterogeneous. Most countries register positive gains from allowing reorganization in 2009, but the distribution has meaningful tails. Full details, figures and construction of the exercise are reported in the Appendix E.

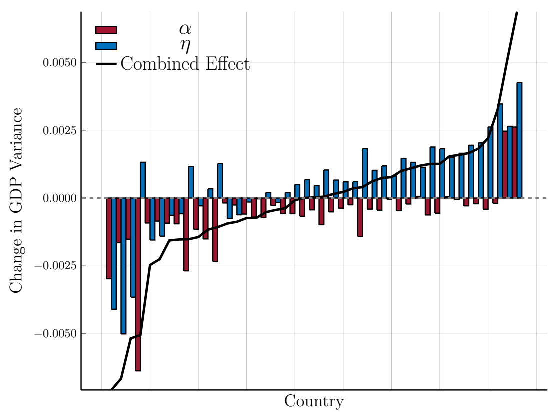
8 Transportation Costs

The analysis on the crisis of 2009 reveals that trade costs constrain firms' ability to reorganize networks during periods of high uncertainty. This raises a policy question: would reducing trade barriers allow better risk-sharing and lower aggregate volatility? I evaluate this by simulating a reduction in transportation costs from their 2011 levels to their 2000 levels, holding all other parameters constant.

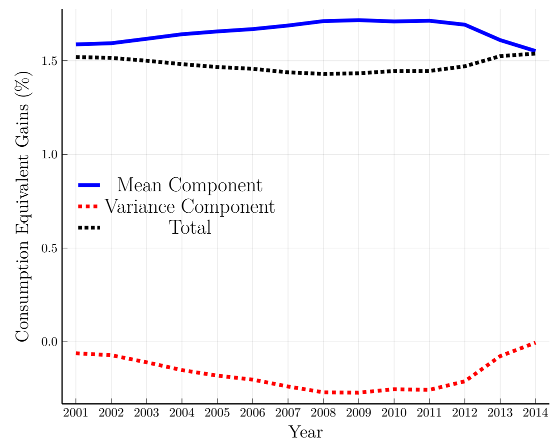
Effect on Aggregate Volatility This exercise revisits the results of Caselli et al., 2020 who quantified the effect of lower distortions on trade on aggregate volatility using the model proposed by Caliendo and Parro, 2015. The key difference from my analysis that in Caliendo and Parro, 2015, producers select the cheapest supplier for each intermediate variety, making inputs from different countries perfect substitutes. This allows complete reallocation away from countries experiencing adverse shocks, generating strong diversification benefits. Instead, my model employs Cobb-Douglas production with fixed expenditure shares. Producers therefore purchase from all suppliers in fixed proportions regardless of shocks, substantially limiting the benefits of diversification. Additionally, in my model, endogenously choose the set of suppliers taking into account the limited possibility of substitution ex-post.

In the counterfactual, I recompute the equilibrium for each year under two alternative trade-cost matrices: the one observed in the first year of the sample (2000) and the one observed in the last year (2011). Although the entire distribution of trade costs shifts between 2000 and 2011, the mean trade cost falls by about 3% over this period. I then ask how the variance of GDP and welfare would change if trade costs were kept fixed at their 2000 level, instead of falling to their observed 2011 level.

Figure 8a reports the effect of lower trade costs on the variance of GDP and decomposes it into contributions from the final and intermediate-goods networks. Under Cobb-Douglas, the direct impact of cheaper trade on aggregate volatility is zero; any change in volatility of GDP arises only through network reorganization. The main result is that, despite some heterogeneity, lower trade costs increase aggregate volatility in most countries. The decomposition shows that this increase is driven primarily by adjustments in the intermediate goods network.



(a) Variance



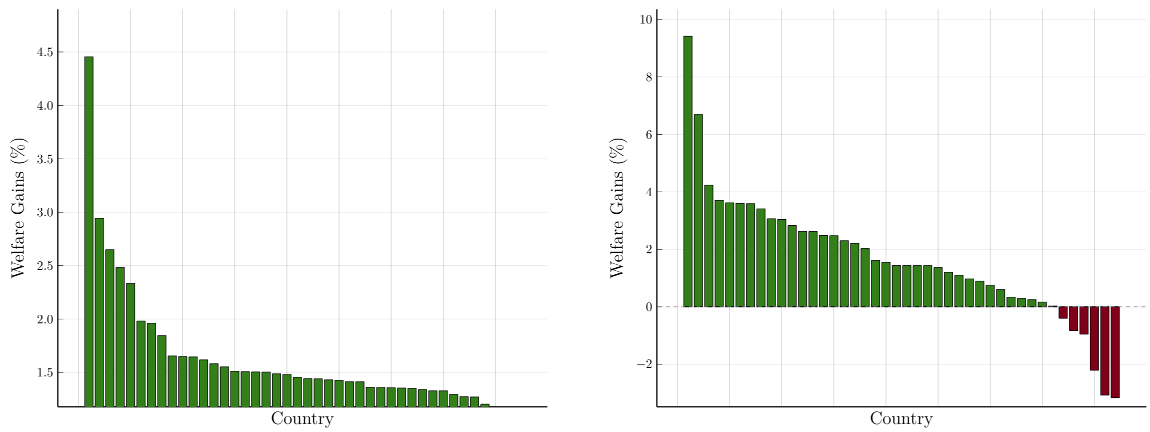
(b) Welfare Gain

Figure 8: Effects of Lower Trade Costs

The Panel (b) of Figure 8b displays the average welfare gains, along with their decomposition into changes due to expected consumption and consumption volatility. On average, cheaper trade increases welfare by increasing expected consumption, whereas higher consumption volatility partially offsets these gains. The negative effect on the welfare of higher risk reaches a peak during the global financial crisis in 2009 and reaches a value close to zero at the end of the sample.

The main lesson of this exercise is that the trade-off that firms face between reducing risk or increasing expected productivity when facing lower trade cost makes them to choose a network with higher risk. So, the combination of limited substitution (Cobb-Douglas) and endogenous choice of the network implies that lower distortions to trade lead to higher aggregate risk. However, since this is compensated for by higher consumption on average, there are welfare gains from the adjustment.

Heterogeneous Welfare Effects Next, I investigate the interaction of changes in the trade cost and the network. In Figure 9, I plot the average effect on welfare for each and the contribution of the network. Each panel shows equivalent welfare gains for consumption at the country level, sorted by magnitude. Green bars indicate countries where 2011 parameters yield higher welfare; red bars indicate countries where 2000 parameters yield higher welfare.



(a) Trade Costs (τ)

(b) Total

Figure 9: Welfare Effect Decomposition

The direct effect of the change in τ, panel 9a is positive for all countries, reflecting the efficiency gains from lower distortions. However, the effect of the network has heterogeneous effects. In panel 9b, I show the total effect. Due to the rich interaction between trade distortions and the network, the total impact is heterogeneous, some countries experience higher welfare gains compared to the direct effect, while others end up experiencing welfare losses due to the the network reorganization.

9 Conclusions

This paper examines how aggregate shocks reshape production networks through endogenous supplier choice, creating a two-way feedback between network formation and shock propagation. I develop a quantitative model where firms balance price efficiency against risk exposure when selecting suppliers before productivity shocks are realized.

The analysis yields three main insights. First, when global uncertainty increases, trade flows redistribute from central sectors, those that both serve many buyers and source from many suppliers, toward peripheral sectors, as firms seek suppliers with less exposure to aggregate shocks. This reallocation reflects a change in comparative advantage: central sectors, precisely because they are well connected, provide the worst insurance when aggregate risk rises. Second, network reorganization accounts for 16-20% of observed GDP growth during and after the 2009 crisis, with heterogeneous effects across countries. Third, I show that lower trade costs can increase aggregate volatility: while reducing transportation costs raises expected welfare through efficiency gains, it also induces firms to concentrate on riskier but more productive suppliers, amplifying exposure to productivity shocks.

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A Proofs

A.1 Lemma 1

Proof. We begin by the resource constraint at each sector i;

$$Y_{ci} = \sum_{\hat{c}} Z_{\hat{c},ci}\tau_{\hat{c},ci} + \sum_{\hat{c}}\sum_{k} Z_{ci,\hat{c}k}\tau_{\hat{c},ci} \tag{25}$$

Multiplying everything by P_{ci} we get:

$$P_{ci}Y_{ci} = \sum_{\hat{c}} P_{ci}Z_{ci,\hat{c}}\tau_{ci,\hat{c}} + \sum_{\hat{c}}\sum_{k} P_{ci}Z_{ci,\hat{c}k}\tau_{ci,\hat{c}} \tag{26}$$

In addition, the optimality conditions for the final and intermediate goods produced are as follows:

$$\tau_{ci,\hat{c}}P_{ci}Z_{ci,\hat{c}} = P_{\hat{c}}Z_{\hat{c}}\alpha_{ci,\hat{c}} \tag{27}$$

$$\tau_{ci,\hat{c}}P_{ci}Z_{ci,\hat{c}k} = P_{\hat{c}k}Y_{\hat{c}k}\eta_{ci,\hat{c}k} \tag{28}$$

Combing them, we have:

$$P_{ci}Y_{ci} = \sum_{\hat{c}} P_{\hat{c}}Z_{\hat{c}}\alpha_{ci,\hat{c}} + \sum_{\hat{c}}\sum_{k}(1-\mu_{\hat{c}k})P_{\hat{c}k}Y_{\hat{c}k}\eta_{ci,\hat{c}k} \tag{29}$$

Next, we use the budget constraint of the households to get:

$$P_{\hat{c}}Z_{\hat{c}} = \sum_{k}\mu_{\hat{c}k}P_{\hat{c}k}Y_{\hat{c}k} \tag{30}$$

So:

$$P_{ci}Y_{ci} = \sum_{\hat{c}}\sum_{k}\mu_{\hat{c}k}\alpha_{ci,\hat{c}}P_{\hat{c}k}Y_{\hat{c}k} + \sum_{\hat{c}}\sum_{k}(1-\mu_{\hat{c}k})P_{\hat{c}k}Y_{\hat{c}k}\eta_{ci,\hat{c}k} \tag{31}$$

In vector notation, we have:

$$\boldsymbol{PY} = (\boldsymbol{\mu\alpha} + (I-\boldsymbol{\mu})\boldsymbol{\eta})\boldsymbol{PY} \tag{32}$$

The solution of the system is the eigenvalue, whit $\lambda = 1$, of the matrix $\boldsymbol{\mu}\boldsymbol{\alpha} + (I - \boldsymbol{\mu})\boldsymbol{\eta}$. We need to solve the following.

$$det(\boldsymbol{\mu}\boldsymbol{\alpha} + (I - \boldsymbol{\mu})\boldsymbol{\eta} - I) = 0 \tag{33}$$

Because the rows of the matrix sum one by construction, we know that $\lambda = 1$ is an eigenvalue of the system, so the solution of the system exist. Finally, I impose normalization $\sum \omega_{ci} = 1$ to identify a unique solution. During this normalization, I choose the global GDP to be numeraire.

□

A.2 Lemma 2

Proof. **A.2.1 Nominal GDP**

The starting point of the proof is the Domar weights that we characterize in Lemma 1.

$$P_{ci} Y_{ci} = \nu_{ci} \tag{34}$$

Taking logs we have:

$$\log P_{ci} + \log Y_{ci} = \log(\nu_{ci}) \tag{35}$$

A.2.2 Labor Demand

The FOC of firms with for labor are as follows:

$$W_c L_{ci} = \mu_{ci} P_{ci} Y_{ci} \tag{36}$$

$$\log P_{ci} + \log Y_{ci} + \log \mu_{ci} = \log W_c + \log L_{ci} \tag{37}$$

Then:

$$\log L_{ci} = -\log W_c + \log \mu_{ci} + \log(\nu_{ci}) \tag{38}$$

A.2.3 Labor supply

Also, FOC of households implies:

$$P_c(L_{ci})^{\frac{1}{\psi}} = W_c \tag{39}$$

$$\log L_{ci} = [\log W_c - \log P_c]\psi \tag{40}$$

Combining both:

$$\log W_c = \frac{\psi}{1+\psi}\log P_c + \frac{1}{1+\psi}[\log \mu_{ci} + \log(\nu_{ci})] \tag{41}$$

A.2.4 Demand of Inputs

From FOC of intermdiate inputs of firms:

$$\log P_{ci} = -\epsilon_i - A(\eta_i^\star) + \mu_{ci}\log W_c + (1-\mu_{ci})\sum_{\hat{c}}\sum_j \eta_{ci,\hat{c}j}[\log P_{\hat{c}j} + \log(\tau_{c,\hat{c}j})] \tag{42}$$

$$\begin{aligned}\log P_{ci} = -\epsilon_i - A(\eta_i^\star) + \mu_{ci}\left[\frac{\psi}{1+\psi}\log P_c + \frac{1}{1+\psi}(\log \mu_{ci} + \log(\nu_{ci}))\right] + \\ (1-\mu_{ci})\sum_{\hat{c}}\sum_j \eta_{ci,\hat{c}j}[\log P_{\hat{c}j} + \log(\tau_{c,\hat{c}j})]\end{aligned} \tag{43}$$

$$\begin{aligned}\log P_{ci} = -\epsilon_i - A(\eta_i^\star) + \mu_{ci}\left[\frac{\psi}{1+\psi}\left[A(\alpha_c^\star) + \sum_{\hat{c}}\sum_j \alpha_{c,\hat{c}j}^\star[\log P_{\hat{c}j} + \log(\tau_{c,\hat{c}j})]\right] + \right. \\ \frac{1}{1+\psi}(\log \mu_{ci} + \log(\nu_{ci}))] + (1-\mu_{ci})\sum_{\hat{c}}\sum_j \eta_{ci,\hat{c}j}[\log P_{\hat{c}j} + \log(\tau_{c,\hat{c}j})]\end{aligned} \tag{44}$$

Turning in matrix notation:

$$\begin{aligned}\log P[I - \frac{\psi}{1+\psi}\mu\alpha^\star - (I-\mu)\eta] = \\ -\epsilon - A(\eta_i^\star) + \frac{\psi}{1+\psi}\mu A(\alpha_c^\star) + \operatorname{diag}\left[\left(\frac{\psi}{1+\psi}\mu\alpha^\star + (I-\mu)\eta\right)\log(\tau)\right] + \frac{1}{1+\psi}(\log\mu + \log(\nu))\end{aligned} \tag{45}$$

Using the definition of the Leontief Inverse:

$$\log P = \mathcal{L}(\alpha^\star)\left(-\epsilon - A(\eta_i^\star) + \frac{\psi}{1+\psi}\mu A(\alpha_c^\star) + \text{diag}\left[\left(\frac{\psi}{1+\psi}\mu\alpha^\star + (I-\mu)\eta\right)\log(\tau)\right] + \frac{1}{1+\psi}(\log\mu + \log(\nu)\right) \tag{46}$$

So, applying the definition in the lemma:

$$\boldsymbol{p} = -\mathcal{L}\,\epsilon + \overline{p}(s) \tag{47}$$

□

A.3 Proof of Lemma 3

Proof. The proof starts form the definition of VA:

$$VA = e^{\epsilon A(\eta_{ci})} L_{ci}^{\mu_{ci}} \tag{48}$$

Taking logs lead to the first part of the Lemma. Also from the proof of Lemma2 we have:

$$l_{ci} = \frac{\psi}{1+\psi}[-p_c + \log\mu_{ci} + \log(\nu_{ci})] \tag{49}$$

□

A.4 Proof of Corollary 5

Proof. The result follows directly from taking derivatives of:

$$\mathbf{E}[p_{ci}] = \mathcal{L}\,(B(s) - \theta) \tag{50}$$

$$\mathbf{V}\,[p_{ci}] = (\,\mathcal{L})^{\mathrm{T}}\,\Sigma\,(\,\mathcal{L}) \tag{51}$$

□

A.5 Proof of Lemma 7

Proof. Let us denote by a star the prices at the initial equilibrium. The real GDP can be expressed as:

$$g_c = \frac{dG_c}{G_c} \tag{52}$$

$$d\log(GDP_c) = \sum_{i\in N_c} \frac{d\, p_i^\star y_i}{G_c} - \frac{d\, p_{xi}^\star x_i}{G_c} \tag{53}$$

$$= \sum_{i\in N_c} \frac{d\, p_i^\star y_i}{p_i^\star y_i}\frac{p_i^\star y_i}{G_c}\frac{\mu_{ci}}{\mu_{ci}} - \frac{d\, p_{xi}^\star x_i}{p_{xi}^\star x_i}\frac{p_{xi}^\star x_i}{G_c} \tag{54}$$

$$= \sum_{i\in N_c} \frac{s_{ci}}{\mu_{ci}}\left(d\log(y_i) - \frac{p_{xi}^\star x_i}{p_i^\star y_i}\, d\log(x_i)\right) \tag{55}$$

In the last line I multiplied and divided by $p_i y_i$ the last term. Also, I use $G_c = \mu_{ci}\sum P_{ci}Y_{ci}$. Then using the production function of sector i, we have:

$$G_c = \sum_{i\in N_c} s_{ci}\left(\epsilon_i + \mu L_i + (1-\mu)\log(x_i) - \frac{p_{xi}x_i}{p_i y_i}\, d\log(x_i)\right) \tag{56}$$

$$= \sum_{i\in N_c} s_{ci}\left(\epsilon_i + \mu L_i\right) \tag{57}$$

Here, the last line uses the FOC of the firm. Now, from (49) and Lemma 2 we get:

$$L = \frac{\psi}{1+\psi}\alpha^\star\, \mathcal{L}\, \epsilon \tag{58}$$

Because the rows of $\hat{\Omega}$ are the same for sectors in the same country, $L_i = L_j$ for any $i, j \in c$. Replace it in (57) to get:

$$G_c = \sum_{i \in N_c} \frac{s_{ci}}{\mu} (\epsilon_i) + L_c \tag{59}$$

$$= \sum_{i \in N_c} \frac{s_{ci}}{\mu} (\epsilon_i) + \frac{\psi}{1+\psi} \sum_{i \in N} \sum_{k \in N} \alpha_{ci} \, \lambda_{ik} \, \epsilon_k \tag{60}$$

□

A.6 Proof of Proposition 1

Proof. The proof has three steps:

Step 1: Firms Objective Starting from equation (6). For any input-bundle shares, the problem is the following.

$$\max \quad E\Big[\Lambda_c \, Y \left(P_{\text{rev}} - K_{\text{cost}}\right)\Big] ,$$

with $(P_{\text{rev}}, K_{\text{cost}}) = (P_c, K_c(\alpha, P))$ for the finals and $(P_{\text{rev}}, K_{\text{cost}}) = (P_{ci}, K_{ci}(\eta, P))$ for the intermediates. Since P_{rev} does not depend on the shares, choosing the shares is equivalent to *minimizing* the discounted expected cost:

$$\min \quad E[\Lambda_c \, Y \, K_{\text{cost}}] \, .$$

Log-moment step. Write $k_{\text{cost}} = \ln K_{\text{cost}}$ and $\lambda_c = \ln \Lambda_c$, $y = \ln Y$. By log-normal identity,

$$\ln E\left[e^{\lambda_c + y + k_{\text{cost}}}\right] \;=\; E[\lambda_c + y + k_{\text{cost}}] \;+\; \tfrac{1}{2}\,\text{Var}(\lambda_c + y + k_{\text{cost}}).$$

Only the part that varies with the shares matters. Using $\frac{1}{2}\partial\,\text{Var}(X) = \text{Cov}(X, \partial X)$ and $\partial k_{\text{cost}} / \partial(\text{share on } (c', k)) = p_{c'k}$, the share-wise index that multiplies each supplier (c', k) is

$$\underbrace{E[p_{c'k}] + \ln \tau_{\bullet,c'k}}_{\text{mean delivered price}} \quad + \quad \underbrace{\text{Cov}\big(\lambda_c + y + k_{\text{cost}},\; p_{c'k}\big)}_{\text{risk term}},$$

where $\tau_{\bullet,c'k}$ denotes the relevant iceberg factor (final or intermediate).

Final-good problem (to get (23)). Here $y = y_c$ and $k_{\text{cost}} = k_c$. Under equilibrium conditions for finals,

$$y_c = -\,p_c, \qquad k_c = p_c.$$

Therefore

$$\mathrm{Cov}(\lambda_c + y_c + k_c,\ p_{c'k}) = \mathrm{Cov}(\lambda_c,\ p_{c'k}) + \underbrace{\mathrm{Cov}(-p_c,\ p_{c'k}) + \mathrm{Cov}(p_c,\ p_{c'k})}_{=0}.$$

Hence the per-supplier index for finals is

$$\boxed{E[p_{c'k}] + \ln \tau_{c,c'k} \ + \ \mathrm{Cov}(\lambda_c,\ p_{c'k})}.$$

Intermediate-good problem. For intermediates we must use sectoral (own-good) demand Y_{ci}; then $y = \ln Y_{ci} = y_{ci}$ and $k_{\text{cost}} = k_{ci}$. Under assumptions for intermediates,

$$y_{ci} = -\,p_{ci}, \qquad k_{ci} = p_{ci}.$$

Thus

$$\mathrm{Cov}(\lambda_c + y_{ci} + k_{ci},\ p_{c'k}) = \mathrm{Cov}(\lambda_c,\ p_{c'k}) + \underbrace{\mathrm{Cov}(-p_{ci},\ p_{c'k}) + \mathrm{Cov}(p_{ci},\ p_{c'k})}_{=0}.$$

Hence the per-supplier index for intermediates is

$$\boxed{E[p_{c'k}] + \ln \tau_{ci,c'k} \ + \ \mathrm{Cov}(\lambda_c,\ p_{c'k})}.$$

Substituting the same pricing-kernel mapping for λ_c (exactly as stated in the text immediately before (24)) yields Eq. (24).

Stacking. Summing these per-supplier indexes with weights $\alpha_{c,c'k}$ (finals) and $\eta_{ci,c'k}$ (intermediates) gives the objective indexes used in (23) and (24), respectively.

Step2, Pricing kernel

Household pricing kernel in terms of labor only. From the labor FOC we have

$$z_c = v'(L_c)\, L_c,$$

and with $\nu(x) = x^{1/\psi+1}$ it follows that

$$\nu'(x) = \Big(\tfrac{1}{\psi} + 1\Big)x^{1/\psi} \quad \Longrightarrow \quad z_c = \Big(\tfrac{1}{\psi} + 1\Big)L_c^{1/\psi+1} = \Big(\tfrac{1}{\psi} + 1\Big)\nu(L_c).$$

Hence

$$z_c - \nu(L_c) = \Big(\tfrac{1}{\psi} + 1\Big)\nu(L_c) - \nu(L_c) = \frac{1}{\psi}\,\nu(L_c).$$

The stochastic discount factor is

$$\Lambda_c = \frac{u'\big(z_c - \nu(L_c)\big)}{P_c} = \frac{\big(z_c - \nu(L_c)\big)^{-\gamma}}{P_c} = \frac{\big(\tfrac{1}{\psi}\nu(L_c)\big)^{-\gamma}}{P_c} = \psi^{\gamma}\,\nu(L_c)^{-\gamma}\,P_c^{-1}.$$

Using $\nu(L_c) = L_c^{1/\psi+1}$, we obtain

$$\log\Lambda_c = -\gamma\log\nu(L_c) + \gamma\log\psi - p_c(s,\epsilon) = -\gamma\Big(1 + \tfrac{1}{\psi}\Big)\log L_c \;+\; \gamma\log\psi \;-\; \log P_c.$$

Also from lemma 3:

$$\log L_c(s,\epsilon) = -\frac{\psi}{1+\psi}\,p_c(s,\epsilon) + \log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right).$$

$$\log L_c(s,\epsilon) = -\frac{\psi}{1+\psi}\,p_c(s,\epsilon) + \log\chi_c.$$

Substituting,

$$\begin{aligned}\log\Lambda_c &= -\gamma\Big(1 + \tfrac{1}{\psi}\Big)\left[-\frac{\psi}{1+\psi}\,p_c + \log\Big(\sum_i (\mu_{ci}\nu_{ci})^{\frac{\psi}{1+\psi}}\Big)\right] + \gamma\log\psi - p_c \\ &= \big[\gamma(1 + \tfrac{1}{\psi})\tfrac{\psi}{1+\psi} - 1\big]\,p_c \;-\; \gamma\Big(1 + \tfrac{1}{\psi}\Big)\log\Big(\sum_i (\mu_{ci}\nu_{ci})^{\frac{\psi}{1+\psi}}\Big) \;+\; \gamma\log\psi.\end{aligned}$$

Since

$$\gamma\Big(1 + \tfrac{1}{\psi}\Big)\tfrac{\psi}{1+\psi} = \gamma,$$

the coefficient on p_c is $\gamma - 1$. Thus

$$\log\Lambda_c(s,\epsilon) = (\gamma - 1)\,p_c(s,\epsilon) + C_c(s),$$

where

$$C_c(s) = -\gamma\Big(1+\tfrac{1}{\psi}\Big)\log\Big(\sum_i (\mu_{ci}\nu_{ci})^{\frac{\psi}{1+\psi}}\Big) + \gamma\log\psi$$

depends only on the network s (through ν_{ci}) and parameters, but not on shocks.

Step3: Combing both For final-good producers in country c, the (log) cost index is

$$k_c(\alpha_c;\alpha^\star) = -A_c(\alpha_c) + \sum_{\hat{c},k}\alpha_{c,\hat{c}k}\big(p_{\hat{c}k} + \log\tau_{c,\hat{c}k}\big),$$

while the equilibrium final-good price index satisfies

$$p_c(\alpha^\star) = -A_c(\alpha_c^\star) + \sum_{\hat{c},k}\alpha^\star_{c,\hat{c}k}\big(p_{\hat{c}k} + \log\tau_{c,\hat{c}k}\big).$$

Since $C_c(s)$, $A_c(\cdot)$ and $\log\tau_{c,\hat{c}k}$ are deterministic, they drop out of covariances, and we obtain

$$\begin{aligned}
\mathrm{Cov}\big(\Lambda_c, k_c(\alpha_c;\alpha^\star)\big) &= (\gamma-1)\,\mathrm{Cov}\Big(p_c(\alpha^\star), k_c(\alpha_c;\alpha^\star)\Big)\\
&= (\gamma-1)\,\mathrm{Cov}\Big(\sum_{\hat{c}',j}\alpha^\star_{c,\hat{c}'j}p_{\hat{c}'j}, \sum_{\hat{c},k}\alpha_{c,\hat{c}k}p_{\hat{c}k}\Big)\\
&= (\gamma-1)\sum_{\hat{c},k}\sum_{\hat{c}',j}\alpha_{c,\hat{c}k}\,\alpha^\star_{c,\hat{c}'j}\,\mathrm{Cov}\big(p_{\hat{c}k}, p_{\hat{c}'j}\big).
\end{aligned}$$

The final-good producer in country c chooses α_c to maximize expected (log) cost adjusted for risk:

$$\max_{\alpha_c\in\mathcal{A}} \ \mathbb{E}\big[k_c(\alpha_c;\alpha^\star)\big] \ + \ \mathrm{Cov}\big(\Lambda_c, k_c(\alpha_c;\alpha^\star)\big).$$

Substituting the expressions above and rearranging,

$$\begin{aligned}
&\mathbb{E}\big[k_c(\alpha_c;s)\big] + \mathrm{Cov}\big(\Lambda_c, k_c(\alpha_c;s)\big)\\
&= -A_c(\alpha_c) + \sum_{\hat{c},k}\alpha_{c,\hat{c}k}\Big(\mathbb{E}[p_{\hat{c}k}] + \log\tau_{c,\hat{c}k} + (\gamma-1)\sum_{\hat{c}',j}\alpha^\star_{c,\hat{c}'j}\,\mathrm{Cov}\big(p_{\hat{c}k}, p_{\hat{c}'j}\big)\Big).
\end{aligned}$$

Defining the *risk-adjusted delivered price* as

$$\mathcal{R}_{c,\hat{c}k}(s) := \mathbb{E}[p_{\hat{c}k}] + \log \tau_{c,\hat{c}k} + (\gamma - 1) \sum_{\hat{c}',j} \alpha^{\star}_{c,\hat{c}'j} \operatorname{Cov}\big(p_{\hat{c}k}, p_{\hat{c}'j}\big),$$

and using $A_c(\alpha_c) = \sum_{\hat{c},k} \kappa_{\hat{c}k}(\alpha_{c,\hat{c}k} - \alpha^0_{\hat{c}k})^2$, the problem of the final-good producer is

$$\alpha^{\star}_c \in \arg\max_{\alpha_c \in \mathcal{A}} \; - \sum_{\hat{c},k} \kappa_{\hat{c}k} \big(\alpha_{c,\hat{c}k} - \alpha^0_{\hat{c}k}\big)^2 \; - \sum_{\hat{c},k} \alpha_{c,\hat{c}k}\, \mathcal{R}^F_{c,\hat{c}k}(s).$$

For an intermediate producer in sector (c, i), the (log) cost index is

$$k_{ci}(\eta_{ci}; s) = -A_{ci}(\eta_{ci}) + \sum_{\hat{c},k} \eta_{ci,\hat{c}k} \big(p_{\hat{c}k} + \log \tau_{ci,\hat{c}k}\big),$$

while the household SDF for country c is the same Λ_c as above. Using again that constants drop from covariances, we obtain

$$\begin{aligned} \operatorname{Cov}\big(\Lambda_c, k_{ci}(\eta_{ci}; s)\big) &= (\gamma - 1) \operatorname{Cov}\Big(p_c(\alpha^{\star}), \sum_{\hat{c},k} \eta_{ci,\hat{c}k} p_{\hat{c}k}\Big) \\ &= (\gamma - 1) \sum_{\hat{c},k} \sum_{\hat{c}',j} \eta_{ci,\hat{c}k}\, \alpha^{\star}_{c,\hat{c}'j} \operatorname{Cov}\big(p_{\hat{c}k}, p_{\hat{c}'j}\big). \end{aligned}$$

Thus the intermediate producer solves

$$\max_{\eta_{ci} \in \mathcal{E}} \; \mathbb{E}\big[k_{ci}(\eta_{ci}; s)\big] \; + \; \operatorname{Cov}\big(\Lambda_c, k_{ci}(\eta_{ci}; \eta^{\star})\big),$$

that is,

$$- A_{ci}(\eta_{ci}) + \sum_{\hat{c},k} \eta_{ci,\hat{c}k} \Big(\mathbb{E}[p_{\hat{c}k}] + \log \tau_{ci,\hat{c}k} + (\gamma - 1) \sum_{\hat{c}',j} \alpha^{\star}_{c,\hat{c}'j} \operatorname{Cov}\big(p_{\hat{c}k}, p_{\hat{c}'j}\big)\Big).$$

Replacing the risk-adjusted prices and writing $A_{ci}(\eta_{ci}) = \sum_{\hat{c},k} \kappa_{\hat{c}k}(\eta_{ci,\hat{c}k} - \eta^0_{ci,\hat{c}k})^2$, the problem of the intermediate producer is

$$\eta^{\star}_{ci} \in \arg\max_{\eta_{ci} \in \mathcal{E}} \; - \sum_{\hat{c},k} \kappa_{\hat{c}k} \big(\eta_{ci,\hat{c}k} - \eta^0_{ci,\hat{c}k}\big)^2 \; - \sum_{\hat{c},k} \eta_{ci,\hat{c}k}\, \mathcal{R}_{ci,\hat{c}k}(s).$$

Which closes the proof

□

A.7 Proof of Lemma 6

We consider the problem of finding

$$\alpha_c^\star \in \arg\max_{\alpha_c \in \mathcal{A}_c} \left\{ a_c(\alpha_c) - \sum_i \alpha_{ci}\, \mathcal{R}_i \right\},$$

where

$$a_c(\alpha_c) = -\sum_i \kappa_{ci} \left(\alpha_{ci} - \alpha_{ci}^0\right)^2.$$

In addition, we impose the normalization constraint

$$\sum_i \alpha_{ci} = 1,$$

and the non-negativity constraint

$$\alpha_{ci} \geq 0 \quad \forall\, i.$$

We assume that

$$\sum_i \alpha_{ci}^0 = 1.$$

Step 1: Forming the Lagrangian We introduce a Lagrange multiplier λ for the equality constraint and additional multipliers $\mu_i \geq 0$ for the inequality constraints. The Lagrangian is given by

$$\mathcal{L}(\alpha_c, \lambda, \mu) = -\sum_i \kappa_{ci} \left(\alpha_{ci} - \alpha_{ci}^0\right)^2 - \sum_i \alpha_{ci}\, \mathcal{R}_i + \lambda \left(\sum_i \alpha_{ci} - 1 \right) - \sum_i \mu_i\, \alpha_{ci}.$$

Step 2: First-Order Conditions Differentiate $\mathcal{L}$ with respect to each α_{ci}:

$$\frac{\partial \mathcal{L}}{\partial \alpha_{ci}} = -2\kappa_{ci} \left(\alpha_{ci} - \alpha_{ci}^0\right) - \mathcal{R}_i + \lambda - \mu_i = 0.$$

This yields

$$-2\kappa_{ci} \left(\alpha_{ci} - \alpha_{ci}^0\right) = \mathcal{R}_i - \lambda + \mu_i,$$

or, equivalently,

$$\alpha_{ci} = \alpha_{ci}^0 + \frac{1}{2\kappa_{ci}} \left(\lambda - \mathcal{R}_i - \mu_i\right).$$

Step 3: Complementary Slackness The Karush-Kuhn-Tucker (KKT) conditions impose that for each i

$$\mu_i \geq 0, \quad \alpha_{ci} \geq 0, \quad \mu_i \, \alpha_{ci} = 0.$$

Thus, for any index i where $\alpha_{ci} > 0$ (the active set I), we have $\mu_i = 0$ and

$$\alpha_{ci} = \alpha_{ci}^0 + \frac{1}{2\kappa_{ci}} \left(\lambda - \mathcal{R}_i\right).$$

For indices where the above expression would yield $\alpha_{ci} < 0$, the optimal solution sets $\alpha_{ci} = 0$ and the corresponding $\mu_i > 0$.

Step 4: Enforcing the Constraint Let I denote the active set of indices for which $\alpha_{ci} > 0$. Then the normalization constraint becomes

$$\sum_{i \in I} \left[\alpha_{ci}^0 + \frac{1}{2\kappa_{ci}} \left(\lambda - \mathcal{R}_i\right) \right] = 1.$$

Expanding the sum, we have

$$\sum_{i \in I} \alpha_{ci}^0 + \lambda \sum_{i \in I} \frac{1}{2\kappa_{ci}} - \sum_{i \in I} \frac{\mathcal{R}_i}{2\kappa_{ci}} = 1.$$

Since $\sum_i \alpha_{ci}^0 = 1$, if all indices are active then $\sum_{i \in I} \alpha_{ci}^0 = 1$ and we obtain

$$1 + \lambda \sum_i \frac{1}{2\kappa_{ci}} - \sum_i \frac{\mathcal{R}_i}{2\kappa_{ci}} = 1.$$

Thus,

$$\lambda \sum_i \frac{1}{2\kappa_{ci}} = \sum_i \frac{\mathcal{R}_i}{2\kappa_{ci}},$$

so that

$$\boxed{\lambda = \sum_i \mathcal{R}_i}$$

In the general case, where only a subset I is active, the normalization condition becomes

$$\lambda \sum_{i \in I} \frac{1}{2\kappa_{ci}} = 1 - \sum_{i \in I} \alpha_{ci}^0 + \sum_{i \in I} \frac{\mathcal{R}_i}{2\kappa_{ci}}.$$

Thus, the formula for λ is

$$\boxed{\lambda = 1 - \sum_{i \in I} \alpha_{ci}^0 + \sum_{i \in I} \frac{\mathcal{R}_i}{2\kappa_{ci}}}$$

Since $\sum_i \alpha_{ci}^0 = 1$, if all indices are active (I is the full index set) then the numerator simplifies to $\sum_i \frac{\mathcal{R}_i}{2\kappa_{ci}}$, recovering the previous expression.

A.8 Proof of Lemma 8

From the household first-order condition we have

$$v(L_c) = \frac{\psi}{1+\psi} Z_c,$$

so that net consumption is a fixed fraction of Z_c:

$$Z_c - v(L_c) = Z_c - v(L_c) = Z_c - \frac{\psi}{1+\psi} Z_c = \frac{1}{1+\psi} Z_c.$$

With CRRA utility,

$$U(Z_c - v(L_c)) = \frac{(Z_c - v(L_c))^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \left(\frac{Z_c}{1+\psi} \right)^{1-\gamma}.$$

Using the relation between consumption and labor,

$$Z_c = \left(1 + \frac{1}{\psi}\right) L_c^{1+\frac{1}{\psi}},$$

and the labor-supply expression from Lemma 3,

$$\log L_c(s,\epsilon) = -\frac{\psi}{1+\psi} p_c(s,\epsilon) + \log \left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}} \right),$$

we obtain

$$\begin{aligned}
\log Z_c(s,\epsilon) &= \log\Big(1+\tfrac{1}{\psi}\Big) + \Big(1+\tfrac{1}{\psi}\Big)\log L_c(s,\epsilon) \\
&= \log\Big(1+\tfrac{1}{\psi}\Big) + \Big(1+\tfrac{1}{\psi}\Big)\left[-\frac{\psi}{1+\psi}\,p_c(s,\epsilon) + \log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right)\right] \\
&= -\,p_c(s,\epsilon) + \log\Big(1+\tfrac{1}{\psi}\Big) + \Big(1+\tfrac{1}{\psi}\Big)\log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right).
\end{aligned}$$

Since $Z_c - v(L_c) = Z_c/(1+\psi)$, we have

$$\begin{aligned}
\log\big(Z_c - v(L_c)\big) &= \log Z_c - \log(1+\psi) \\
&= -\,p_c(s,\epsilon) + \log\Big(1+\tfrac{1}{\psi}\Big) + \Big(1+\tfrac{1}{\psi}\Big)\log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right) - \log(1+\psi).
\end{aligned}$$

Assume $\log(Z_c - v(L_c))$ is (approximately) normal with mean $\mathbb{E}[\log(Z_c - v(L_c))]$ and variance $\mathrm{Var}(\log(Z_c - v(L_c)))$. By the lognormal identity,

$$\log\mathbb{E}\big[U(Z_c - v(L_c))\big] = \log\Big(\tfrac{1}{1-\gamma}\Big) + (1-\gamma)\,\mathbb{E}\big[\log(Z_c - v(L_c))\big] + \frac{1}{2}(1-\gamma)^2\ \mathrm{Var}\big(\log(Z_c - v(L_c))\big).$$

From the expression above,

$$\mathbb{E}\big[\log(Z_c - v(L_c))\big] = -\,\mathbb{E}[p_c(s,\epsilon)] + \log\Big(1+\tfrac{1}{\psi}\Big) + \Big(1+\tfrac{1}{\psi}\Big)\log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right) - \log(1+\psi),$$

and, since the remaining terms are deterministic,

$$\mathrm{Var}\big(\log(Z_c - v(L_c))\big) = \mathrm{Var}\big(p_c(s,\epsilon)\big).$$

Putting these together,

$$\begin{aligned}
\log\mathbb{E}\big[U(Z_c - v(L_c))\big] = \underbrace{\log\Big(\tfrac{1}{1-\gamma}\Big) + (1-\gamma)\left[\log\Big(1+\tfrac{1}{\psi}\Big) - \log(1+\psi)\right]}_{=:\,c} + \Big(1+\tfrac{1}{\psi}\Big)\log\left(\sum_i (\nu_{ci}\mu_{ci})^{\frac{\psi}{1+\psi}}\right) + \\
-\,(1-\gamma)\,\mathbb{E}[p_c(s,\epsilon)] + \frac{1}{2}(1-\gamma)^2\ \mathrm{Var}\big(p_c(s,\epsilon)\big).
\end{aligned}$$

Thus welfare for country c can be written as

$$\log \mathbb{E}\big[U(Z_c - v(L_c))\big] = C_c(\mu, \nu, \psi) \; - \; (1-\gamma)\, \mathbb{E}[p_c(s,\epsilon)] \; + \; \frac{1}{2}(1-\gamma)^2 \; \mathrm{Var}\big(p_c(s,\epsilon)\big),$$

where $C_c(\mu, \nu, \psi)$ collects all terms that depend on the labor-share parameters μ_{ci}, Domar weights ν_{ci}, and the Frisch elasticity ψ, but not on the realization of shocks.

B Existence and Uniqueness

This appendix provides detailed derivations for existences and uniqueness of the equilibrium. The strategy is cast the equilibrium as an static game (MFG) to analyze existence and uniqueness of equilibrium.

Assumptions Lets us begin by state the assumptions we require in the proof

Assumption 3 (Assumption)**.**

1. *Assumption 2*
2. *Input shares satisfy* $1 > \mu > 0$.
3. ***Curvature:*** $\kappa^I_k, \kappa^F_c > 0$
4. ***Finite frictions:*** $\tau_{ki} \geq 1$.
5. *There exists* $\bar{A} > -\infty$ *such that, for all feasible* α_c, $A(\alpha_c) \;\geq\; \bar{A}$

It is important to highlight that no new assumptions are introduced in here. This statement serves merely as a summary of previously established assumptions needed for this proof.

Notation for uniform bounds. For any vector x, let $\bar{x} := \|x\|_\infty$ denote its ℓ_∞ norm. Also note:

$$\|x\|_\infty \;=\; \max_i |x_i| \;\leq\; \max\{\,|\bar{x}|,\; |\underline{x}|\,\},$$

Then, I define

$$\underline{\mu} := \min_k \mu_k, \overline{\mu} := \max_k \mu_k, \overline{\log \tau} := \| \log \tau \|_\infty, \overline{\log \mu} := \| \log \mu \|_\infty, \overline{\log \nu} := \| \log \nu \|_\infty.$$

B.1 Game Formulation

Players, Strategies I formulate the equilibrium as a game. Using the notation of my model, the players are intermediate goods producers $k = 1, \ldots, N$ and final goods producers $c = 1, \ldots, C$. Since there is a representative firm for each sector, each k and each c can be understood as a type of producer. The action of each producer are the share vectors $\eta_k \in \mathcal{A}$ (intermediate goods) and $\alpha_c \in \mathcal{A}$ (final goods), respectively. The strategy space is

$$\mathcal{M} := \left(\prod_{k=1}^{N} \mathcal{A} \right) \times \left(\prod_{c=1}^{C} \mathcal{A} \right),$$

Given that $\mathcal{A}$ is the simplex (assumption 2) $\mathcal{M}$ is nonempty, compact, and convex. The state is the matrix of risk-adjusted prices $R(s)$.

Aggregate State For any $s = (\eta, \alpha) \in \mathcal{M}$. As Lemma 2 proves, the Leontief inverse exists up to normalization. That guarantees the function $p(s, \epsilon)$ denote the unique normalized solution to the prices. $E[p(s, \epsilon)]$ and $\text{Cov}[p_i(s, \epsilon), p_i(s, \epsilon)]$ are unique functions. Thus, the aggregate state $R(s)$ from Proposition 1 is a unique function of the aggregate network.

Payoff As shown in Proposition 1, the cost of each type of producer can be defined as:

$$J(c, \alpha_c, s) = -\sum_j \kappa^F (\alpha_{c,j} - \alpha^0_{c,j})^2 - \sum_j \alpha_{c,j} R_{cj}(s),$$

$$J(k, \eta_k, s) = -\sum_j \kappa^I (\eta_{k,j} - \eta^0_{k,j})^2 - \sum_j \eta_{k,j} R_{kj}(s),$$

Best Response Given s, let $\eta^*(s)$ and $\alpha^*(s)$ be the solution of Lemma 6. Then, best-response map is:

$$\Phi : \mathcal{M} \to \mathcal{M}, \qquad \Phi(s) := (\eta^*(s), \alpha^*(s)).$$

B.2 Existence via Fixed Point

We are ready to proof the first claim of the proposition, existence:

Claim (Continuity of Φ). For every $s \in \mathcal{M}$: (i) $R(s)$ is well defined and continuous in s; (ii) Lemma 6 has a unique solution in $R(s)$; (iii) the map $s \mapsto (\eta^*(s), \alpha^*(s))$ is continuous. Hence Φ is a continuous self-map of $\mathcal{M}$.

> *Proof.* (i) By Lemma 1 the normalized solution $p(s, \epsilon)$ exists, is unique, and continuously depends on s. Then $R(s)$ is also unique and continuous. (ii) The quadratic penalty makes each objective strongly convex on the compact convex constraint set, so a unique minimizer exists. (iii) The continuity of $R(s)$ and strong convexity of the cost J yield the continuity of the best responses according to Berge's maximum theorem. Therefore, Φ is continuous and maps $\mathcal{M}$ to itself. □

Claim (Existence via fixed point). There exists a competitive equilibrium $(s^\star, R(s^\star))$ with $s^\star = (\eta^*, \alpha^*) \in \mathcal{M}$.

> *Proof.* By Claim 1, $\Phi : \mathcal{M} \to \mathcal{M}$ is a continuous self-map of a compact convex non-empty set. By Brouwer's fixed point theorem, there exists $s^* = (\eta^*, \alpha^*) \in \mathcal{M}$ with $\Phi(s^*) = s^*$. Given s^*, solve Lemma 2 to obtain the unique normalized price vector $p(s^\star, \epsilon)$. By construction, each η_k^* and α_c^* optimize their respective problems at $R(s^*)$ and prices satisfy unit costs; hence $(s^*, p(s^*, \epsilon))$ is a competitive equilibrium. □

B.3 GE Effects

To prove uniqueness, we show that the best response map Φ is a contraction in the space of networks. Here, prices, covariances and the risk adjusted costs depend on the whole network. Instead of relying on monotonicity, I bound how much mean prices and covariances can change when the network moves. The maximum change in expected prices and covariances are defined as:

$$L_{pp} := \sup_{s_1 \neq s_2} \frac{\| \operatorname{Cov}(p(s_1), p(s_1)) - \operatorname{Cov}(p(s_2), p(s_2)) \|_\infty}{\|s_1 - s_2\|_\infty} \tag{61}$$

$$L_p := \sup_{s_1 \neq s_2} \frac{\left\| \mathbb{E}[p(s_1)] - \mathbb{E}[p(s_2)] \right\|_\infty}{\|s_1 - s_2\|_\infty}, \qquad s = (\alpha, \eta). \tag{62}$$

Bound on the Change of Price covariance. Recall from Corollary 1.

$$\mathrm{Cov}(p(s_1), p(s_1)) = \mathcal{L}\Sigma_\epsilon\mathcal{L}^\top \tag{63}$$

Claim. There exists a finite constant L_p such that (61) holds. Moreover, one can choose

$$L_{pp} \;\le\; \frac{2\,\bar{\sigma}^2(1+\psi)^3}{\underline{\mu}^3}\sqrt{(1-\underline{\mu})^2 + \left(\tfrac{\psi}{1+\psi}\,\overline{\mu}\right)^2} \tag{64}$$

Proof. Let $s_1 = (\alpha_1, \eta_1)$ and $s_2 = (\alpha_2, \eta_2)$ be two feasible networks, and denote

$$\Delta s := s_1 - s_2, \quad \Delta\mathcal{L} := \mathcal{L}(s_1) - \mathcal{L}(s_2), \quad \Delta B := B(s_1) - B(s_2).$$

Then,

$$\mathrm{Cov}(p(s_1), p(s_1)) - \mathrm{Cov}(p(s_2), p(s_2)) = \mathcal{L}_1\Sigma_\epsilon(\Delta\mathcal{L})^\top + (\Delta\mathcal{L})\Sigma_\epsilon\mathcal{L}_1^\top, \tag{65}$$

Step 1: Bound on $\mathcal{L}(s)$ Recall

$$\mathcal{L}(s) = \left(I - (I-\mu)\eta - \tfrac{\psi}{1+\psi}\,\mu\alpha\right)^{-1}, \qquad s = (\eta, \alpha),$$

and define

$$H(s) := (I-\mu)\eta + \tfrac{\psi}{1+\psi}\,\mu\alpha, \qquad \mathcal{L}(s) = (I - H(s))^{-1}.$$

The row sum of row k of $H(s)$ is

$$\sum_j H_{kj}(s) = (1-\mu_k) + \tfrac{\psi}{1+\psi}\mu_k = 1 - \tfrac{\psi}{1+\psi}\mu_k.$$

Hence, using $\mu_k \ge \underline{\mu}$,

$$\|H(s)\|_\infty = \max_k \sum_j |H_{kj}(s)| = \max_k \sum_j H_{kj}(s) \le 1 - \tfrac{\psi}{1+\psi}\underline{\mu} =: x < 1.$$

By the Neumann series,

$$\mathcal{L}(s) = (I - H(s))^{-1} = \sum_{n=0}^{\infty} H(s)^n,$$

and thus

$$\|\mathcal{L}(s)\|_\infty \le \sum_{n=0}^{\infty} \|H(s)\|_\infty^n \le \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{1}{\frac{\psi}{1+\psi}\underline{\mu}} = \frac{1+\psi}{\underline{\mu}\,\psi},$$

Step 2: Bounds on $\Delta\mathcal{L}$. Write $H_\ell := H(s_\ell)$ and $\mathcal{L}_\ell := \mathcal{L}(s_\ell)$ for $\ell = 1, 2$. Using the matrix identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1},$$

with $A = I - H_1$ and $B = I - H_2$, we obtain

$$\mathcal{L}_1 - \mathcal{L}_2 = (I - H_1)^{-1} - (I - H_2)^{-1} = \mathcal{L}_1\,(H_1 - H_2)\,\mathcal{L}_2.$$

Taking $\|\cdot\|_\infty$ norms and applying the submultiplicativity of the operator norm,

$$\|\mathcal{L}_1 - \mathcal{L}_2\|_\infty \le \|\mathcal{L}_1\|_\infty\,\|H_1 - H_2\|_\infty\,\|\mathcal{L}_2\|_\infty.$$

By step1,

$$\|\mathcal{L}_1\|_\infty,\ \|\mathcal{L}_2\|_\infty \le \left(\frac{1+\psi}{\underline{\mu}\,\psi}\right)^2,$$

so

$$\|\mathcal{L}_1 - \mathcal{L}_2\|_\infty \le \left(\frac{1+\psi}{\underline{\mu}\,\psi}\right)^2 \|H_1 - H_2\|_\infty. \tag{66}$$

We now bound $\|H_1 - H_2\|_\infty$ in terms of Δs. By definition,

$$H_1 - H_2 = (I - \mu)(\eta_1 - \eta_2) + \tfrac{\psi}{1+\psi}\,\mu(\alpha_1 - \alpha_2) = (I - \mu)\Delta\eta + \tfrac{\psi}{1+\psi}\,\mu\Delta\alpha.$$

For any row k,

$$\sum_j |(H_1 - H_2)_{kj}| \le (1 - \mu_k)\sum_j |\Delta\eta_{kj}| + \tfrac{\psi}{1+\psi}\mu_k \sum_j |\Delta\alpha_{kj}|.$$

Thus,

$$\|H_1 - H_2\|_\infty \le (1 - \underline{\mu})\,\|\Delta\eta\|_\infty + \tfrac{\psi}{1+\psi}\,\overline{\mu}\,\|\Delta\alpha\|_\infty.$$

If we define

$$\|\Delta s\|_\infty := \max\{\|\Delta\eta\|_\infty,\ \|\Delta\alpha\|_\infty\},$$

then

$$\|H_1 - H_2\|_\infty \le \left[(1-\underline{\mu}) + \tfrac{\psi}{1+\psi}\,\overline{\mu}\right] \|\Delta s\|_\infty = K_H\,\|\Delta s\|_\infty.$$

Substituting this into (66) yields

$$\|\mathcal{L}(s_1) - \mathcal{L}(s_2)\|_\infty \le \left(\frac{1+\psi}{\underline{\mu}\,\psi}\right)^2 K_H\,\|\Delta s\|_\infty,$$

Replace into 65 to get the result □

Bound on the Change of Mean Prices Recall from Corollary 1. that expected prices can be written as

$$\mathbb{E}[p(s)] = -\,\mathcal{L}(s)\,\theta + \mathcal{L}(s)\,B(s). \tag{67}$$

where

$$B(s) = \frac{\psi}{1+\psi}\,\mu\,A(\alpha^\star) + \mathrm{diag}\Big(\tfrac{\psi}{1+\psi}\,[\mu\alpha^\star + (I-\mu)\eta^\star]\Big)\log\tau + \frac{1}{1+\psi}(\log\mu + \log\nu) \tag{68}$$

Claim (Bounded sensitivity of mean prices). There exists a finite constant L_p such that (62) holds. Moreover, one can choose

$$L_p \;\le\; \left(\overline{\theta} + \bar{B}\right)\left(\frac{1+\psi}{\underline{\mu}}\right)^2 \sqrt{(1-\underline{\mu})^2 + \left(\tfrac{\psi\,\overline{\mu}}{1+\psi}\right)^2}, \tag{69}$$

where

$$\bar{B} := \frac{\psi}{1+\psi}\,\overline{\mu}\,\bar{A} + \frac{\psi}{1+\psi}\,\log\overline{\tau} + \frac{1}{1+\psi}\big(\log\overline{\mu} + \log\overline{\nu}\big). \tag{70}$$

Proof. Let $s_1 = (\alpha_1, \eta_1)$ and $s_2 = (\alpha_2, \eta_2)$ be two feasible networks, and denote

$$\Delta s := s_1 - s_2, \quad \Delta\mathcal{L} := \mathcal{L}(s_1) - \mathcal{L}(s_2), \quad \Delta B := B(s_1) - B(s_2).$$

From (67),

$$\mathbb{E}[p(s_1)] - \mathbb{E}[p(s_2)] = -\,\Delta\mathcal{L}\,\theta + \Delta\mathcal{L}\,B(s_2) + \mathcal{L}(s_1)\,\Delta B. \tag{71}$$

Taking $\|\cdot\|_\infty$ norms and using the triangle inequality,

$$\big\|\mathbb{E}[p(s_1)] - \mathbb{E}[p(s_2)]\big\|_\infty \le \|\Delta\mathcal{L}\|_\infty\left(\|\theta\|_\infty + \|B(s_2)\|_\infty\right) + \|\mathcal{L}(s_1)\|_\infty\,\|\Delta B\|_\infty. \tag{72}$$

Step 1 and 2: Bounds on $\mathcal{L}(s)$ and $\Delta\mathcal{L}$. Take steps 1 and 2 from the proof of claim B.3

Step 3: uniform bound on $B(s)$. From (68)

$$\|\mu A(\alpha^\star)\|_\infty \leq \overline{\mu}\, \|A(\alpha^\star)\|_\infty \leq \overline{\mu}\, \bar{A}.$$

The diagonal term is bounded by

$$\left\| \operatorname{diag}\Big(\tfrac{\psi}{1+\psi}\left[\mu\alpha^\star + (I-\mu)\eta^\star\right]\Big) \log\tau \right\|_\infty \leq \frac{\psi}{1+\psi}\, \|\log\tau\|_\infty,$$

since $\mu\alpha^\star + (I-\mu)\eta^\star$ is a vector of shares in $[0,1]$. Finally,

$$\left\| \frac{1}{1+\psi}(\log\mu + \log\nu) \right\|_\infty \leq \frac{1}{1+\psi}\big(\|\log\mu\|_\infty + \|\log\nu\|_\infty\big).$$

Combining these, we obtain the uniform bound

$$\|B(s)\|_\infty \leq \bar{B} \quad \text{for all } s, \tag{73}$$

with $\bar{B}$ given by (70).

Step 3: bound on ΔB. The mapping $s \mapsto B(s)$ in (68) is affine in (α, η), up to the adjustment term $A(\alpha^\star)$, which is bounded. Hence there exists a finite constant C_B such that

$$\|\Delta B\|_\infty \leq C_B\, \|\Delta s\|_\infty, \tag{74}$$

and we can choose the crude bound $C_B \leq \bar{B}$.

Step 4: collecting bounds. Substituting (73), and (74) into (72), we obtain

$$\left\| \mathbb{E}[p(s_1)] - \mathbb{E}[p(s_2)] \right\|_\infty \leq \left(\frac{1+\psi}{\underline{\mu}}\right)^2 D\big(\|\theta\|_\infty + \bar{B}\big)\|\Delta s\|_\infty + \frac{1+\psi}{\underline{\mu}\psi}\, C_B\, \|\Delta s\|_\infty.$$

Absorbing the second term into the first by enlarging $\bar{B}$ if necessary, we obtain the simpler bound

$$\left\|\mathbb{E}[p(s_1)] - \mathbb{E}[p(s_2)]\right\|_\infty \le \left(\overline{\theta} + \bar{B}\right)\left(\frac{1+\psi}{\underline{\mu}}\right)^2 \sqrt{(1-\underline{\mu})^2 + \left(\frac{\psi\,\overline{\mu}}{1+\psi}\right)^2},$$

which implies (69) by the definition (62) of L_p. □

B.4 Contraction and Uniqueness

The dependence of risk adjusted costs on the network s is summarized by

$$\|R(s_1) - R(s_2)\|_\infty \;\le\; \Lambda\,\|s_1 - s_2\|_\infty, \qquad \Lambda := |1-\gamma|\,L_{pp} + L_p,$$

for all $s_1, s_2 \in \mathcal{M}$, where $\|\cdot\|_\infty$ denotes the sup-norm and $\|s\|_\infty := \max\{\|\eta\|_\infty, \|\alpha\|_\infty\}$ for $s = (\eta, \alpha)$.

Claim (Uniqueness). If

$$\min\{\kappa^F, \kappa^I\} > \Lambda,$$

then the best response operator Φ is a contraction on $\mathcal{M}$. Hence, $s^* = (\eta^*, \alpha^*)$ and the corresponding normalized price vector p^* are unique.

Proof. Step 1 (Inner problems via Lemma 6). By Lemma 6, for each $m \in \{I, F\}$ and each pair (c,k) the optimal share $s^m_{ck}(s)$ satisfies

$$s^m_{ck}(s) = \max\left\{s^{0,m}_{ck} - \frac{1}{2\kappa^m}\big(\lambda^m(s) - R_{ck}(s)\big),\ 0\right\},$$

where $\lambda^m(s)$ is the average of $R_{ck}(s)$ over the active set and is linear in $R(s)$. Define

$$x^m_{ck}(s) := s^{0,m}_{ck} - \frac{1}{2\kappa^m}\big(\lambda^m(s) - R_{ck}(s)\big).$$

Then, for any s_1, s_2,

$$|x^m_{ck}(s_1) - x^m_{ck}(s_2)| \le \frac{1}{2\kappa^m}\Big(|\lambda^m(s_1) - \lambda^m(s_2)| + |R_{ck}(s_1) - R_{ck}(s_2)|\Big) \le \frac{1}{\kappa^m}\,\|R(s_1) - R(s_2)\|_\infty,$$

since λ^m is an average of components of R. The scalar map $z \mapsto \max\{z, 0\}$ is 1-Lipschitz, so

$$|s^m_{ck}(s_1) - s^m_{ck}(s_2)| \leq |x^m_{ck}(s_1) - x^m_{ck}(s_2)|.$$

Taking the sup over (c, k) gives

$$\|s^m(s_1) - s^m(s_2)\|_\infty \;\leq\; \frac{1}{\kappa^m}\,\|R(s_1) - R(s_2)\|_\infty, \qquad m \in \{I, F\}. \tag{75}$$

Identifying $s^F(s) \equiv \alpha^*(s)$ and $s^I(s) \equiv \eta^*(s)$, (75) becomes

$$\|\alpha^*(s_1)-\alpha^*(s_2)\|_\infty \leq \frac{1}{\kappa^F}\,\|R(s_1)-R(s_2)\|_\infty, \qquad \|\eta^*(s_1)-\eta^*(s_2)\|_\infty \leq \frac{1}{\kappa^I}\,\|R(s_1)-R(s_2)\|_\infty.$$

Step 2 (Sensitivity of risk-adjusted costs). By the bounds on mean prices and covariances,

$$\|R(s_1) - R(s_2)\|_\infty \;\leq\; \Lambda\,\|s_1 - s_2\|_\infty \qquad \text{for all } s_1, s_2 \in \mathcal{M}.$$

Step 3 (Contraction of Φ). The aggregate best response operator is

$$\Phi(s) := (\eta^*(s), \alpha^*(s)).$$

Using the product sup-norm on $\mathcal{M}$ and the bounds above,

$$\begin{aligned}
\|\Phi(s_1) - \Phi(s_2)\|_\infty &= \max\left\{\|\eta^*(s_1) - \eta^*(s_2)\|_\infty,\ \|\alpha^*(s_1) - \alpha^*(s_2)\|_\infty\right\} \\
&\leq \max\left\{\frac{1}{\kappa^I}, \frac{1}{\kappa^F}\right\}\|R(s_1) - R(s_2)\|_\infty \\
&\leq \frac{\Lambda}{\min\{\kappa^F, \kappa^I\}}\,\|s_1 - s_2\|_\infty.
\end{aligned}$$

If $\min\{\kappa^F, \kappa^I\} > \Lambda$,

$$\frac{\Lambda}{\min\{\kappa^F, \kappa^I\}} < 1,$$

so Φ is a contraction on the complete metric space $(\mathcal{M}, \|\cdot\|_\infty)$. By Banach's fixed-point theorem, Φ admits a unique fixed point s^*. The corresponding normalized price vector $p^* = p(s^*, \epsilon)$ is unique by Lemma 2, which yields uniqueness of the competitive equilibrium. □

C Effect of Aggregate TFP Shocks

C.1 Proof of Proposition 3

Part 1: Contemporaneous Effect At time t, the network is predetermined (chosen at $t-1$). The value-added is:

$$a_{ci,t} = \epsilon_{ci,t} + A(\eta^{\star}_{ci,t}) + \mu_{ci} l_{ci,t} \tag{76}$$

Taking the derivative with respect to g_t:

$$\frac{\partial a_{ci,t}}{\partial g_t} = \frac{\partial \epsilon_{ci,t}}{\partial g_t} + \frac{\partial A(\eta^{\star}_{ci,t})}{\partial g_t} + \mu_{ci} \frac{\partial l_{ci,t}}{\partial g_t} \tag{77}$$

$$= 1 + 0 + \mu_{ci} \cdot \frac{\psi}{1+\psi} \cdot \ell_c \tag{78}$$

$$= 1 + \mu_{ci} \frac{\psi}{1+\psi} \ell_c \tag{79}$$

where we used:

- $\frac{\partial \epsilon_{ci,t}}{\partial g_t} = 1$ (direct effect)
- $\frac{\partial A(\eta^{\star}_{ci,t})}{\partial g_t} = 0$ (network predetermined)
- $\frac{\partial l_{ci,t}}{\partial g_t} = \frac{\psi}{1+\psi} \ell_c$ (from $\frac{\partial p_{c,t}}{\partial g_t} = -\ell_c$)

Part 2: Lagged Effect At $t+1$, both the persistent shock and the adjusted network affect value-added:

$$a_{ci,t+1} = \epsilon_{ci,t+1} + A(\eta^{\star}_{ci,t+1}) + \mu_{ci} l_{ci,t+1} \tag{80}$$

The total derivative with respect to θ_g includes:

Direct persistence: Through $g_{t+1} = \rho_g g_t + \theta_{g,t+1}$:

$$\frac{\partial \epsilon_{ci,t+1}}{\partial \theta_g} = \rho_g \tag{81}$$

Labor supply from persistent shock:

$$\mu_{ci} \frac{\partial l_{ci,t+1}}{\partial g_{t+1}} \cdot \frac{\partial g_{t+1}}{\partial \theta_g} = \mu_{ci} \frac{\psi}{1+\psi} \ell_c \cdot \rho_g \tag{82}$$

Network adjustment effects: The network chosen at t for use at $t+1$ responds to the shock. This includes:

- Changes in own TFP: $\frac{\partial A(\eta^{\star}_{ci,t+1})}{\partial \eta_{ci}} \cdot \frac{d\eta^{*}_{ci}}{d\theta_g}$
- Price effects from all sectors' adjustments: $\sum_{c'i'} \frac{\partial p_{ci,t+1}}{\partial \eta_{c'i'}} \cdot \frac{d\eta^{*}_{c'i'}}{d\theta_g}$
- Demand effects from final goods adjustments: $\sum_{c'} \frac{\partial \nu_{ci,t+1}}{\partial \alpha_{c'}} \cdot \frac{d\alpha^{*}_{c'}}{d\theta_g}$

These are summarized as the network adjustment term. □

C.2 Proof of Proposition 4

Intermediate Goods Network (η) Intermediate producer (ci) chooses $\eta_{ci,c\hat{k}}$ to minimize expected costs:

$$\min_{\eta_{ci}} E_t \left[\exp\left(\frac{\kappa_I}{2} \sum_{c\hat{k}} (\eta_{ci,c\hat{k}} - \eta^0_{ci,c\hat{k}})^2 + \sum_{c\hat{k}} \eta_{ci,c\hat{k}} p_{c\hat{k},t+1} \right) \right] \tag{83}$$

The first-order condition yields:

$$\eta^{*}_{ci,c\hat{k}} = \eta^0_{ci,c\hat{k}} - \frac{1}{2\kappa_I} (R_{ci,c\hat{k}} - \bar{R}_{ci}) \tag{84}$$

where the risk-adjusted price is:

$$R_{ci,c\hat{k}} = E_t[p_{c\hat{k},t+1}] + \log \tau_{ci,c\hat{k}} - \mathrm{Cov}_t[p_{c\hat{k},t+1}, m_{ci,t+1}] \tag{85}$$

Given:

$$\frac{\partial E_t[p_{c\hat{k},t+1}]}{\partial \theta_g} = -\rho_g \ell_{c\hat{k}} \tag{86}$$

$$\frac{\partial \mathrm{Cov}_t[p_{c\hat{k},t+1}, m_{ci,t+1}]}{\partial \theta_g} = -\rho_g \ell_{c\hat{k}} \ell_c \frac{\psi}{1+\psi} \tag{87}$$

We obtain:

$$\frac{\partial R_{ci,c\hat{k}}}{\partial \theta_g} = -\rho_g \ell_{c\hat{k}} \left(1 - \ell_c \frac{\psi}{1+\psi}\right) \tag{88}$$

Similarly for the average: $\frac{\partial \bar{R}_{ci}}{\partial \theta_g} = -\rho_g \bar{\ell}_{ci}(1 - \ell_c \frac{\psi}{1+\psi})$

Therefore:

$$\frac{d\eta^*_{ci,c\hat{k}}}{d\theta_g} = \frac{\rho_g}{2\kappa_I}(\ell_{c\hat{k}} - \bar{\ell}_{ci}) \left(1 - \ell_c \frac{\psi}{1+\psi}\right) \tag{89}$$

Final Goods Network (α) The final goods aggregator in country c faces an analogous problem with adjustment cost κ_F. Following the same steps:

$$\frac{d\alpha^*_{c,c\hat{k}}}{d\theta_g} = \frac{\rho_g}{2\kappa_F}(\ell_{c\hat{k}} - \bar{\ell}_c) \left(1 - \ell_c \frac{\psi}{1+\psi}\right) \quad \square \tag{90}$$

C.3 Proof of Proposition 5

Part 1: Contemporaneous Effect At time t, the network is predetermined and value-added depends on realized shocks:

$$a_{ci,t} = \epsilon_{ci,t} + A(\eta^{\star}_{ci,t}) + \mu_{ci} l_{ci,t} \tag{91}$$

Since σ_g^2 affects only the second moment of shocks, not their realization:

$$\frac{\partial a_{ci,t}}{\partial \sigma_g^2} = 0 \tag{92}$$

Part 2: Lagged Effect Changes in risk at t affect network choices for $t+1$. Since risk doesn't affect expected productivity ($E[\epsilon_{ci,t+1}]$ unchanged), the entire effect operates through network reorganization and its impact on prices and demand. □

C.4 Proof of Proposition 6

Step 1: Common Risk Structure

Both intermediate and final goods producers in country c face the same consumption price index:

$$p_{c,t+1} = \sum_{c'j} \alpha_{c,c'j} p_{c'j,t+1} \tag{93}$$

Under aggregate shocks: $p_{c,t+1} = -g_{t+1}\ell_c$ + other shocks, where $\ell_c = \sum_{c'j} \alpha_{c,c'j}\ell_{c'j}$.

Step 2: Risk-Adjusted Prices

For any producer in country c, the risk-adjusted price of supplier $(c\hat{k})$ is:

$$R_{c,c\hat{k}} = E_t[p_{c\hat{k},t+1} + \log \tau_{c,c\hat{k}}] - (\gamma - 1)\text{Cov}_t[p_{c\hat{k},t+1}, p_{c,t+1}]\frac{\psi}{1+\psi} \tag{94}$$

The covariance with consumption prices:

$$\text{Cov}_t[p_{c\hat{k},t+1}, p_{c,t+1}] = \ell_{c\hat{k}}\ell_c\sigma_g^2 \tag{95}$$

Step 3: Effect of Risk on Covariance

Taking the derivative:

$$\frac{\partial \text{Cov}_t[p_{c\hat{k},t+1}, p_{c,t+1}]}{\partial \sigma_g^2} = \ell_{c\hat{k}}\ell_c \tag{96}$$

Therefore:

$$\frac{\partial R_{c,c\hat{k}}}{\partial \sigma_g^2} = -(\gamma - 1)\ell_{c\hat{k}}\ell_c\frac{\psi}{1+\psi} \tag{97}$$

Similarly for the average: $\frac{\partial \bar{R}_c}{\partial \sigma_g^2} = -(\gamma - 1)\bar{\ell}_c\ell_c\frac{\psi}{1+\psi}$

Step 4: Network Optimization

For intermediate goods with adjustment cost κ_I:

$$\frac{d\eta^*_{ci,c\hat{k}}}{d\sigma_g^2} = -\frac{1}{2\kappa_I}\left[\frac{\partial R_{c,c\hat{k}}}{\partial \sigma_g^2} - \frac{\partial \bar{R}_c}{\partial \sigma_g^2}\right] = -\frac{(\gamma - 1)}{2\kappa_I}(\ell_{c\hat{k}} - \bar{\ell}_c)\ell_c\frac{\psi}{1+\psi} \tag{98}$$

For final goods with adjustment cost κ_F, the identical risk structure yields:

$$\frac{d\alpha^*_{c,c\hat{k}}}{d\sigma_g^2} = -\frac{(\gamma - 1)}{2\kappa_F}(\ell_{c\hat{k}} - \bar{\ell}_c)\ell_c\frac{\psi}{1+\psi} \quad \square \tag{99}$$

D Variance Estimation Full Model

Let i index country–sector units, with $i = (c(i), j(i))$, and t index time. I use the following factor structure for demeaned growth innovations:

$$\varepsilon_{i,t} \;=\; g_t \;+\; a_{c(i),t} \;+\; b_{j(i),t} \;+\; u_{i,t}, \qquad u_{i,t} \sim \mathcal{N}(0, \sigma_{ci}), \tag{100}$$

where $(g_t, a_{c,t}, b_{j,t})$ are orthogonal Gaussian factors with time-varying variances $\sigma_g(t),\ \sigma_a(c,t),\ \sigma_b(j,t)$, and σ_{ci} is a time-invariant idiosyncratic variance. Define the total variance for unit i in period t as

$$\sigma^2_{i,t} \equiv \sigma_g(t) + \sigma_a(c(i), t) + \sigma_b(j(i), t) + \sigma_{ci}, \quad \tilde{\varepsilon}_{i,t} \equiv \varepsilon_{i,t}/\sigma_{i,t}.$$

Distributional moments (generalized tail comovement). For a grid of quantile probabilities $\mathcal{T} = \{\tau_k\}$ with $z_\tau \equiv \Phi^{-1}(\tau)$, I form three families of empirical moments:

$$\text{Global: } B^{\text{emp}}_t(\tau) \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\left\{\tilde{\varepsilon}_{i,t} \le z_\tau\right\}, \tag{101}$$

$$\text{Country } c\text{: } B^{\text{emp}}_{c,t}(\tau) \equiv \frac{1}{J}\sum_{j=1}^{J} \mathbf{1}\left\{\tilde{\varepsilon}_{(c,j),t} \le z_\tau\right\}, \tag{102}$$

$$\text{Sector } j\text{: } B^{\text{emp}}_{j,t}(\tau) \equiv \frac{1}{C}\sum_{c=1}^{C} \mathbf{1}\left\{\tilde{\varepsilon}_{(c,j),t} \le z_\tau\right\}. \tag{103}$$

These are distributional versions of tail-comovement statistics: they use the *full set of quantiles* (not only a single lower tail) and can be computed on all units, within a country, or within a sector.

Model-implied moments under normality. Under (100) and Gaussianity, the conditional distribution of $\tilde{\varepsilon}_{i,t}$ given the factors is normal with mean a weighted sum of standardized factors and variance equal to one minus the corresponding variance shares. Let

$$R^2_{g,i,t} \equiv \frac{\sigma_g(t)}{\sigma^2_{i,t}}, \quad R^2_{a,i,t} \equiv \frac{\sigma_a(c(i), t)}{\sigma^2_{i,t}}, \quad R^2_{b,i,t} \equiv \frac{\sigma_b(j(i), t)}{\sigma^2_{i,t}}, \quad z_{g,t} \equiv \frac{g_t}{\sqrt{\sigma_g(t)}},\ z_{a,c,t} \equiv \frac{a_{c,t}}{\sqrt{\sigma_a(c,t)}},\ z_{b,j,t} \equiv \frac{b_{j,t}}{\sqrt{\sigma_b(j,t)}}.$$

Then, averaging across the relevant cross-sections,

$$B_t^{\text{mod}}(\tau;\theta) \equiv \frac{1}{N}\sum_{i=1}^{N}\Phi\left(\frac{z_\tau - \sqrt{R^2_{g,i,t}}\,z_{g,t} - \sqrt{R^2_{a,i,t}}\,z_{a,c(i),t} - \sqrt{R^2_{b,i,t}}\,z_{b,j(i),t}}{\sqrt{1 - R^2_{g,i,t} - R^2_{a,i,t} - R^2_{b,i,t}}}\right), \tag{104}$$

$$B_{c,t}^{\text{mod}}(\tau;\theta) \equiv \frac{1}{J}\sum_{j=1}^{J}\Phi\left(\frac{z_\tau - \sqrt{R^2_{g,(c,j),t}}\,z_{g,t} - \sqrt{R^2_{a,(c,j),t}}\,z_{a,c,t} - \sqrt{R^2_{b,(c,j),t}}\,z_{b,j,t}}{\sqrt{1 - R^2_{g,(c,j),t} - R^2_{a,(c,j),t} - R^2_{b,(c,j),t}}}\right), \tag{105}$$

$$B_{j,t}^{\text{mod}}(\tau;\theta) \equiv \frac{1}{C}\sum_{c=1}^{C}\Phi\left(\frac{z_\tau - \sqrt{R^2_{g,(c,j),t}}\,z_{g,t} - \sqrt{R^2_{a,(c,j),t}}\,z_{a,c,t} - \sqrt{R^2_{b,(c,j),t}}\,z_{b,j,t}}{\sqrt{1 - R^2_{g,(c,j),t} - R^2_{a,(c,j),t} - R^2_{b,(c,j),t}}}\right), \tag{106}$$

where $\theta = \{\sigma_g(t), \sigma_a(c,t), \sigma_b(j,t), \sigma_{ci}\}$ collects the variance components.

Why only variance ratios matter. The arguments of $\Phi(\cdot)$ in (104)–(106) involve the *standardized* factors $z_{g,t}, z_{a,c,t}, z_{b,j,t}$ and the *shares* $R^2_{\cdot,i,t}$. Hence, the model-implied moments depend on the *ratios* $\sigma_g(t)/\sigma^2_{i,t}$, $\sigma_a(c,t)/\sigma^2_{i,t}$, and $\sigma_b(j,t)/\sigma^2_{i,t}$, not on levels. This invariance motivates estimating *time-varying variances* by GMM.

GMM objective for time-varying variances. Let $\mathcal{T}$ be the set of quantiles and $\{w_\tau\}$ be weights. I estimate θ by minimizing the squared distance between empirical and model moments across time, quantiles, and cross-sections:

$$\min_\theta \sum_t \sum_{\tau\in\mathcal{T}} w_\tau \Big\{\big[B_t^{\text{emp}}(\tau) - B_t^{\text{mod}}(\tau;\theta)\big]^2 + \sum_c \big[B_{c,t}^{\text{emp}}(\tau) - B_{c,t}^{\text{mod}}(\tau;\theta)\big]^2 + \sum_j \big[B_{j,t}^{\text{emp}}(\tau) - B_{j,t}^{\text{mod}}(\tau;\theta)\big]^2\Big\}. \tag{107}$$

Operationally, I update one block of variances at a time (global, country, sector) holding the others fixed, and rescale to match average target variances; convergence is declared when changes in all variance paths fall below a tolerance.

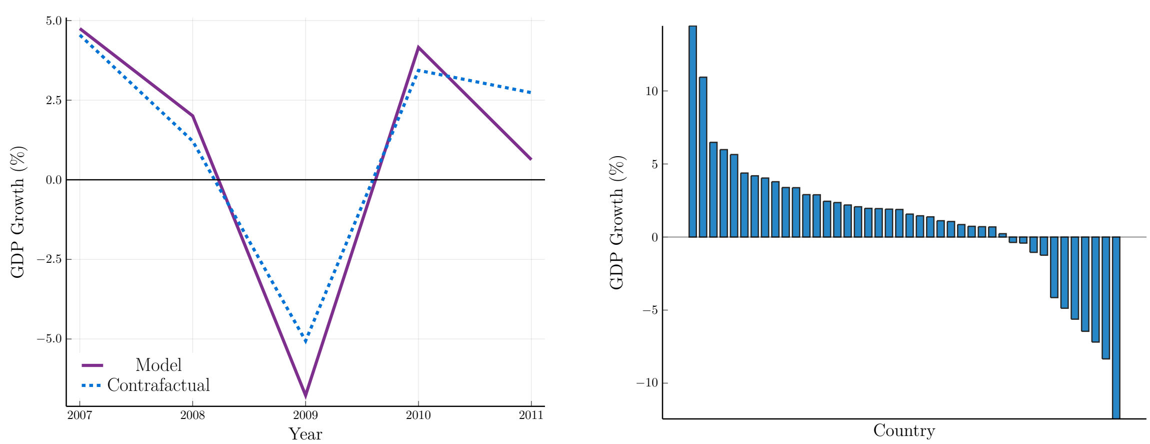
E Fixed Expectations

In this subsection, I present the result of the first contrafactual of the paper. I keep expectations at their 2007 levels, which means that firms do not adjust the network during the crisis. Next, I study (i) GDP growth, (ii) trade, and (iii) welfare.

Change in GDP Figure 10 compares the GDP growth in the model and in the counterfactual.

Panel (a) shows that the contrafactual scenario shows a significantly stronger impact of crises, with an average negative GDP growth that is 2% more severe. However, the distribution is asymmetric: some countries have significant positive effects, many show negligible effects near zero, and some experience large negative impacts. Panel (b) ranks countries according to the impact on GDP growth. Thus, network adjustments had a heterogeneous effect in terms of GDP growth, negligible for many, and contractionary for some others.

In the contrafactual scenario with a fixed network, the model predicts a distribution of GDP outcomes with extreme values. This implies that the effects of crises could be more pronounced. Leading a distribution with a wider spread of GDP outcomes across countries compared to the data.

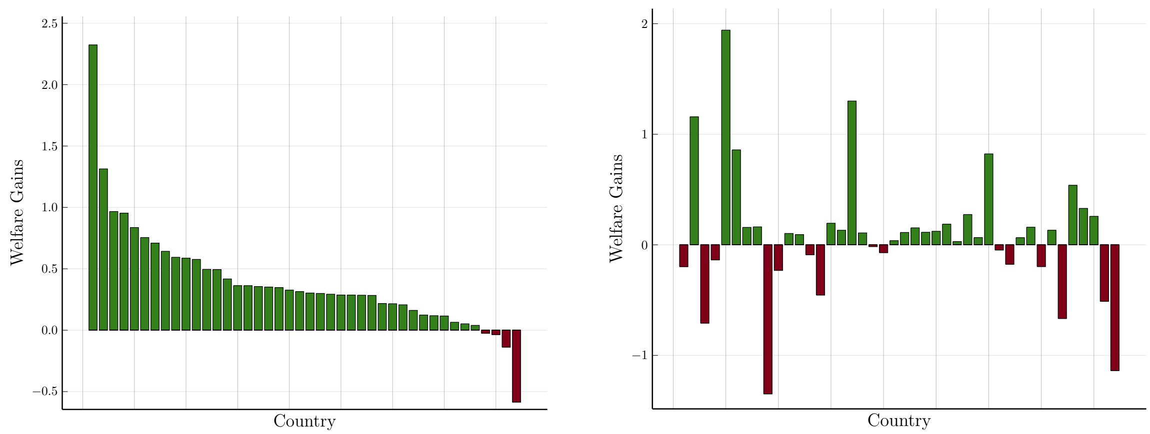


(a) Mean GDP Growth

(b) GDP Growth by Country

Figure 10: Effect on GDP Growth

Welfare Figure 11 compares the' welfare changes of countries from the adjustments of the expenditure-share in 2009 and 2010. Countries are classified according to the impact of welfare in 2009. Both years show an asymmetric distribution: most countries see welfare gains in 2009, while a few face welfare losses from the adjustment of the network. In 2010, the distribution showed smaller gains and more countries experiencing welfare losses.



(a) 2009

(b) 2010

Figure 11: Welfare Impact

Welfare losses reflect two characteristics of the model, one is that the equilibrium is not efficient due to the incomplete markets across countries, and that there are spillovers of the choices of the network between countries that are not internalized in the choice of firms.

F Calibration

F.1 Calibration of risk aversion and adjustment costs

This appendix describes how I discipline the risk aversion parameter γ and the adjustment costs κ^I and κ^F from a demand-system regression on input shares.

Step 1: Structural relationship. For a buyer n (country industry) and a supplier m (country-industry), let $s_{mn,t}$ denote the share of m in n's input bundle at date t, and s^0_{mn} the corresponding share under the ideal technology (α^0, η^0). Define excess shares

$$\tilde{s}_{mn,t} \equiv s_{mn,t} - s^0_{mn}.$$

Let $\bar{p}_m \equiv \mathbb{E}[p^D_m]$ be the supplier's factory gate expected price (common across buyers), and let

$$\sigma^p_{cm} \equiv \mathrm{Cov}\big(p^D_c, p^D_m\big)$$

be the covariance between the buyer country's final good price and the supplier's price. For each buyer $n = (c, i)$, I demean these characteristics over the set of active suppliers $\mathcal{S}_n$:

$$\tilde{p}_{mn,t} \equiv \bar{p}_m - \bar{p}_n, \qquad \bar{p}_n \equiv \frac{1}{|\mathcal{S}_n|} \sum_{m' \in \mathcal{S}_n} \bar{p}_{m'},$$

$$\tilde{\sigma}^p_{mn} \equiv \sigma^p_{cm} - \bar{\sigma}^p_n, \qquad \bar{\sigma}^p_n \equiv \frac{1}{|\mathcal{S}_n|} \sum_{m' \in \mathcal{S}_n} \sigma^p_{cm'}.$$

The FOC condition for optimal input shares implies, for each network type $\tau \in \{I, F\}$ (intermediate or final),

$$\tilde{s}_{mn,t} = a_n - \frac{1}{2\kappa^\tau} \tilde{p}_{mn,t} - \frac{1-\gamma}{2\kappa^\tau} \tilde{\sigma}^p_{mn}, \tag{108}$$

where a_n collects terms that are constant across suppliers for buyer n, and κ^τ is the adjustment cost for network type τ.

Taking time averages over t in (108) yields

$$\bar{\tilde{s}}_{mn} = a_n - \frac{1}{2\kappa^\tau} \bar{\tilde{p}}_{mn} - \frac{1-\gamma}{2\kappa^\tau} \tilde{\sigma}^p_{mn}, \tag{109}$$

so the average deviations of the network from its ideal counterpart are linearly related to the average relative prices and covariances.

Step 2: Empirical analogue. In the data, I construct empirical counterparts $\widehat{\tilde{p}}_{mn}$ and $\widehat{\tilde{\sigma}}^{p}_{mn}$, and estimate a pooled regression for intermediate and final networks:

$$\bar{\tilde{s}}_{mn} = a_n + \beta^I_\mu D^I_n \widehat{\tilde{p}}_{mn} + \beta^F_\mu D^F_n \widehat{\tilde{p}}_{mn} + \beta_\sigma \widehat{\tilde{\sigma}}^{p}_{mn} + \varepsilon_{mn}, \tag{110}$$

where D^I_n and D^F_n are dummies that indicate whether buyer n belongs to the intermediate or final network, respectively, and a_n are buyer fixed effects.

$$X = \left[D^I_n \widehat{\tilde{p}}_{mn},\ D^F_n \widehat{\tilde{p}}_{mn},\ \widehat{\tilde{\sigma}}^{p}_{mn}\right],$$

β^I_μ and β^F_μ capture the sensitivity of shares to expected prices in the intermediate and final networks, while β_σ captures the common sensitivity to risk (covariance) across both networks.

Step 3: Mapping to structural parameters. Comparing (109) with (110) yields the moment conditions

$$\beta^I_\mu = -\frac{1}{2\kappa^I}, \qquad \beta^F_\mu = -\frac{1}{2\kappa^F},$$

$$\beta_\sigma = \frac{1-\gamma}{2\kappa^I} = \frac{1-\gamma}{2\kappa^F},$$

so that

$$\kappa^I = -\frac{1}{2\beta^I_\mu}, \qquad \kappa^F = -\frac{1}{2\beta^F_\mu}, \qquad \gamma = 1 + \frac{\beta_\sigma}{\beta^I_\mu} = 1 + \frac{\beta_\sigma}{\beta^F_\mu}.$$

In practice, I compute γ using both expressions and verify that the two implied values are very similar; the reported value $\gamma = 6.3$ is the baseline. The calibrated κ^I and κ^F implied by β^I_μ and β^F_μ generate average deviations of the endogenous networks $(\alpha^\star, \eta^\star)$ from the ideal technology (α^0, η^0) that match the empirical dispersion in observed networks.

F.2 Algorithm

Fixed-point network computation.

- **Inputs:** ε (shocks), Σ_ε (shock covariance), α_0, η_0 (ideal networks), κ^F, κ^I (adjustment costs), τ (trade costs), γ (risk aversion), μ (labor shares), ψ (Frisch elasticity).

- **Output:** Equilibrium networks (α^*, η^*).
- **Initialization:**
  - Compute initial $\mathcal{R}$ using (α_0, η_0).
  - Set $\alpha^{(0)} \leftarrow \text{UpdateNetwork}(\alpha_0, \kappa^F, \mathcal{R})$, $\quad \eta^{(0)} \leftarrow \text{UpdateNetwork}(\eta_0, \kappa^I, \mathcal{R})$.
  - Set $t \leftarrow 0$ and *converged* $\leftarrow$ False.
- **Iterate while not converged and $t <$ maxiter:**
  1. Increment iteration: $t \leftarrow t + 1$.
  2. **Leontief inverse:**
  $$\mathcal{L}^{(t)} \leftarrow \left[I - \tfrac{\psi}{1+\psi}(I - D_\mu)\alpha^{(t-1)} - D_\mu \eta^{(t-1)}\right]^{-1}.$$
  3. **Influence matrix:** $M^{(t)} \leftarrow \text{ComputeInfluence}(D_\mu, \alpha^{(t-1)}, \mathcal{L}^{(t)}, \psi)$.
  4. **Prices and covariances:**
     - $P^{(t)} \leftarrow -\mathcal{L}^{(t)}\varepsilon$ (price realizations).
     - $E[P^{(t)}] \leftarrow \text{mean}(P^{(t)})$ (expected prices).
     - $\text{Cov}(P, P)^{(t)} \leftarrow (\mathcal{L}^{(t)})^\top \Sigma_\varepsilon \mathcal{L}^{(t)}$ (price covariances).
     - $\text{Cov}(g, P)^{(t)} \leftarrow \text{ComputeGDPPriceCov}(M^{(t)}, \mathcal{L}^{(t)}, \Sigma_\varepsilon)$.
  5. **Risk-adjusted prices:**
  $$\mathcal{R}^{(t)}_{c\hat{k}} \leftarrow E[\ln P^{(t)}_{\hat{c}k}] + \ln \tau_{c,\hat{c}k} \; + \; (1-\gamma)\,\text{Cov}(p_c, P_{\hat{c}k})^{(t)},$$
  where p_c is the final-good price index for country c.
  6. **Network updates with damping:**
     - $\tilde{\alpha}^{(t)} \leftarrow \text{UpdateNetwork}(\alpha_0, \kappa^F, \mathcal{R}^{(t)})$, $\quad \tilde{\eta}^{(t)} \leftarrow \text{UpdateNetwork}(\eta_0, \kappa^I, \mathcal{R}^{(t)})$.
  7. **Convergence check:**
     - $\text{diff}_\alpha \leftarrow \max|\alpha^{(t)} - \alpha^{(t-1)}|$, $\quad \text{diff}_\eta \leftarrow \max|\eta^{(t)} - \eta^{(t-1)}|$.
     - $\text{diff} \leftarrow \max(\text{diff}_\alpha, \text{diff}_\eta)$; set *converged* $\leftarrow$ True if diff $<$ tol.
- **Return:** $\alpha^{(t)}, \eta^{(t)}$.

UpdateNetwork: constrained simplex problem.

- **Inputs:** m_0 (ideal network), κ (adjustment cost), $\mathcal{R}$ (risk-adjusted prices).
- **Output:** m^* (optimal network satisfying the simplex constraint).
- **Procedure (for each sector/country i):**

 1. Initialize the active set $\mathcal{A}_i \leftarrow \{1, \ldots, N\}$ and set *converged* $\leftarrow$ False.
 2. While not *converged*:

    (a) **Lagrange multiplier:**

$$\lambda_i \leftarrow \frac{1 - \sum_{j\in\mathcal{A}_i} m_{0,ij} + \sum_{j\in\mathcal{A}_i} \frac{\mathcal{R}_{ij}}{2\kappa}}{\sum_{j\in\mathcal{A}_i} \frac{1}{2\kappa}}.$$

    (b) **Update coefficients (and enforce nonnegativity):** For each $j \in \mathcal{A}_i$,

$$m^*_{ij} \leftarrow m_{0,ij} + \frac{1}{2\kappa}(\lambda_i - \mathcal{R}_{ij}),$$

    set $m^*_{ij} \leftarrow 0$ if this value is negative, and remove such j from $\mathcal{A}_i$.

    (c) If no new violations occur, set *converged* $\leftarrow$ True.

- **Return:** m^*.

Elasticity of substitution: CES and Network Problem

Definition. Consider a buyer that allocates expenditure shares s_i across two suppliers $i = 1, 2$ as a function of generalized prices q_i. We define the (Hicks–Allen) elasticity of substitution between 1 and 2 as

$$\sigma_{12} \equiv -\frac{d\log(s_1/s_2)}{d\log(q_1/q_2)}. \tag{111}$$

CES with trade costs

Let delivered prices be $P_i = \tau_i p_i$, where p_i is the factory-gate price and τ_i is an iceberg trade cost. With a CES aggregator of elasticity σ and weights ω_i, Hicksian demand implies expenditure shares

$$s_i = \frac{\omega_i P_i^{1-\sigma}}{\omega_1 P_1^{1-\sigma} + \omega_2 P_2^{1-\sigma}}, \qquad i = 1, 2. \tag{112}$$

The relative expenditure share is therefore

$$\frac{s_1}{s_2} = \frac{\omega_1 P_1^{1-\sigma}}{\omega_2 P_2^{1-\sigma}} = \frac{\omega_1}{\omega_2}\left(\frac{P_1}{P_2}\right)^{1-\sigma} = \frac{\omega_1}{\omega_2}\left(\frac{\tau_1 p_1}{\tau_2 p_2}\right)^{1-\sigma}. \tag{113}$$

Taking logs,

$$\log\left(\frac{s_1}{s_2}\right) = \log\left(\frac{\omega_1}{\omega_2}\right) + (1-\sigma)\log\left(\frac{P_1}{P_2}\right). \tag{114}$$

Using the definition (111) with $q_i = P_i$,

$$\frac{d\log(s_1/s_2)}{d\log(P_1/P_2)} = 1 - \sigma \quad \Rightarrow \quad \sigma_{12}^{\text{CES}} = \sigma. \tag{115}$$

Thus in the CES case the elasticity of substitution is a global constant, independent of trade costs τ_i, prices p_i and the current allocation s. Trade costs only affect the *level* of the shares by changing delivered prices P_i, not the elasticity.

Quadratic network problem with risk-adjusted prices

Now consider the two-supplier version of the first-stage network problem. The buyer chooses expenditure shares s_1, s_2 solving

$$\min_{s_1,s_2} \quad \kappa(s_1 - s_1^0)^2 + \kappa(s_2 - s_2^0)^2 + s_1 R_1 + s_2 R_2 \tag{116}$$

$$\text{s.t.} \quad s_1 \geq 0,\ s_2 \geq 0,\ s_1 + s_2 = 1, \tag{117}$$

where s_i^0 is a baseline portfolio and R_i is the risk-adjusted price. We assume here that $s_1^0 + s_2^0 = 1$ and that both suppliers are active (interior solution), so the non-negativity constraints are slack.

Let λ be the Lagrange multiplier on $s_1 + s_2 = 1$. The first-order conditions are

$$2\kappa(s_1 - s_1^0) + R_1 - \lambda = 0, \tag{118}$$

$$2\kappa(s_2 - s_2^0) + R_2 - \lambda = 0, \tag{119}$$

$$s_1 + s_2 = 1. \tag{120}$$

Adding the first two equations and using $s_1^0 + s_2^0 = 1$ and $s_1 + s_2 = 1$ gives

$$R_1 + R_2 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{R_1 + R_2}{2} \equiv \bar{R}. \tag{121}$$

Substituting back yields

$$s_1 = s_1^0 - \frac{1}{2\kappa}(R_1 - \bar{R}) = s_1^0 - \frac{1}{4\kappa}(R_1 - R_2), \tag{122}$$

$$s_2 = s_2^0 - \frac{1}{2\kappa}(R_2 - \bar{R}) = s_2^0 + \frac{1}{4\kappa}(R_1 - R_2). \tag{123}$$

Let $\Delta R \equiv R_1 - R_2$. Then

$$s_1 = s_1^0 - \frac{1}{4\kappa}\Delta R, \qquad s_2 = s_2^0 + \frac{1}{4\kappa}\Delta R. \tag{124}$$

Hence the relative share is

$$\frac{s_1}{s_2} = \frac{s_1^0 - \frac{1}{4\kappa}\Delta R}{s_2^0 + \frac{1}{4\kappa}\Delta R}. \tag{125}$$

To compute the elasticity of substitution, we take the generalized prices to be $q_i = e^{R_i}$, so that

$$\log\left(\frac{q_1}{q_2}\right) = R_1 - R_2 = \Delta R. \tag{126}$$

Differentiating (125) with respect to ΔR,

$$\frac{d}{d\Delta R}\log\left(\frac{s_1}{s_2}\right) = \frac{4\kappa(s_1^0 + s_2^0)}{(\Delta R - 4\kappa s_1^0)(\Delta R + 4\kappa s_2^0)}. \tag{127}$$

Using $s_1^0 + s_2^0 = 1$, and noting that $d\log(q_1/q_2)/d\Delta R = 1$, the elasticity (111) becomes

$$\sigma_{12}^{\text{net}}(\Delta R) = -\frac{d\log(s_1/s_2)}{d\log(q_1/q_2)} = -\frac{4\kappa}{(\Delta R - 4\kappa s_1^0)(\Delta R + 4\kappa s_2^0)}. \tag{128}$$

This elasticity is *state-dependent*: it varies with ΔR, the baseline portfolio (s_1^0, s_2^0), and the adjustment cost κ.

Comparison

In the CES case with trade costs, the relative expenditure share is constant equal to σ, independent of trade costs, prices, or the current allocation.

In the quadratic network problem, by contrast, input shares adjust *linearly* around a baseline portfolio s^0,

$$s_i = s_i^0 - \frac{1}{2\kappa}(R_i - \bar{R}),$$

so the relative share depends both on risk-adjusted prices and on the baseline composition. The intuition is that the elasticity of substitution depends on the initial distance to the ideal technology which in turn depends on transportation cost and the equilibrium network.