Debt Sustainability, Confidence Risk and International Reserves *

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Abstract

This paper analyzes how a government can use reserves to prevent a self-fulfilling crisis, as described in Lorenzoni and Werning (2019), where confidence-driven fluctuations affect bond prices. We propose a three-period model in which the government follows a fixed fiscal surplus rule and chooses the optimal reserve accumulation policy. Our analysis reveals a new mechanism for which debt-financed reserves provide insurance against self-fulfilling crises in the presence of long-term bonds. We also present empirical evidence that governments tend to accumulate reserves during periods of exceptionally high spreads and show how our theoretical framework could help explain this empirical pattern.

Keywords: Self-fulfilling debt crisis, sovereign debt, multiple equilibria, default risk, debt sustainability

JEL classification: F21,F61, E62, F64, F34, F38, F41, P41, P43.

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1 Introduction

Since the European debt crisis, a growing body of literature has studied macroeconomic conditions that make an economy vulnerable to self-fulfilling crises. The primary concern at the time was that nervous investors were demanding high interest rates, which could increase future debt payments and compromise fiscal sustainability, thereby becoming self-fulfilling. The argument is that high interest rates contribute to the rise in debt over time, eventually driving countries to insolvency, thus justifying high interest rates in the first place. This dynamic increases concerns about confidence-driven fluctuations in sovereign debt prices.

While investors were casting doubt on the rising debt levels of European countries, some governments were accumulating foreign reserves that paid lower interest rates than the newly issued debt. In Figure 1, we plot the levels of debt and reserves as a percentage of GDP for countries that experienced a significant increase in bond spreads during the crisis.[1] In particular, the policies of Spain, Italy, and Portugal involved increasing both reserves and debt during a period characterized by exceptionally high sovereign bond spreads.[2]

(a) Total Debt (b) Foreign Reserves

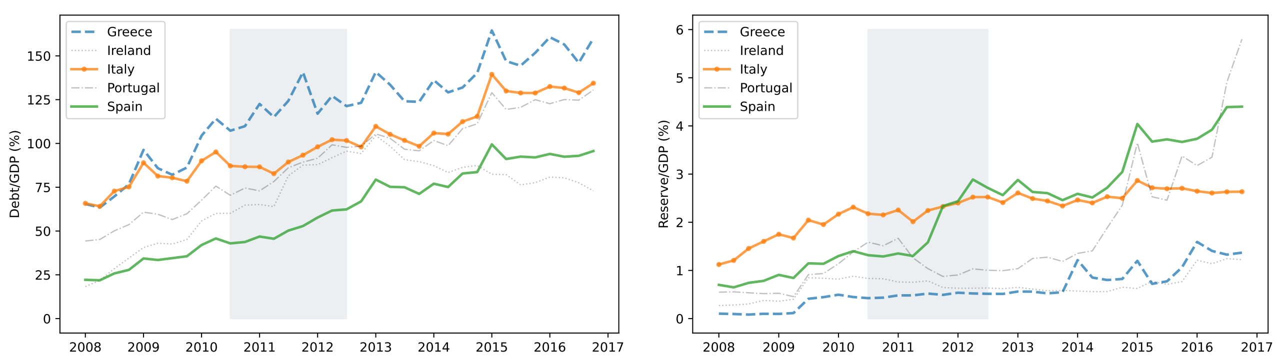


Figure 1: European Debt Crises

Note: Data of Total General Government Gross Debt is from IMF's Sovereign Debt Investor Base. Reserves are from IMF's International Financial Statistics. GDP is from the World Bank's World Development Indicators.

A key argument in the sovereign debt literature to explain why governments simultaneously issue debt and accumulate reserves is that these actions provide insurance against

[1]Reserves include the monetary authorities' holdings of SDRs, reserve position in the Fund, and foreign exchange, including financial derivative claims on non-euro area countries. They exclude claims among euro-area countries and all euro-denominated claims among non-euro-area countries.

[2]See Aizenman and Sun (2012) and Section 7 for additional empirical evidence that countries seem to keep reserves even during crises.

fundamental risk.[3] According to this view, countries accumulate reserves during periods of low spreads and utilize them during crises to avoid issuing debt when spreads are high, thus obtaining insurance. However, we observed during the European crises that some countries found it optimal to both accumulate reserves and issue debt at high spreads, challenging this conventional view.

More recently, the literature has explored the idea that reserves can help eliminate the possibility of multiple equilibria, thus reducing confidence-driven fluctuations in bond spreads (e.g., Barbosa-Alves et al. (2024)). In this paper, we provide a novel mechanism through which, in the presence of long-term bonds, the government can effectively reduce these fluctuations by issuing debt to accumulate reserves. Effectively, debt-financed reserve accumulation extends the maturity of the government's portfolio, which in turn improves the insurance provided by long-term bonds. This improved insurance relaxes the government's budget constraint in states where lenders coordinate on low bond prices. The intuition is that the government exchanges risk-free foreign reserves for government bonds, which have low value during a crisis. As a result, the combination of higher reserves and long-term bonds provides insurance against a *bad equilibrium*, in which lenders panic and coordinate on low bond prices, thereby eliminating the possibility of multiple equilibria.

In our analysis, we consider a three-period version of the model in Lorenzoni and Werning (2019). The government follows a fiscal policy rule that determines a stochastic process for the primary surplus. It finances deficits through long-term bonds and defaults whenever repaying the existing debt is not feasible. At the beginning of each period, lenders bid and determine bond prices, and the government adjusts its debt issuance based on the prevailing price to satisfy its budget constraint. Importantly, we assume the government cannot adjust the surplus in response to bond price fluctuations, reflecting that, in practice, fiscal policy faces rigidities of fiscal policy due to political and administrative constraints, see for example Conesa and Kehoe (2024).

This friction, combined with the assumption that lenders move first by bidding on government bonds, creates the potential for multiple equilibria. The key contribution of the paper is incorporating foreign reserves into the analysis and studying their role in debt sustainability when the government faces the risk of multiple equilibria. We establish that the government can eliminate the possibility of multiple equilibria by re-balancing its portfolio towards higher levels of debt and reserves.

The multiplicity in the model is dynamic. If investors expect low future borrowing, they anticipate high bond prices in the next period. With long-term bonds, this expectation raises

[3] See, for example, Bianchi et al. (2018) and Bianchi and Sosa-Padilla (2023).

the current value of government bonds, allowing the government to finance its deficit with less debt. Lower current debt relaxes the fiscal budget in the next period, leading to low future borrowing and confirming the initial expectation. Conversely, if investors expect high future borrowing, they anticipate greater dilution, lowering current bond prices. This forces the government to issue more debt, tighten future fiscal budgets, and reinforce expectations of high future borrowing.

The government's only choice is the composition of its foreign portfolio. It issues debt to finance an exogenous fiscal deficit but can also issue additional debt to accumulate reserves. This financial operation provides insurance against self-fulling crises for intermediate deficit levels. The mechanism works as follows. The government carries higher debt and assets into the future by issuing debt to finance reserves. If lenders were to coordinate on low future bond prices, the government would hold a larger stock of debt with a low real value due to pessimistic expectations. However, it would also hold a higher stock of risk-free assets. Then, the government could buy back the increased debt stock and get extra revenue from the operation. As a result, debt-financed reserves relax the fiscal budget in states with low bond prices. Critically, by relaxing the budget constraint, debt-fianced reserves reduce future borrowing needs, undermining the high borrowing expectation and preventing the self-fulfilling crisis. On the other hand, by choosing a foreign portfolio with higher levels of debt and reserves, the government reduces future risks. Lower future risks, in turn, lead to an efficiency gain that increases current prices in the presence of long-term bonds, which helps the government further reduce the debt and default risk.

The most closely related paper is Barbosa-Alves et al. (2024), but it differs from ours in two crucial ways: insurance mechanisms and theory implications. First, we study a model with multiple equilibria à la Calvo, in which the bond market for government bonds remains open during a confidence-driven crisis. This allows the government to trade bonds at low prices when investors panic, enabling debt-financed reserves to buy back bonds and relax budget constraints in such states. In our model, insurance is provided through the state contingency of the real value of long-term bonds. In contrast, Barbosa-Alves et al. (2024) analyzes a model with multiple equilibria à la Cole-Kehoe, where the government can face runs. In that setting, the state contingency of bond prices is irrelevant during a confidence-driven crisis because the bond market shuts down. Instead, debt-financed reserves provide insurance by increasing liquidity in a crisis. In practice, both mechanisms are likely complementary.

The second difference lies in the theoretical implications of the optimal policy under a confidence-driven crisis. As in our model, Barbosa-Alves et al. (2024) finds that debt-financed reserves could positively affect the current price, but only when NFA is sufficiently high. When NFA is too low, increasing debt and reserves does not raise prices, and the government finds it optimal to reduce debt and hold zero reserves. In contrast, our model suggests that the government should issue debt to accumulate reserves even when NFA is low because it always improves insurance and reduces economic risk.

A key insight of these theoretical contributions is that it may be optimal for the government to accumulate reserves even when bond prices are low. Moreover, reserves are particularly valuable when low bond prices result from the risk of a future self-fulfilling crisis.[4] This finding contrasts with earlier contributions, such as Bianchi et al. (2018), which show that governments typically use international reserves as a hedge by accumulating them in good times and depleting them during turbulent times when spreads are high (bond prices are low).

We present new empirical evidence that during typical episodes of *Fiscal Stress*, defined as quarters when government bond yields are particularly elevated, governments tend to accumulate reserves rather than using them to avoid rolling over high-spread debt. This behavior is consistent with the theory that governments should accumulate reserves when fundamentals are weak to prevent self-fulfilling crises. We also analyze the dynamics of debt and reserve accumulation during the 2008 global final crisis, where we observe a similar pattern.

One critical challenge in quantitatively assessing the contribution of this mechanism to the observed empirical patterns is that emerging market economies may accumulate reserves for different motives, such as exchange rate management. To partially address this concern, we conduct a robustness check by excluding countries with fixed exchange rates, for which reserve accumulation is more likely to be influenced by exchange rate policies. However, we leave it for future research to quantify the role of our proposed mechanism in explaining the empirical pattern presented in this paper.

Related Literature. Our paper belongs to the literature on self-fulfilling crises. Two canonical examples of multiplicity in sovereign debt models are Cole and Kehoe (2000) and Calvo (1988). Similar to Cole and Kehoe (2000), the timing is key for generating multiplicity in our model. Specifically, we assume that the lenders bid on bond prices after the government commits to a fiscal surplus. For further research on the Cole-Kehoe-type runs, see Bocola and Dovis (2019), Conesa and Kehoe (2017), Bianchi and Mondragon (2022) and Barbosa-Alves et al. (2024). Additionally, Aguiar et al. (2022) examines a related form of multiplicity, where the government issues debt before choosing whether to repay or default.

[4] See Corsetti and Maeng (2023) or Hur and Kondo (2016) for related results in models with multiple equilibria.

The main precursor to our paper is Lorenzoni and Werning (2019). We extend their model by analyzing the role of reserves in dynamic settings with multiple equilibria. As in our framework, they assume the government follows a fiscal rule and show how Calvo-style multiplicity can arise. Their analysis focuses on the impact of debt maturity, showing that economies with longer-maturity debt are less vulnerable to multiple equilibria. In contrast, our paper explores how reserves can help prevent self-fulfilling crises. Our main contribution is to show that the government can use debt-financed reserves to reduce the risk of multiplicity. In effect, this financial operation extends the maturity of the government's portfolio.

Our paper also contributes to the literature on sovereign default and international reserve accumulation. For a recent survey, see Bianchi and Lorenzoni (2022). Some notable examples are Bianchi et al. (2018), who examine the role of international reserves in a model with long-term bonds, where the government faces rollover risk due to changes in international investors' risk aversion. Similarly, Bianchi and Sosa-Padilla (2023) explore how international reserves provide a hedge for the government in a model with nominal rigidities and the possibility of unemployment. Alfaro and Kanczuk (2009) study the joint accumulation of international reserves and defaultable debt in a benchmark model with *one-period bonds*. They find that the sovereign should not accumulate reserves when debt is short-term.

Closely related to our analysis, Barbosa-Alves et al. (2024) and Corsetti and Maeng (2023) study the role of international reserves in the presence of self-fulfilling crises. Barbosa-Alves et al. (2024) analyze a model with self-fulfilling runs, as in Cole and Kehoe (2000), and show that issuing debt to accumulate reserves is optimal only when the government's NFA position is sufficiently high. Corsetti and Maeng (2023) study joint accumulation of reserves and debt in a model of belief-driven sovereign risk crises proposed by Aguiar et al. (2022). Their findings suggest that accumulating reserves and one-period debt is desirable because it mitigates the intra-period risk that the government may refuse to repay if the bond auction fails. In contrast to these studies, we focus on a setting with dynamic multiplicity, as in Lorenzoni and Werning (2019), and show that debt-financed reserves are optimal even when the NFA position is low.

We also contribute to the empirical literature on the evolution of debt and reserves around crises. Two important studies in this area are Gourinchas and Obstfeld (2012) and Aizenman and Sun (2012). While Gourinchas and Obstfeld (2012) focus on default episodes, our analysis centers on episodes where governments choose to repay, but face unusually high bond yields. Meanwhile, Aizenman and Sun (2012) examine the global financial crises; we complement their analysis by identifying countries that were particularly affected by crises and experienced government bond yields above typical levels.

Finally, our paper builds on the literature on debt sustainability (see Bohn (1995) and Bohn (2005)). Like this literature, we are interested in analyzing policies that promote debt sustainability. In our case, we explore the role of foreign reserves in reducing the risk of default in environments with multiple equilibria.

Outline. The paper is organized as follows. Section 2 presents the model. Section 3 describes the equilibrium conditions. Section 4 studies multiplicity in period one and the role of reserves in mitigating the risk of self-fulfilling crises. Section 5 explores multiplicity in period zero and welfare implications. Section 6 discusses the relation of our results with existing literature. Section 7 presents the empirical analysis. Section 8 concludes.

2 Model

We consider a small open economy that lasts for three periods, indexed by $t \in \{0, 1, 2\}$. The economy is populated by the government and a continuum of risk-neutral foreign investors with discount factor β. The main innovation of the paper is to add international reserves into a model where the government is exposed to self-fulfilling crises as in Lorenzoni and Werning (2019). The government follows an exogenous fiscal surplus rule, and default occurs whenever it cannot raise enough revenue from new debt issuance to cover its current financing needs.

The timing is as follows. At the beginning of each period, the government generates a fiscal surplus z_t, representing total taxes collected minus total government spending on purchases and transfers. A negative value of z_t corresponds to a primary deficit. Next, lenders bid on the price of government bonds. The variable ω_t is a sunspot that acts as an equilibrium selection device, choosing the price whenever multiple equilibria are possible. Given the price, the government issues debt or defaults at the end of the period.

2.1 Stochastic process

Uncertainty is modeled as follows. The government has a fixed surplus in periods zero and one, but z_2 is randomly drawn from a continuous distribution $F(z_2)$ with support $[\underline{Z}, \overline{Z}]$.

Another source of uncertainty in the model is confidence risk. In the event of multiplicity in periods zero or one, the government faces uncertainty regarding the sunspot variable that selects the equilibrium. In such cases, with probability π, the government faces a *good sunspot* associated with a high price, and $\omega_t = 1$. On the other hand, with probability $1 - \pi$, it encounters a *bad sunspot* linked to a low price, and $\omega_t = 0$.

The exogenous state in period zero is denoted by $s_0 = \omega_0$. In period one, it is $s_1 = \{\omega_0, \omega_1\}$, and in period two, it is $s_2 = \{\omega_0, \omega_1, z_2\}$.

2.2 The Government

The government's preferences are given by

$$U = \mathbb{E}[u(z_2 - b_2) - \gamma d_2], \tag{1}$$

where u represents the utility function, d_2 is an indicator variable that takes the value of one if the government defaults in the last period, and γ denotes the default cost. We assume γ is sufficiently high, so this preference structure resembles the objective function in the literature on debt sustainability (see, for example, Bohn (1995)), where the government wants to finance an exogenous fiscal surplus while avoiding default.

In period zero, the government issues a two-period bond b_1, which pays a sequence of coupons $\kappa_1, (1-\delta)\kappa_2$ in periods one and two, respectively. Additionally, in period one, it issues a new bond, i_2, which pays κ_2 in period two. The level of debt in period two is defined as $b_2 = i_2 + (1-\delta)b_1$.

We normalize the parameters by setting $\{\kappa_1 = \frac{1}{\beta} - (1-\delta), \kappa_2 = \frac{1}{\beta}\}$, ensuring that the bond prices of b_1 and b_2 are equal to one when there is no risk of default. This structure mimics the quantitative literature on long-term bonds in sovereign debt models (see, for example, Arellano and Ramanarayanan (2012), Hatchondo and Martinez (2009)), where $\delta = 0$ corresponds to a consol, while $\delta = 1$ corresponds to a short-term bond.

The government's only choice variable is the level of international reserves. We model these reserves as a one-period risk-free bond, denoted by a_1. The price of reserves is $q_a = \frac{1}{\beta}$, and they yield one unit of the endowment upon maturity. In the event of repayment, the government's budget constraints are as follows:

$$z_0 + q_0(s_0)(b_1(s_0)) = q_a a_1(s_0), \tag{2}$$

$$z_1 + q_1(s_1)(b_2(s_1) - (1-\delta)b_1(s_0)) + a_1(s_0) = \kappa_1 b_1(s_0), \tag{3}$$

$$z_2 \geq \kappa_2 b_2(s_1), \tag{4}$$

where q_0 and q_1 denote the prices of government bonds. We assume that the government always honors its debt whenever feasible. That is, if there exists a level of debt b_2 that satisfies (3), the government will repay the outstanding debt $\kappa_1 b_1$. Similarly, the government will repay in period two if the fiscal surplus z_2 is sufficient to cover the outstanding debt

$\kappa_2 b_2(s_1)$.

Let d_t be an indicator variable that takes the value of one if the government defaults and zero otherwise. After a default, investors receive a recovery value given by

$$\nu_t = \phi z_t, \tag{5}$$

where $\phi \in [0, 1]$ is the recovery rate. If default occurs, the government is permanently excluded from the financial market.

2.3 International Investors

We assume there is a continuum of risk-neutral international investors, who collectively have enough resources to buy any amount of government bonds. Given the government's default policy, the non-arbitrage condition of international investors implies:

$$q_t = \beta \mathbb{E}\left[(1 - d_{t+1})(\kappa_{t+1} + (1 - \delta)q_{t+1}) + d_{t+1}\frac{\nu_{t+1}}{b_{t+1}}|\ s_t\right]. \tag{6}$$

2.4 Competitive Equilibrium

We are ready to define the equilibrium:

Definition 1. (Competitive Equilibrium) A competitive equilibrium consists of a sequence of prices $\{q_0, q_1\}$; debt $\{b_1, b_2\}$; reserves $\{a_1\}$; and default rules $\{d_t\}_{t=1,2}$, such that:

i Given prices, debt, reserves, and default rules are consistent with the government's budget constraints (2), (3) and (4).

ii Given debt, reserves and default rules, the non-arbitrage condition of international investors (6) holds.

2.5 Discussion

Fiscal rules of this kind are standard in the literature on debt sustainability (see, for example, Bohn (1995), Bohn (2005)). One key insight from this literature is that governments cannot immediately adjust the fiscal surplus in response to bond prices fluctuations. Instead, we observe in practice that governments tend to adjust debt levels when prices are low because adjusting the surplus is typically costly and time-consuming.

For simplicity, we adopt the limiting assumption that the fiscal rule does not react to the level of government debt. As highlighted by Lorenzoni and Werning (2019), this rule makes the government more vulnerable to self-fulfilling crises. This assumption underscores reserves' role in preventing such crises.

Alternatively, one could assume that the government's fiscal rule allows for some adjustment of the fiscal surplus in response to bond prices, while still constraining it in the short run. As discussed by Lorenzoni and Werning (2019), the critical aspect for our analysis is that the government cannot freely adjust the surplus in the short run. This lack of flexibility opens the door for multiplicity, as the government cannot commit to issuing a certain level of debt before the auction of bonds. To understand the intuition, consider a case where the government commits to a fixed level of debt before lenders submit their bids. If, out of equilibrium, the bond price were to decrease, the government would need to adjust the surplus to meet its fiscal budget. However, in this environment, the government cannot make such adjustments because it cannot freely control the fiscal surplus.

As in Lorenzoni and Werning (2019), our model assumes that the government responds to any reduction in bond prices by increasing debt in order to maintain the total revenue generated from borrowing. This inability to adjust expenditures in response to bond price changes creates a key friction, which is the source of multiple equilibria in the model.

Without loss of generality, we also assume that the government does not accumulate reserves in period one for expositional convenience. If the government could choose reserves in period one, it would pick zero reserves. This aligns with a well-known result in the literature on foreign reserves accumulation, which states that reserves play no role if the government issues debt using short-term bonds (see Alfaro and Kanczuk (2009), Bianchi et al. (2018)).

In our setting, this result can be understood as follows. In period one, the government issues only short-term bonds, which pay high interest rates due to default risk. The relevant state in period two can be simplified to $(b_2 - a_2)$. If the government were to accumulate reserves, it could relax the fiscal budget in period one by selling those reserves, which have a higher price, and repurchasing debt with a lower price, while keeping $(b_2 - a_2)$ constant. Therefore, the optimal choice for the government is to hold zero reserves.

3 Equilibrium

This section solves the model by backward induction and describes the equilibrium.

3.1 Period Two

In period two, the government begins with an initial debt position of b_2 and does not issue new bonds. The relevant state for the government is $\{b_2, z_2\}$. Two cases are possible. If $z_2 \geq \kappa_2 b_2$, the government can repay, and current bondholders receive κ_2 per bond. If instead $z_2 < \kappa_2 b_2$, the government defaults, and each bondholder receives:

$$\frac{\phi z_2}{b_2} < \kappa_2. \tag{7}$$

This implies that the government's default rule is:

$$D_2(b_2, z_2) = \begin{cases} 1 & \text{if} \quad z_2 < \kappa_2 b_2, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Using the default function and the recovery value, the equilibrium bond price function in period one can be expressed as:

$$Q_1(b_2) = \beta \left[(1 - F(\kappa_2 b_2))\kappa_2 + \frac{\phi}{b_2} \int_{\underline{Z}}^{\kappa_2 b_2} z_2 dF(z_2) \right]. \tag{9}$$

Note that the price function in period one is unique.

3.2 Period One

The relevant state in period one is government's initial portfolio $\{b_1, a_1\}$ and the realization of the sunspot $\{\omega_1\}$ that selects the price in case of multiplicity.

Default Function. In period one, the budget constraint becomes:

$$z_1 + Q_1(b_2)(b_2 - (1 - \delta)b_1) + a_1 = \kappa_1 b_1. \tag{10}$$

We define $\mathcal{L}_1(b_2) = Q_1(b_2)(b_2 - (1 - \delta)b_1) + a_1 - \kappa_1 b_1$ as the debt Laffer curve including reserves less coupon payments. We also define the peak of the Laffer curve including reserves as a function of the initial portfolio as follows:

$$m_1(b_1, a_1) = \max_{b_2} \left\{ Q_1(b_2)(b_2 - (1-\delta)b_1) + a_1 \right\}. \tag{11}$$

Using the peak of the Laffer curve, we know that the government repays if:

$$z_1 + m_1(b_1, a_1) \geq \kappa_1 b_1. \tag{12}$$

The default function of the government in period one can be characterized using $m_1(b_1, a_1)$ as follows:

$$D_1(b_1, a_1) = \begin{cases} 1 & \text{if} \quad z_1 + m_1(b_1, a_1) < \kappa_1 b_1, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Note that both the peak of the Laffer curve and the default function are independent of the sunspot's realization. Next, we characterize the debt policy in period one.

Debt Function. We define $B_2(b_1, a_1, \omega_1)$ as the equilibrium debt as a function of the relevant state. When $D_1(b_1, a_1) = 1$, the government is excluded from the financial markets, so $B_2(b_1, a_1, \omega_1) = 0$. Conversely, when $D_1(b_1, a_1) = 0$, $B_2(b_1, a_1, \omega_1)$ selects the equilibrium debt from the set of candidates b_2 that satisfy the government's budget constraint. Formally can use equation (10) to define:

$$\mathbb{B}_2(b_1, a_1) = \left\{ b_2 : b_1 = \frac{z_1 + Q_1(b_2)b_2 + a_1}{\kappa_1 + (1-\delta)Q_1(b_2)} \mid b_1, a_1 \right\}. \tag{14}$$

This set contains all possible debt values that can be part of the equilibrium for any pair of $\{b_1, a_1\}$. For some initial portfolios, $\mathbb{B}_2(b_1, a_1)$ contains only one value, in which case the equilibrium is unique. However, the set contains three values for some combinations of debt and reserves.

If multiplicity arises, we select an equilibrium using the sunspot variable ω_1. We denote a *good sunspot* when $\omega_1 = 1$, in which case lenders coordinate on a price consistent with the lowest value of b_2 that satisfies the equilibrium conditions. Conversely, when $\omega_1 = 0$, we refer to a *bad sunspot*, where debt takes the highest value consistent with equilibrium conditions. We follow Ayres et al. (2018), among others, in discarding the intermediate case as a possible equilibrium. The main argument for discarding the intermediate equilibrium is that, as shown by Ayres et al. (2018), a debt value that falls within the decreasing segment of the Laffer curve is not part of equilibrium in a perturbed game where lenders exhibit an arbitrarily low level of coordination. Another argument for eliminating the intermediate equilibrium is that it yields counterintuitive comparative statics, wherein higher deficits result in lower debt in equilibrium. Formally, the debt function in period one is:

$$B_2(b_1, a_1, \omega_1) = \begin{cases} \max \mathbb{B}_2(b_1, a_1) & \text{if} \quad z_1 + m_1(b_1, a_1) \geq \kappa_1 b_1 \quad \text{and} \quad \omega_1 = 0, \\ \min \mathbb{B}_2(b_1, a_1) & \text{if} \quad z_1 + m_1(b_1, a_1) \geq \kappa_1 b_1 \quad \text{and} \quad \omega_1 = 1, \\ 0 \quad \text{otherwise.} \end{cases} \tag{15}$$

Price Function. Finally, we use the break-even condition of international investors and the default and debt functions of the government to define the equilibrium price in period zero as:

$$Q_0(b_1, a_1) = \begin{cases} \beta \left(\kappa_1 + (1-\delta)\mathbb{E}\left[Q_1(B_2(b_1, a_1, \omega_1))\right]\right) & \text{if} \quad z_1 + m_1(b_1, a_1) \geq \kappa_1 b_1, \\ \max\left\{\frac{\beta \phi z_1}{b_1}, 0\right\} & \text{otherwise.} \end{cases} \tag{16}$$

As highlighted by Lorenzoni and Werning (2019), the existence of multiple equilibria in period one means that multiple price functions are consistent with equilibrium conditions in period zero. By imposing a structure on the sunspot variable, we select a price function in which lenders use the probability of a *good sunspot* in period one (π) to price the bond in period zero.

3.3 Period Zero

Next, we describe the equilibrium in period zero for a given state ω_0.

Default Function. In period zero, the budget constraint becomes:

$$z_0 + Q_0(b_1, a_1)(b_1) = q_a a_1. \tag{17}$$

Similar to period one, we define $\mathcal{L}_0(b_1, a_1) = Q_0(b_1, a_1)b_1 - q_a a_1$ as the debt Laffer curve minus reserves. Additionally, the default function of the government becomes

$$D_0 = \begin{cases} 1 & \text{if} \quad z_0 + m_0 < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where m_0 is the peak of the Laffer curve defined as:

$$m_0 = \max_{b_1, a_1} \left\{ Q_0(b_1, a_1) b_1 - q_a a_1 \right\}. \tag{19}$$

Note the peak of the Laffer curve and the default function are not affected by the realization of the sunspot in period zero. However, they depend on the probability of a bad sunspot in period one because of its effect on the price function.

Debt Function. Analogous to period one, we use the budget constraint in period zero to define:

$$\mathbb{B}_1(a_1) = \left\{ b_1 : z_0 + Q_0(b_1, a_1)(b_1) = q_a a_1 |\ a_1 \right\}. \tag{20}$$

These are the candidates value of debt in period zero that could be part of the equilibrium for any choice of foreign reserves (a_1). We define the equilibrium debt as:

$$B_1(a_1, \omega_0) = \begin{cases} \max \mathbb{B}_1(a_1) & \text{if} \quad z_0 + m_0 \geq 0 \quad \text{and} \quad \omega_0 = 0, \\ \min \mathbb{B}_1(a_1) & \text{if} \quad z_0 + m_0 \geq 0 \quad \text{and} \quad \omega_0 = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Optimal Policy. Finally, the government's problem in period zero is:

$$V = \max_{a_1} \mathbb{E}[u(z_2 - B_2(B_1(a_1, \omega_0), a_1, \omega_1)) - \gamma D_2(B_2(B_1(a_1, \omega_0), a_1, \omega_1)), z_2)] \tag{22}$$

3.4 Equilibrium

We close the description of the model with the definition of a Continuation Equilibrium of the economy.

Definition 2. (Continuation Equilibrium) A Continuation equilibrium consists of a set of policies $\mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_2, \mathcal{D}_1, \mathcal{D}_2$, a value function for the government V and price schedules Q_0, Q_1, such that:

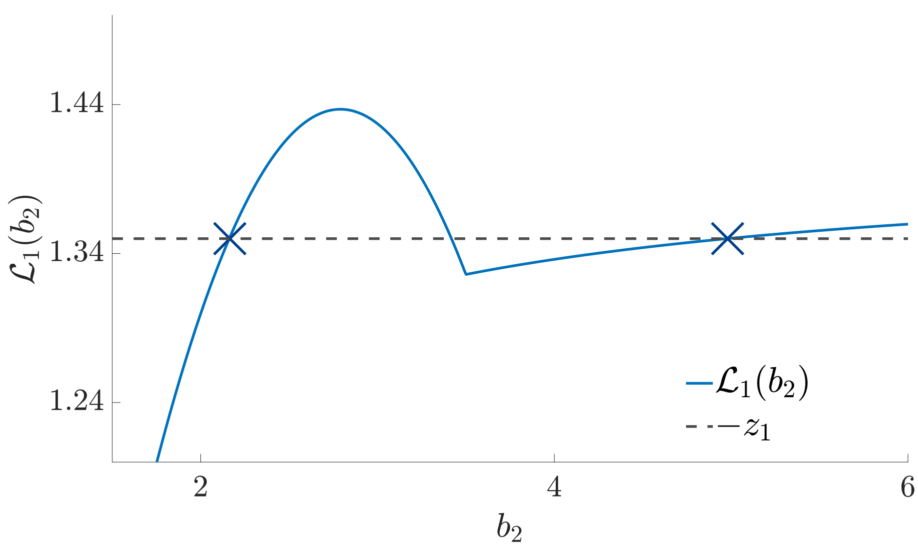
i Given price schedules and default and debt policy rules, $\mathcal{A}_1$ solves the government problem, and V attains the maximum.

4 Multiplicity in Period One

We begin our analysis by examining the equilibrium in period one. Figure 2 illustrates the equilibrium conditions. The solid line depicts the debt Laffer curve including reserves less coupon payments, $\mathcal{L}_1(b_2)$. The dashed line represents the constant fiscal deficit $-z_1$. The points where these curves intersect correspond to debt values b_2 consistent with equilibrium conditions. In this example, two equilibria are possible.[5]

The possibility of multiple equilibria is related to the presence of dilution. As b_2 increases, three forces affect the Laffer curve. First, the government issues more bonds, increasing income. Second, higher debt increases the probability of default and lowers bond prices, reducing income. Third, as bond prices decline, the value of outstanding debt $(1-\delta)b_1$ goes down due to a diluting effect.[6] While the first and the third effects increase debt revenue, the second effect decreases it.

Figure 2: Equilibrium Condition in Period One



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0$, $\phi = 0.7$, $b_1 = 0.55$, $a_1 = 0$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

The interaction of these three forces leads to a Laffer curve with three distinct regions. In

[5]We discard the intermediate equilibrium as discussed in the last section.

[6]See for example Hatchondo et al. (2016), and Lorenzoni and Werning (2019) for discussions on the role of dilution in sovereign debt models.

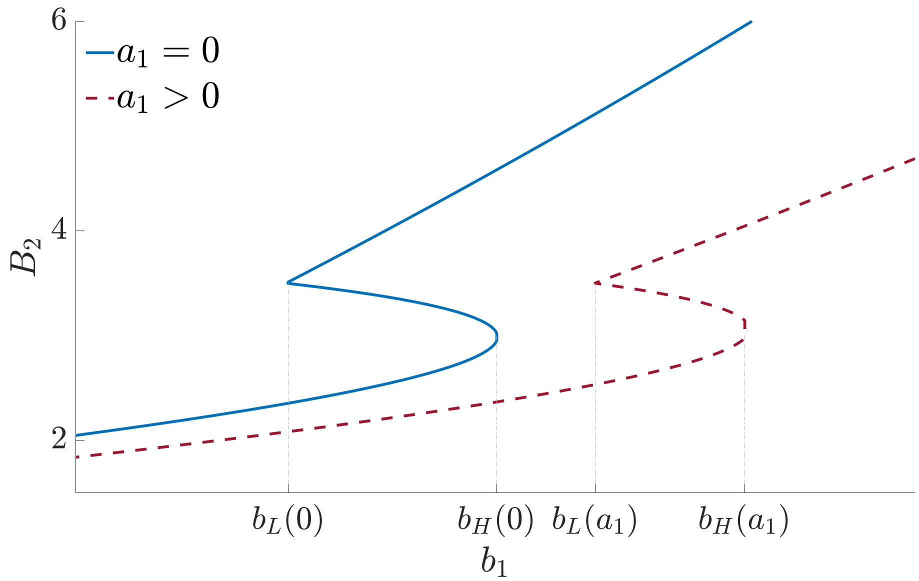
the first region, the risk of default is low, making bond price relatively inelastic to borrowing. Here, the quantity effect of increased borrowing is stronger than the negative effect on prices, resulting in an upward-sloping Laffer curve. As borrowing approaches the first peak of the curve, the price effect becomes stronger. Eventually, this leads to the second region, where the Laffer curve slopes downward. However, the presence of dilution creates a third region. In the last region, the probability of default in period two is one, causing bond prices to decline linearly. At these borrowing levels, the additional revenue from issuing new bonds remains constant. Nevertheless, as b_2 increases, the dilution effect further reduces the value of the existing debt stock $(1-\delta)b_1$. Consequently, at very high borrowing levels, the Laffer curve begins to rise again.

If the government faces a Laffer curve with these three regions, multiple equilibria may arise. The intuition behind this multiplicity is as follows. If investors anticipate that the government will issue a high b_2, they expect a high probability of default in period two and, as a result, bid a low price for bonds. Given these low prices, the government must issue a large b_2 to finance the deficit, confirming the initial expectation. On the other hand, if investors expect the government to issue a low b_2, the expected probability of default is low, bond prices stay high, and the government issues a lower level of debt, again confirming the initial expectation.

4.1 Debt Function

In Figure 3, we plot the equilibrium borrowing as a function of the initial debt b_1, comparing two different levels of reserves a_1. The solid line represents an economy without reserves, while the dashed line corresponds to an economy with a positive level of reserves.

Figure 3: Borrowing and Reserve



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

This construction identifies three regions for the initial debt b_1, given a level of reserves a_1. In the first region, denoted as $\mathcal{B}_{\mathcal{L}}$, the initial debt is low, and only one equilibrium exists, characterized by low new borrowing and high bond prices. In the second region, $\mathcal{B}_{\mathcal{H}}$, where the initial debt is high, the equilibrium is also unique but features high borrowing and low bond prices. In this region, the probability of default in the last period is one. Finally, the multiplicity region $\mathcal{B}_{\mathcal{M}}$ allows for multiple equilibria. At these intermediate debt levels, two stable equilibria are possible: one with high borrowing and low prices and another with low borrowing and high prices.[7] Formally, we define:

Definition 3. (Portfolio Regions) Let $\{b_1, a_1\}$ be the initial levels of debt and reserves in period one, and $B_2(b_1, a_1, \omega_1)$ be the debt function. It is possible to break the space of (b_1, a_1) as follows:

$$\mathcal{B}_{\mathcal{L}} = \{(b_1, a_1) : B_2(b_1, a_1, \omega_1) = \mathbb{B}_2(b_1, a_1), F(B_2) < 1\}$$
$$\mathcal{B}_{\mathcal{M}} = \{(b_1, a_1) : B_2(\omega_1 = 0) = \max \mathbb{B}_2(b_1, a_1), F(B_2) = 1; B_2(\omega_1 = 1) = \min \mathbb{B}_2(b_1, a_1), F(B_2) < 1\}$$
$$\mathcal{B}_{\mathcal{H}} = \{(b_1, a_1) : B_2(b_1, a_1, \omega_1) = \mathbb{B}_2(b_1, a_1), F(B_2) = 1\}$$

[7]This classification assumes that the government repays in period one. Formally, there exists another region where $D_1(b_1, a_1) = 1$, in which any b_2 satisfies the government's budget constraint.

We characterize the boundaries of those regions in the following Lemma:

Lemma 1. *(Thresholds of Debt) Suppose there exists $x < \overline{Z}$ such that*

$$z_0 + \beta \left(z_1 + (1 - F(x))\, x + \beta\phi \int_{\underline{Z}}^{x} z_2 dF(z_2) \right) = 0.$$

Then, given a_1, there exist two levels of debt in the period one $b_H(a_1), b_L(a_1)$ such that; $b_1 \in \mathcal{B}_{\mathcal{M}}$ if and only if $b_H(a_1) \geq b_1 \geq b_L(a_1)$

Proof. The proof is in the Appendix A □

Lemma 1 establishes a condition under which an equilibrium with low borrowing exists. Intuitively, if the sum of fiscal surpluses across periods zero, one, and two is negative, then no equilibrium exists in which the government can avoid default. This is because the government's expected income is too low.

Lemma 1 also underscores the importance of the initial portfolio in determining the possibility of multiple equilibria in period one, which occurs only for intermediate values of initial debt ($b_H(a_1) \geq b_1 \geq b_L(a_1)$). For a given level of reserves, if the initial debt is low, and the government borrows at a level such that the default probability in the last period is one, it generates more revenue than necessary to finance the fiscal surplus. Consequently, only one equilibrium exists, characterized by low borrowing and high bond prices. On the other hand, if the initial debt is too high, the government can only repay its debt by borrowing at a level such that the probability of default in the second period is one. However, when $b_H(a_1) > b_L(a_1)$, intermediate values of b_1 lead to two possible equilibria.

Moreover, as shown in Figure 3, as the level of reserves increases, the boundaries of the multiplicity region expand because the government can finance the surplus with lower borrowing levels. We formally establish this relationship as follows:

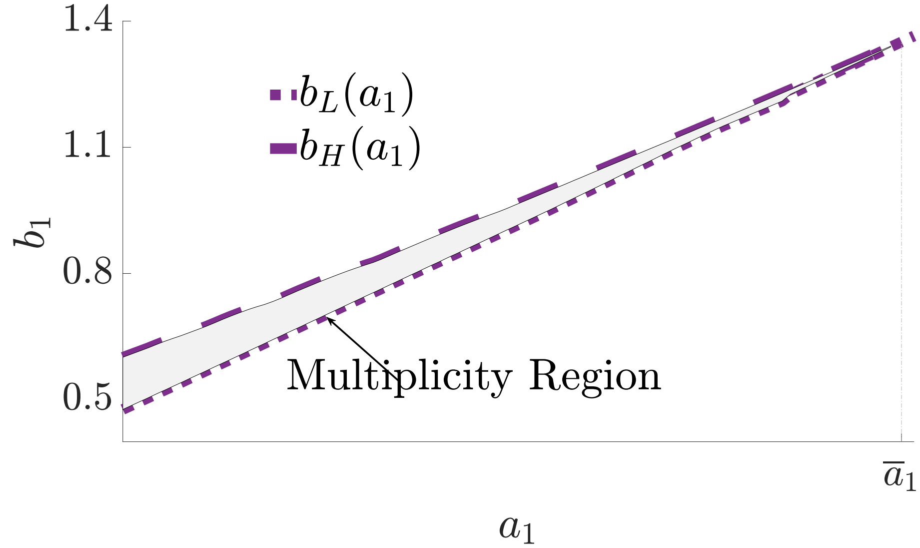
Lemma 2. *(Thresholds of Debt and Reserve) Suppose $\hat{a}_1 > \tilde{a}_1 \geq 0$. Then:*

1. $b_L(\hat{a}_1) > b_L(\tilde{a}_1)$
2. $b_H(\hat{a}_1) > b_H(\tilde{a}_1)$.

Proof. The proof is in the Appendix A □

Lemma 2 compares the boundaries of the debt regions for economies with different levels of reserves. In particular, it establishes that these boundaries are increasing functions of international reserves. As the level of reserves increases, the threshold of initial debt b_1 for which there are multiple solutions for b_2 rises. Similarly, the threshold of b_1 beyond which the equilibrium b_2 delivers a default probability of one in the last period also increases. In Figure 4, we plot the boundaries of the multiplicity region as a function of reserves.

Figure 4: Boundaries of the Multiplicity Region



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

The dashed line represents $b_L(a_1)$, and the dotted line represents $b_H(a_1)$. Notably, there exist a level of reserves $\bar{a}_1$ such that for any $a_1 > \bar{a}_1$, $b_L(a_1) = b_H(a_1)$. At this level of reserves, the government eliminates the possibility of multiple equilibria in period one. We formalize this result in the following proposition.

Proposition 1. *There exists a level of reserves $\overline{a}_1$ such that $b_L(\overline{a}_1) = b_H(\overline{a}_1)$ and the equilibrium in period one is unique.*

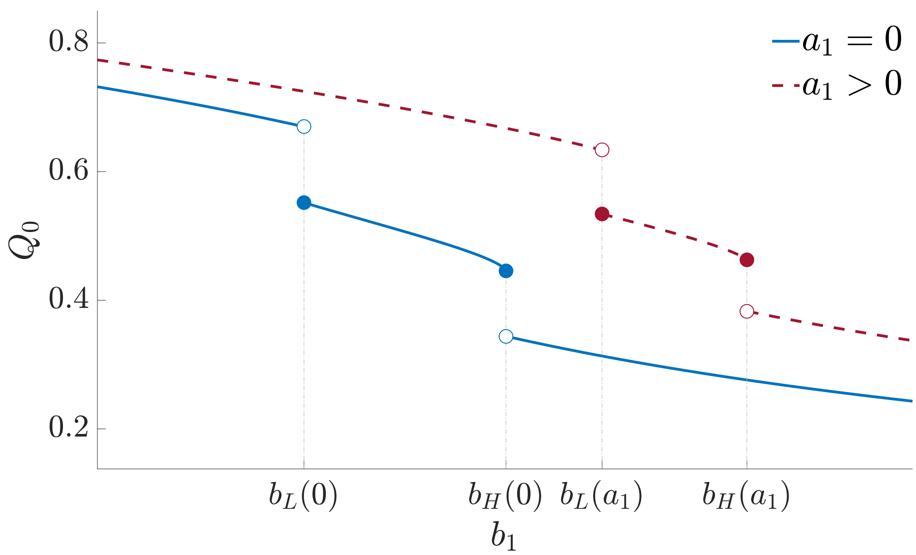
Proof. Proof in the Appendix A □

If the government chooses $\overline{a}$, it effectively eliminates the possibility of multiple equilibria. The intuition of this result is as follows. At $\overline{a}$, the government chooses to issue a high level of debt in the first period to accumulate a significant amount of reserves. When the stock of initial debt is high, the dilution effect from increasing borrowing is always stronger than the income effect of reducing the value of new borrowing. As a result, the Laffer curve is always increasing in new borrowing. Since the Laffer curve is always increasing, there is no possibility of having multiple equilibria because higher borrowing will always lead to higher income.

4.2 Price Function

We now analyze the implications of multiplicity on the price function in period zero. In Figure 5, we replace the equilibrium debt function $B_2(b_1, a_1, \omega_1)$ in equation (16) to determine the price of government bonds as a function of debt and reserves in period zero.

Figure 5: Period Zero Price Function



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

The red dashed lines represent the price function when the government accumulates positive reserves, while the solid blue lines represent a government without reserves. The price function displays two jumps at the boundaries of the multiplicity region, which directly results from the multiplicity in period one. When the government issues a level of debt b_1 and accumulates reserves a_1 such that it is just below the boundary of the multiplicity region $b_L(a_1)$, future borrowing $B_2(b_1, a_1, \omega_1)$ is unique and low. Consequently, expected dilution is low, and the bond price in period zero is high.

On the other hand, if the government increases b_1 by an arbitrarily small amount while keeping a_1 fixed, it enters the multiplicity region. In this case, there exists a *good equilibrium* with low future borrowing $B_2(b_1, a_1, \omega_1 = 1)$, close to the borrowing level just below the boundary. However, there is also an equilibrium with high future borrowing $B_2(b_1, a_1, \omega_1 =$

0). As a result, the price in this region is a weighted expectation of these two possible outcomes: one with low future borrowing and one with high future borrowing. The possibility of high future borrowing implies that future prices are lower in some states, which leads to a reduction in the current price of bonds, as lenders discount the expected dilution.

Similarly, the price function experiences another jump around the boundary of the high-risk region $b_H(a_1)$. If debt is increased from a level below $b_H(a_1)$ by an arbitrarily small amount while keeping reserves fixed, it eliminates the possibility of a *good equilibrium* in period one. In this case, the only equilibrium involves high future borrowing, which leads to high dilution. Consequently, the price in period zero drops.

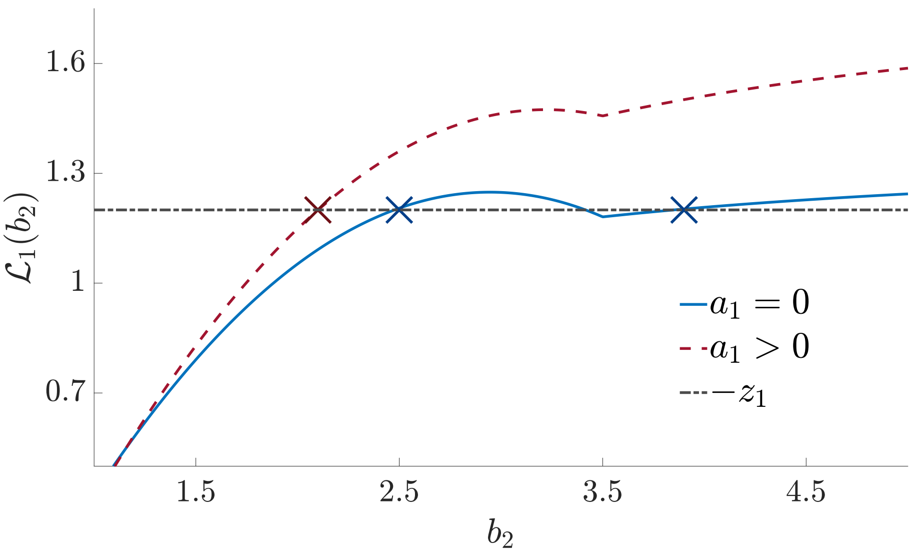
4.3 Debt-Financed Reserves

In our previous analysis, we established that the boundaries of initial debt levels, which determine whether the model exhibits multiple equilibria, increase with the level of reserves. To further inspect the trade-offs in portfolio composition, we study a financial operation in which the government issues debt in period zero to finance reserve accumulation.

In Figure 6, we compare the period-one equilibrium of two economies that have the same market value of the government portfolio at period-zero prices but differ in their levels of reserve accumulation. The solid blue line represents the debt Laffer curve including reserves less coupon payments for an economy that issues just enough debt in period zero to finance its deficit. In constrast, the red dashed line represents the debt Laffer curve including reserves less coupon payments for an economy that issues additional debt in period zero to both finance the deficit and accumulate positive reserves. In this example, the economy with positive reserves exhibits a unique equilibrium in period one, whereas the economy without reserves experiences multiple equilibria.

To understand the intuition behind this result, we decompose the effect of reserve accumulation into two components. First, note that the red dashed line is positioned to the left of the blue solid line. The fact that the equilibrium is unique with reserves represents an increase in efficiency. In the absence of multiplicity in period one, government bond prices in period zero increase, allowing it to acquire a portfolio with a higher market value. This higher portfolio value implies that the government needs to issue less b_2 to finance any deficit $-z_0$, which in turn shifts the red dashed line to the left.

Figure 6: Equilibrium Condition in Period One



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

The second effect is that the slope of the Laffer curve is steeper with reserves. Recall that when the government increases borrowing b_2, three forces come into play: the stock of borrowing increases, the price of debt decreases, and the value of outstanding debt $(1 - \delta)b_1$ decreases. The first two forces operate similarly regardless of the levels of b_1 and a_1. However, in the economy with reserves, a higher b_1 amplifies the dilution effect, allowing the government to increase its revenue faster when increasing b_2. We formalize this argument in the following lemma:

Lemma 3. *(Elasticity of the Debt Laffer Curve) Consider two economies with* $\{\hat{b_1}, \hat{a_1}\}$ *and* $\{\tilde{b}_1, \tilde{a}_1\}$ *which have the same NFA position and* $\hat{a_1} > \tilde{a}_1$*. Let* $\hat{\mathcal{L}}, \tilde{\mathcal{L}}$ *be the Laffer curve of those economies. Then:*

$$\frac{\partial \hat{\mathcal{L}}/\partial b_2}{\hat{\mathcal{L}}} b_2 > \frac{\partial \tilde{\mathcal{L}}/\partial b_2}{\tilde{\mathcal{L}}} b_2 \quad \textit{if} \quad \partial \hat{\mathcal{L}}/\partial b_2 > 0$$
$$\frac{\partial \hat{\mathcal{L}}/\partial b_2}{\hat{\mathcal{L}}} b_2 < \frac{\partial \tilde{\mathcal{L}}/\partial b_2}{\tilde{\mathcal{L}}} b_2 \quad \textit{if} \quad \partial \hat{\mathcal{L}}/\partial b_2 < 0$$

Proof. The proof is in the Appendix A □

Lemma 3 establishes that the elasticity of the Laffer curve is higher in the presence of reserve accumulation, specifically in regions where the Laffer curve increases with b_2. In other words, in a borrowing range where the government generates more income by increasing b_2, this income grows at a faster rate in an economy with higher initial debt and reserves. Conversely, in the region where the Laffer curve declines with borrowing, it does so at a slower rate, as the elasticity is lower in this region. Both results come from a stronger dilution effect in the economy with reserves accumulation.

A key implication of Lemma 3 is that multiple equilibria emerge only for higher deficit levels $-z_0$ in an economy where the government chooses to accumulate reserves. Suppose that z_0 is such that, in the absence of reserve accumulation, the economy exhibits multiplicity in period one. In this economy, if the government instead increases debt in period zero to acquire reserves, the equilibrium becomes unique. We formalize this result in the following proposition.

Proposition 2. *Assume z_1 is such that the economy with $a_1 = 0$ has multiple equilibria. There is a level of reserves a_1^* such that $a_1^* > 0$ for which the economy has a unique equilibrium associated with a lower expected b_2.*

Proof. Proof in the Appendix A

□

Proposition 2 underscores one of the paper's key results, which can be understood as follows. A higher level of reserves provides the government with additional resources in period one, regardless of the bond prices. However, if lenders were to coordinate on a low bond price, the value of the higher initial debt issued to finance reserve accumulation would also be low. As a result, the government can relax the budget constraint in states where lenders coordinate on low prices by issuing more debt and accumulating reserves in period zero.

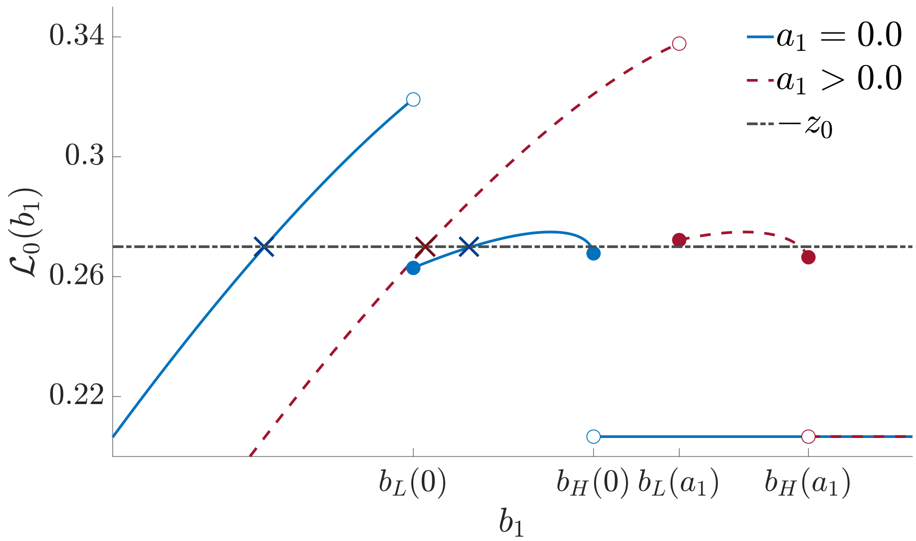
By relaxing the budget constraint under the *bad equilibrium*, the government generates more revenue than needed to finance the surplus when bond prices are low. Consequently, an equilibrium where lenders coordinate on low bond price is unsustainable in an economy where the government holds both higher debt and higher reserves.

In states where lenders coordinate on low bond prices, the dilution effect on investors who buy government bonds in period zero is strong. A necessary condition for Proposition 2 is that period-zero bond prices do not fully adjust to account for the stronger dilution effect in states where lenders coordinate on low prices. This condition is satisfied because there is a chance that the government gets the *good equilibrium*, in which case bond price stays high and dilution remains low.

5 Multiplicity in Period Zero

Multiplicity in period one carries over to period zero through its impact on the price function. In Figure 7, we illustrate this relationship by plotting the equilibrium condition in period zero.

Figure 7: Equilibrium Condition in Period Zero



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

The Laffer curve minus reserves in period zero exhibits two drops at the boundaries of the multiplicity region due to jumps in period-zero bond price at these points, as discussed in Figure 5. Consequently, for some levels of z_0, multiple equilibria emerge in period zero.

We begin by analyzing the case where the government does not accumulate reserves, represented by the solid blue line. In Figure 7, we pick a level of z_0 that allows for multiple equilibria. In this case, the sunspot in period zero may select an equilibrium with a low level of debt. In this first equilibrium, period one has no uncertainty because future borrowing is unique, given the low initial debt level. As a result, this equilibrium is characterized by low debt, low future borrowing, and high bond prices.

On the other hand, a second equilibrium exists in period zero. In this equilibrium, borrowing in period one could be high or low, depending on the realization of the sunspot in that period. As a result, this second equilibrium is characterized by higher debt level, uncertainty about future borrowing, and lower bond prices.

However, the government can issue a higher level of debt b_1 and accumulate reserves

a_1, which we illustrate using a red dashed line. In this case, the equilibrium with reserves accumulation is unique. This result is related to the analysis in the previous section, where higher levels of debt and reserves help prevent multiplicity in period one. By financing the deficit in period zero with a portfolio that ensures a unique equilibrium in period one, the government faces high bond prices in period zero and avoids multiplicity. In the following subsection, we formalize this analysis and characterize the properties of the equilibria for all combinations of $\{b_1, a_1\}$.

It is important to note that it is optimal for the government to accumulate reserves when fundamentals are weak and multiple equilibria are possible. In such cases, increasing reserves restores uniqueness and eliminates confidence-driven fluctuation in prices. Critically, in the presence of long-term debt, eliminating the bad equilibrium has a positive impact on current prices, which helps finance the current surplus while keeping higher NFA position in period one.

Lastly, note that the peak of the Laffer curve when the government accumulates reserves is higher than when it does not. This implies that the government can avoid immediate default for some values of z_0. The intuition of the result is that by avoiding multiplicity in period one, the government increases efficiency, which in turn raises prices in period zero. This enables the government to finance higher deficits without immediate default.

5.1 Self-Fulfilling Crises and Reserves

In this section, we analyze the government's vulnerability to a self-fulfilling crisis for different combinations of $\{b_1, a_1\}$. Figure 8 illustrates the relationship between the government's portfolio and equilibrium outcomes. Similar to Figure 4, we plot the boundaries of the multiplicity regions in purple. Additionally, we plot the combination of debt and reserves consistent with the government's budget constraint in period zero.

Panel 8a shows an economy with an intermediate initial surplus z_0. In this economy, fundamentals are weak. While the government can finance the period-zero surplus without default, it remains vulnerable to a self-fulfilling crisis that increases the default probability in the last period. The solid yellow line represents the combinations of reserves and debt that allow the government to finance the deficit in period zero when the sunspot selects the *bad equilibrium*. As discussed in the previous section, when reserve accumulation is low, the level of debt required to finance the first-period deficit falls within the multiplicity region of period one.

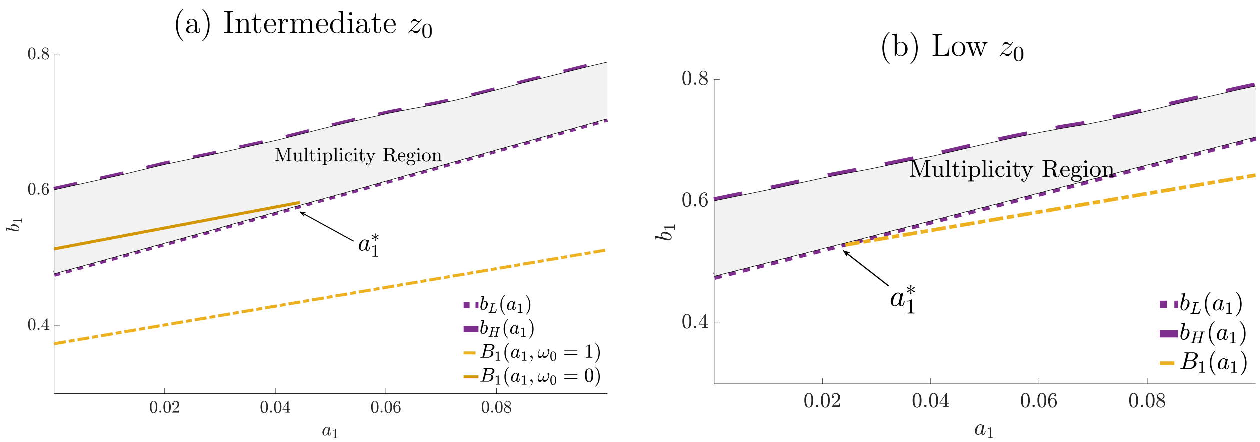


Figure 8: Portfolio Composition and Multiplicity

The dash-dotted yellow line represents the equilibrium condition when the sunspot in period zero is the *good sunspot*. In this case, for all levels of reserve accumulation, the government can finance the surplus with a combination of debt and reserves that ensures the period-one equilibrium remains outside the multiplicity region. As a result, the equilibrium in period one is unique, with both period-zero debt issuance and future borrowing remaining low.

With zero reserves and under the *bad sunspot*, the initial level of debt is high, and the government faces multiple possible levels of future borrowing. However, by increasing debt and accumulating reserves, the government moves along the solid yellow line. Eventually, it reaches the boundary of the multiplicity region at a^*, where multiplicity is effectively eliminate multiplicity. Also note that at a^*, the positive effect on prices of eliminating multiplicity makes the debt jump into the dashed yellow line, where the government can finance the deficit in period zero with low debt and low future borrowing.

The formal proof that $\overline{a}_1 > a_1^*$ is unavailable. Nonetheless, it is always the case in the numerical examples we study. The importance of this result is that the government does not need to accumulate reserves to the point where the boundaries of the multiplicity region coincide to eliminate multiplicity. As reserve accumulation increases bond prices, the portfolio's value in period zero rises, allowing the government to prevent multiplicity with a lower level of reserves.

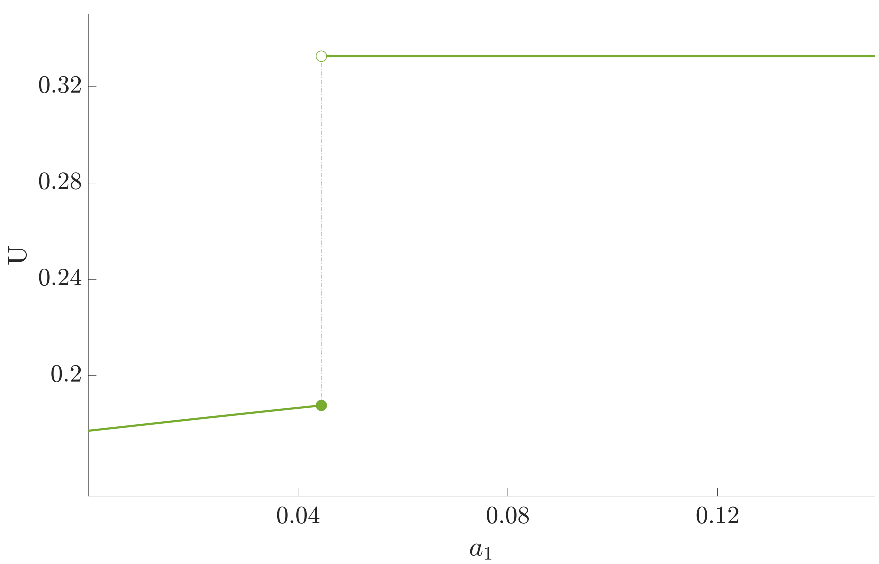
Panel 8b illustrates the equilibrium for an economy with a low z_0. In this case, we do not plot a solid yellow line at zero reserves because no level of debt satisfies the budget constraint in period zero. Consequently, if the government chooses not to accumulate reserves, it will default in the first period. In fact, there is no equilibrium within the multiplicity region because the risk of future crises and its impact on bond prices prevent the government from avoiding default in period zero.

However, there exists a level of reserves at which the government can finance the surplus, with debt at the boundary of the low-risk zone. With this portfolio, bond prices are high enough to ensure that the government avoids default in period zero. The dotted yellow line represents all combinations of reserves and debt that satisfy the budget constraint in period zero, allowing the government to avoid immediate default.

5.2 Welfare and Debt Sustainability

We now assess the welfare implications of reserves and government's optimal reserve holdings. Figure 9 illustrates welfare as a function of reserves. When the government holds zero reserves, multiple equilibria may arise. However, as reserves increase, debt and future borrowing in the event of self-fulfilling crises decrease, leading to higher welfare. In this example, where utility is linear, welfare increases proportionally with reserves. Eventually, the government accumulates enough reserves to ensure a unique equilibrium. At this threshold, the elimination of multiple equilibria results in a discrete jump in welfare.

Figure 9: Welfare



Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_0 = -0.27$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

Notably, all reserve levels that implement a unique equilibrium deliver the same welfare for the government. At these levels, the relative price of reserves and debt in period zero ensures that portfolio composition does not affect borrowing in period one b_2. Consequently, both b_2 and welfare remain unchanged across these reserve levels, making the government indifferent to any reserve level that guarantees a unique equilibrium.

6 Relation to the Literature

This section discusses the relationship between our model and key findings in the existing literature.

6.1 Maturity Structure

Our results are closely related to those in Lorenzoni and Werning (2019). They establish that a longer maturity structure—achieved by lowering δ—pushes the model toward uniqueness. Their mechanism is similar to the one we described in this paper. Specifically, if the government rebalances the maturity structure towards longer-term debt by decreasing δ, it reduces government payments in period one and increases the stock of debt that matures in period two $(1-\delta)b_1$. This effectively provides insurance in period one against the risk of a self-fulfilling crisis, as lower prices in period one would dilute the increased stock of existing debt $(1-\delta)b_1$. Similar to the mechanism in our model, better insurance in period one pushes the model toward uniqueness.

A key insight of Lorenzoni and Werning (2019) is that increasing the average maturity of the government's debt portfolio reduces the risk of self-fulling crises. In our model, debt-financed reserves serve as another way to effectively increase the maturity of the government's portfolio.

6.2 Incentives vs Insurance

We establish that increasing reserves and debt provides insurance for the government against a type of self-fulling crisis that arises from the government's resource constraints. The effect of reserves on the government's incentives to strategically default is beyond the scope of the paper. A promising area for future research is to study how reserves could expose the government to another type of self-fulling crisis, one that operates through the government's incentives to borrow, as described by Aguiar and Amador (2020).

6.3 Insurance Against Fundamental Risk

Bianchi et al. (2018), among others, have established that reserves provide insurance against fundamental risk. In this view, the government should accumulate reserves when bond prices are high and draw them down when bond prices are low, typically due to a high risk of default or high lenders' risk aversion. In constrast, we find that it is optimal for the government to accumulate reserves even when bond prices are low. The reason is that reserves help increase current prices when the government issues long-term bonds and faces the risk of self-fulling crises, as they reduce expectations of future borrowing.

6.4 Self-Fulfilling Runs

Our result is related to Barbosa-Alves et al. (2024). As in our model, they find that debt-financed reserves reduce the risk of self-fulfilling runs by providing liquidity to the government when investors panic. The intuition is that in their model, during a run, if the government chooses to repay, it must generate a high surplus to pay the outstanding debt, as rolling over part of the debt is not feasible. In such states, increasing both debt and reserves improves liquidity because the government can use reserves to pay back the debt, while only needing to pay a fraction δ of the increased debt.

In contrast, in our model, the government faces no liquidity problems in period one. Even when lenders panic, the government can still trade bonds in the market without needing to adjust the surplus. As a result, in our model, reserves do not provide liquidity. Instead, debt-financed reserves allow the government to take advantage of low prices in states where lenders panic by buying back debt using reserves. The main implication of a *bad sunspot* in our model is that the government is forced to accumulate high debt in period one. In these states, reserves help mitigate the consequences of the *bad sunspot* by allowing the government to buy back debt at lower prices.

Another key difference is that Barbosa-Alves et al. (2024) argues that the government should only issuing debt to accumulate reserves when it manages to exit the crisis zone in the next period. In their model, debt-financed reserves increase prices as the government approches the safe zone. In contrast, in our model, debt finance reserves provide improved insurance for the government by reducing borrowing in crisis states. This mechanism positively impacts prices, regardless of the government's ability to exit the crisis zone.

7 Empirical Evidence

One key implication of the theory is that governments should accumulate reserves during periods when bond prices are low. In these periods, the confidence risk stemming from the possibility of multiple equilibria plays a crucial role, making it optimal for the government to accumulate reserves to reduce this risk.[8] This prediction contrasts with another strand of the literature, which argues that governments choose to accumulate reserves as insurance against fundamental risk. In this view, governments issue debt to finance reserves accumulation when bond prices are high and use reserves to avoid rolling over debt when bond prices are low, due to the high risk of default.[9]

In this section, we empirically test whether governments choose to use reserves when bond prices are low (or equivalently, when bond yields are high). To do so, we examine the empirical regularities of episodes of *Fiscal Stress*. We also analyze the dynamics of key variables of interest during the 2008 financial crisis.

7.1 Episodes of Fiscal Stress

Our goal is to identify periods when the cost of financing deficits through debt issuance is exceptionally high relative to a country's historical average. To maintain our focus on the role of reserves in debt sustainability, we exclude default episodes from our analysis, as the paper is not concerned with the role of reserves during such events.[10]

In Figure 10, we illustrate an episode of *Fiscal Stress* by plotting the yield of Colombian sovereign bonds. The gray-shaded area marks the second quarter of 2008, during which the yield of local currency Colombian bonds spiked to a historically high level. A key feature of this episode is that Colombia did not default during this period.

Formally, we define an episode of *Fiscal Stress* as a quarter where governments did not default and the bond yields were at least two standard deviations above the historical mean. This definition is inspired by the literature on financial crises, which defines a Sudden Stop episode as a quarter in which the current account declines by more than two standard deviations below the mean.[11]

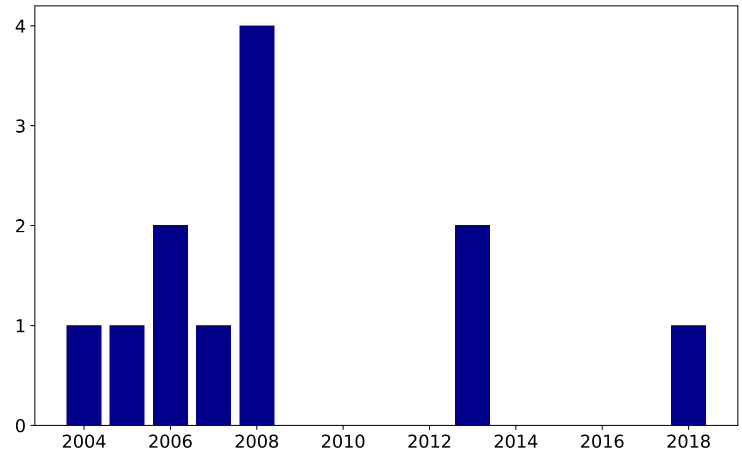
[8] See Corsetti and Maeng (2023) and Hur and Kondo (2016) for related results

[9] See Bianchi et al. (2018).

[10] See Gourinchas and Obstfeld (2012) for an analysis of NFA during default episodes.

[11] See for example Calvo et al. (2006)

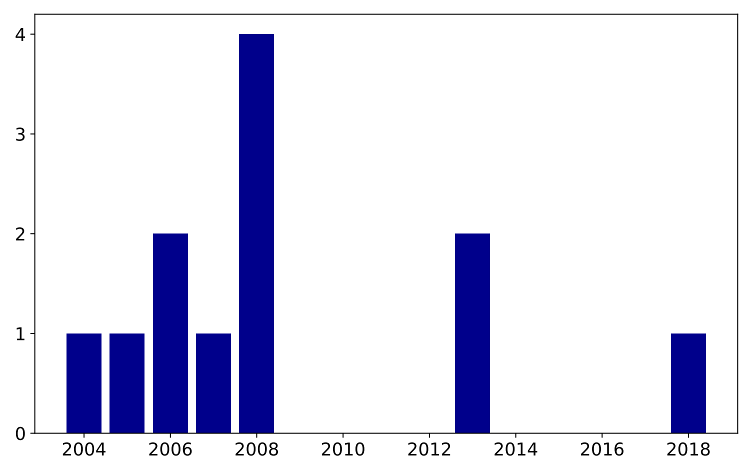
Figure 10: Yield Sovereign Bonds Colombian Government



Source: Bloomberg.

We apply this filter to identify the first quarter in which the *Fiscal Stress* criteria are met in a panel of 17 Emerging Market (EM) economies. Our sample includes all EMs for which quarterly government bond yield data are available. In Figure 11, we plot the number of identified episodes each year based on this methodology. We identify two periods during which many countries experienced *Fiscal Stress*: the first in 2004 and the second during the U.S. financial crisis in the third quarter of 2008. Moreover, we identify 14 distinct episodes with no country appearing more than once in the sample.

Figure 11: Number of Episodes



Source: Bloomberg.

In Figure 12, we conduct an event analysis centered around the beginning of episodes of *Fiscal Stress*. We plot the mean and median of all key variables over a window that spans four quarters before and after each episode. We analyze all variables as a percentage of GDP and normalize their values at time t to zero. In addition, we apply a four-quarter moving average to smooth the series, a technique commonly used in empirical studies on financial crises to mitigate excess volatility in the raw data. The analysis includes only those episodes for which we have complete data on debt and reserves. In Appendix B, we provide an additional plot analyzing the behavior of reserves for all episodes in the sample.

Panel 12a illustrates the movement of yields around these episodes. By construction, the yields experience a sharp increase at time t. By examining a nine-quarter window, we capture both the gradual rise in yields that typically begins four quarters before the spike and the subsequent decline, which generally returns them to near-average levels within four quarters after the peak.

Panel 12b depicts the dynamics of the net foreign asset (NFA) position over the same nine-quarter window. The data show that countries tend to consolidate their NFA positions during these episodes. In particular, at the beginning of the window, the median of NFA is 1 percentage point below its level at the episode's peak. It continues to rise throughout the window, ultimately reaching a 0.7 percentage point above its value at the event's peak. The mean of NFA follows a similar pattern, displaying a steady upward trend over the period. Next, we assess whether governments achieve this NFA consolidation by reducing debt or increasing reserves.

Panel 12c reveals that a typical episode features an increase in foreign reserves. This finding is the main result of this section, indicating that countries tend to accumulate reserves during periods of *Fiscal Stress*. The median trajectory shows a brief decline in reserves at both the beginning and the end of the window. However, the overall cumulative effect across the quarters is positive, with reserves rising from 0.4 percentage points below their peak value to 0.2 percentage points above it. This trend is even more pronounced when considering the mean.

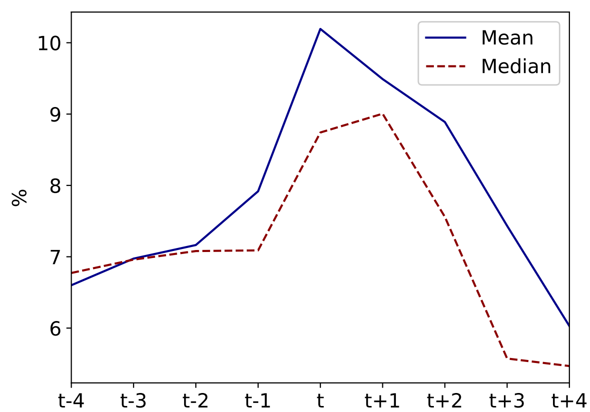
Finally, Panel 12d illustrates the dynamics of public debt. Both the mean and median debt levels decrease, indicating that governments also reduce debt during *Fiscal Stress* episodes. In summary, governments adopt a strategy of simultaneously increasing reserves and reducing debt to strengthen the public sector's NFA position.

In Appendix B, we analyze foreign reserves dynamics across the full sample of identified episodes, including those where debt data are unavailable. The results remain consistent, confirming the robustness of our main findings.

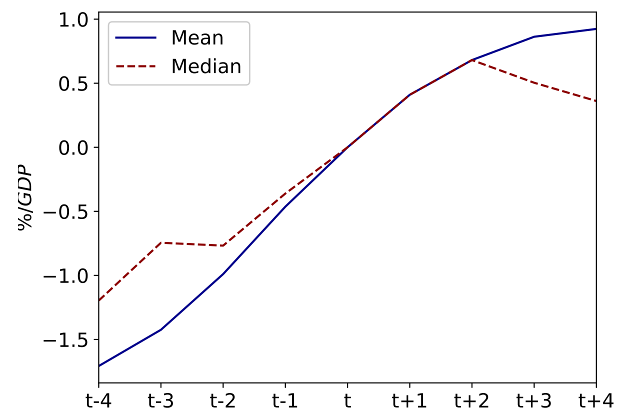
We also conduct the analysis excluding countries with fixed exchange rates, as their reserve accumulation policies are more likely influenced by their exchange rate regimes. The results presented in this section remain robust even after excluding those countries.

We emphasize that this exercise does not completely address a critical challenge for the empirical analysis of this section: in practice, countries accumulate reserves for various reasons. Assessing the marginal contribution of the mechanism proposed in this paper to the observed empirical patterns is ultimately a quantitative question. We leave this for future research.

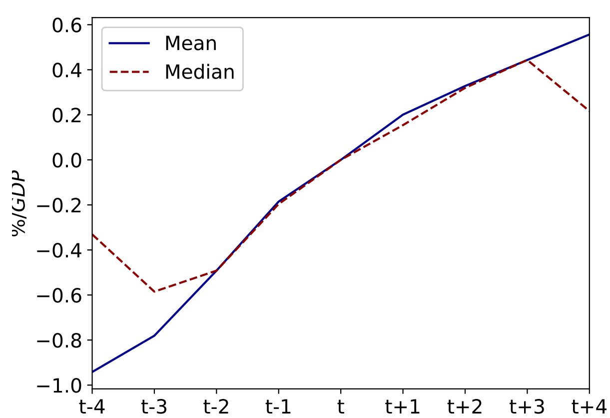
(a) Yield



(b) Net Foreign Assets



(c) Foreign Reserves



(d) Total Debt

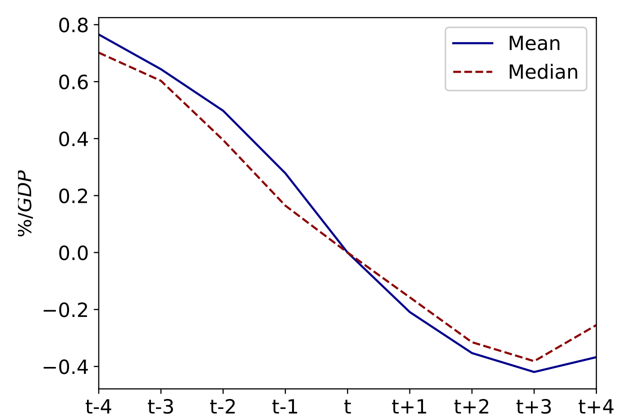


Figure 12: Fiscal Stress Dynamics

Note: Data of Total General Government Gross Debt is from IMF's Sovereign Debt Investor Base. Reserves are from IMF's International Financial Statistics. GDP is from the World Bank's World Development Indicators. Data on Yields are from Bloomberg.

7.2 Global Financial Crises

We now turn to an analysis of the global financial crisis. The first quarter of 2008 is a well-studied period in which governments faced exceptionally high yields due to unfavorable international conditions.[12] We contribute to the literature by identifying the countries that were particularly affected by these adverse global conditions and examining how the dynamics of key variables in these countries differ from those in other emerging markets.

In Figure 13, we plot the dynamics of NFA, debt, and reserves following the methodology described in the last subsection. In this case, we plot the dynamics around the peak of the global financial crises in the third quarter of 2008.

We divide the sample into two groups: (i) countries that experienced an episode of *Fiscal Stress*, as defined in the previous subsection (4 countries), and (ii) those where yields did not rise to an unusual level. We then analyze the median of the key variables for both groups.

Panel 13a shows that countries experiencing *Fiscal Stress* exhibited a more pronounced spike in yields. However, the typical country experienced a yield increase during the financial crisis.

Panel 13b reveals that both group consolidated their NFA positions during this period. Similarly, both groups increased their reserves, as shown in Panel 13c. The difference lies in their debt management strategies. Panel 13d indicates that countries experiencing an unusual spike in sovereign bond yields opted to reduce their debt, whereas those that did not experience a significant yield increase chose to increase their debt. Although both groups strengthened their NFA positions, countries less affected by the crisis chose to increase both debt and reserves, even as yields rose.

This analysis builds on the work of Aizenman and Sun (2012) by identifying the countries most affected by the global financial crisis. We extend their findings by demonstrating that, during this period, the typical country increased its foreign reserves. Moreover, even among the countries most impacted—those experiencing a surge in sovereign yields—reserve accumulation played a crucial role in their response to the crisis.

[12] See, for example, Aizenman and Sun (2012)

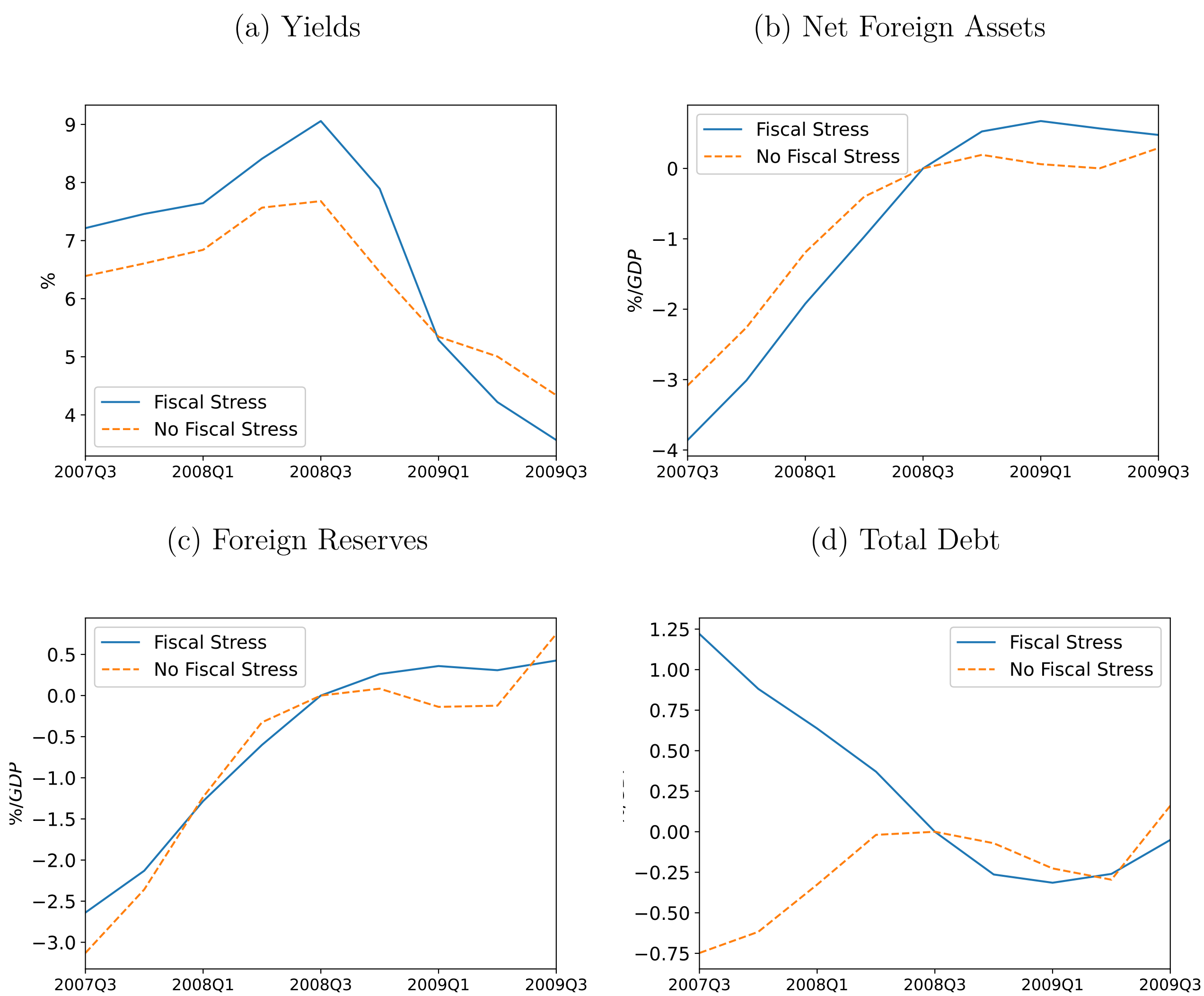


Figure 13: Global Financial Crisis Dynamics

Note: Data of Total General Government Gross Debt is from IMF's Sovereign Debt Investor Base. Reserves are from IMF's International Financial Statistics. GDP is from the World Bank's World Development Indicators. Data on Yields are from Bloomberg.

7.3 Illustration of Reserves Accumulation During Crises

We close this section by illustrating how the model's mechanisms can account for the empirical patterns we identified. This exercise serves as a proof of concept, illustrating how the model can explain why a government might find it optimal to accumulate reserves during periods of high sovereign bond yields.

In Figure 14, we present a comparative static exercise in which we vary the government's initial deficit z_0. This exercise follows the same parameterization as the numerical examples of Section 3, with the initial deficit being the only variable changed. We plot the equilibrium levels of debt and reserves in the first period. Since the government is indifferent to any level of reserves that implements the unique equilibrium, we assume it chooses the lowest level of reserves consistent with uniqueness.

In this exercise, a higher initial deficit increases risk in the first period. Panel 14a shows that, in equilibrium, the government responds by issuing more debt. Consequently, as illustrated in Panel 14b, the equilibrium yield on government bonds rises due to the increased default risk.

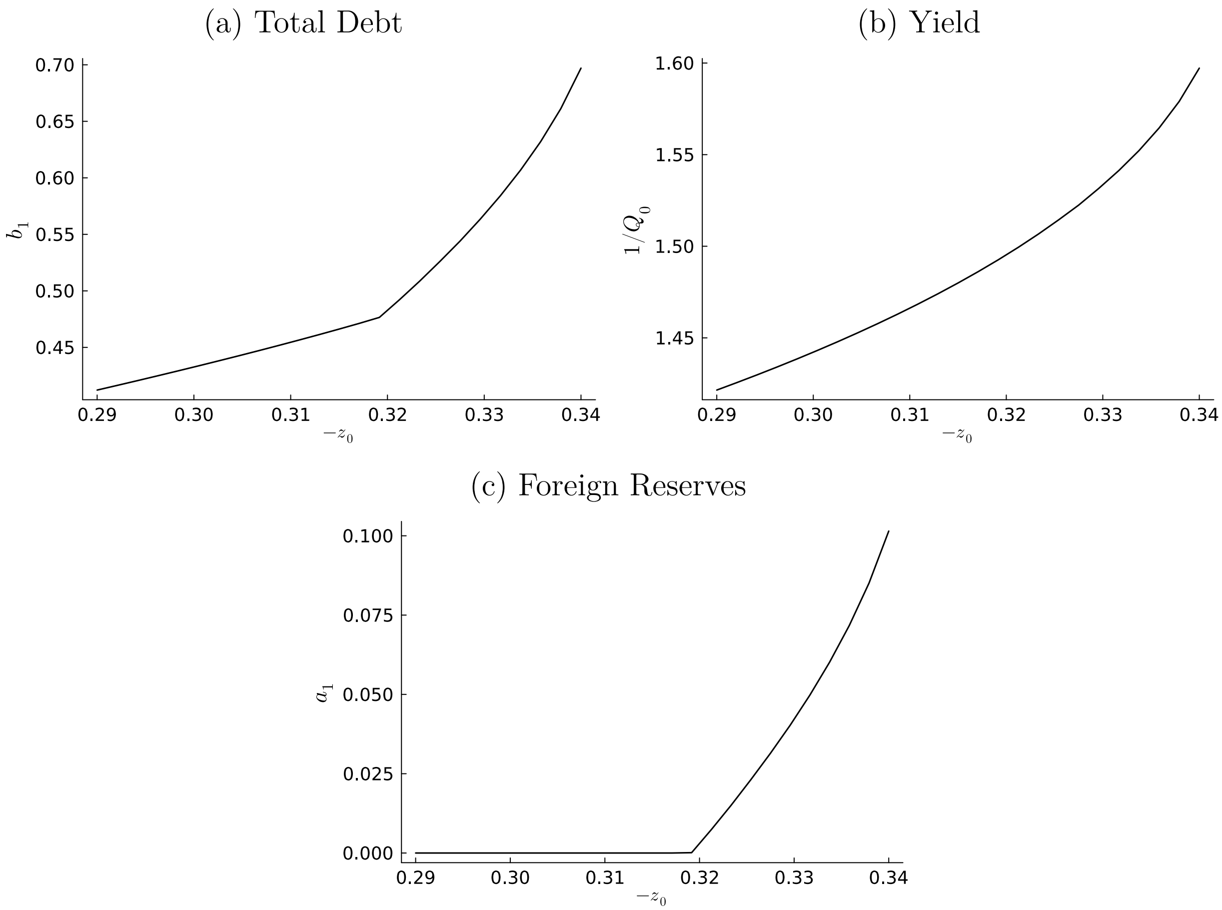


Figure 14: Fiscal Stress in the Model

Note: This figure is computed assuming $\beta = 0.95$, $\delta = 0.0$, $\phi = 0.7$, $z_1 = -1.2$, $z_2 \sim Unif(0.55, 3.5)$.

Crucially, when the risk increases due to the need to finance higher deficits, the levels of reserves required to eliminate the risk of self-fulfilling crises also increases. As a result, the government accumulates higher levels of reserves. In the model, when prices exhibit confidence-driven fluctuations due to multiple equilibria, the government finds it optimal to accumulate reserves even when bond yields are high.

This counterfactual exercise can replicate the behavior observed in some European countries around 2012, as well as the behavior of EMs that were not under *Fiscal Stress* during the 2008 financial crises. However, to capture the behavior of countries experiencing *Fiscal Stress*, which reduces debt to consolidate their NFA position, we would need to consider fiscal rules that respond to increases in debt. We leave this extension for future research. Nevertheless, we highlight that the main mechanism of the model, in which governments accumulate reserves to reduce confidence-driven fluctuations in prices, is consistent with the main empirical regularity identified in this section: that governments tend to accumulate reserves during periods of *Fiscal Stress*.

Summary. In this section, we present empirical evidence that governments tend to accumulate reserves rather than spend them when bond yields are exceptionally high. This evidence complements the empirical regularities outlined in the introduction, which highlight the debt and reserve policies observed in some European countries around 2012.

Furthermore, we conduct a comparative statics exercise to illustrate how the model's mechanism, which is that the government uses reserves to mitigate confidence risk, can help explain this observed pattern.

8 Conclusions

This paper presents a model in which the government may face a self-fulfilling crisis driven by investors' expectations of high future borrowing. Lack of investor confidence reduces current bond prices, forcing the government to take on more debt. Higher debt, in turn, leads to higher future borrowing, confirming the initial expectation. We extend the existing literature by introducing foreign reserves as risk-free assets that the government can accumulate to strengthen its financial position.

We establish that the government can restore uniqueness by issuing debt to accumulate reserves. This financial operation provides insurance against future crises, improving current bond prices and eliminating the possibility of multiple equilibria. Crucially, we find that it is optimal for the government to accumulate reserves even when economic fundamentals are weak. As a result, our model rationalizes the behavior of certain European countries that accumulated reserves during the sovereign debt crisis in 2012.

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A Appendix

Proof of Lemma 1 (Thresholds of Debt)

Proof. Recall that the government's budget constraint in period one is

$$z_1 + Q_1(b_2)(b_2 - (1-\delta)b_1) + a_1 = \kappa_1 b_1. \tag{23}$$

Taking total derivative with respect to b_1, we obtain

$$0 + \frac{\partial Q_1(b_2)}{\partial b_2}\frac{\partial b_2}{\partial b_1}[b_2 - (1-\delta)b_1] + Q_1(b_2)\left[\frac{\partial b_2}{\partial b_1} - (1-\delta)\right] + 0 = \kappa_1,$$
$$\frac{\partial b_2}{\partial b_1}\left[\frac{\partial Q_1(b_2)}{\partial b_2}(b_2 - (1-\delta)b_1) + Q_1(b_2)\right] = \kappa_1 + (1-\delta)Q_1(b_2). \tag{24}$$

Rearranging terms to get:

$$\frac{\partial b_2}{\partial b_1} = \frac{[\kappa_1 + (1-\delta)Q_1(b_2)]}{\left[\frac{\partial Q_1(b_2)}{\partial b_2}(b_2 - (1-\delta)b_1) + Q_1(b_2)\right]} \tag{25}$$
$$= \frac{\left[\frac{z_1 + a_1 + Q_1(b_2)b_2}{b_1}\right]}{\left[\frac{\partial Q_1(b_2)}{\partial b_2}(b_2 - (1-\delta)b_1) + Q_1(b_2)\right]}. \tag{26}$$

Then notice that for all $b_2 \geq \overline{Z}$, we have

$$Q_1(b_2) = \left[\frac{\beta\phi}{b_2}\int_{\underline{Z}}^{\overline{Z}} z_2 dF(z_2)\right] = \frac{\beta\phi\mathbb{E}[z_2]}{b_2}, \tag{27}$$
$$Q_1(b_2)b_2 = \beta\phi\mathbb{E}[z_2], \tag{28}$$
$$\frac{\partial Q_1(b_2)}{\partial b_2} = -\frac{\beta\phi\mathbb{E}[z_2]}{b_2^2}. \tag{29}$$

We then denote $b_1 = b_L$ such that $B_2 = \overline{Z}$. That is,

$$b_L(a_1) = \frac{z_1 + Q_1(\overline{Z})\,\overline{Z} + a_1}{\kappa_1 + (1-\delta)Q_1(\overline{Z})}. \tag{30}$$

Given $b_2 \geq \overline{Z}$, we can simplify

$$\frac{\partial b_2}{\partial b_1} = \frac{\left[\frac{z_1 + a_1 + \beta\phi\mathbb{E}[z_2]}{b_1}\right]}{-\frac{\beta\phi\mathbb{E}[z_2]b_2}{b_2^2} - \frac{\partial Q_1(b_2)}{\partial b_2}(1-\delta)b_1 + \frac{\beta\phi\mathbb{E}[z_2]}{b_2}} = -\frac{\left[\frac{z_1 + a_1 + \beta\phi\mathbb{E}[z_2]}{b_1}\right]}{\frac{\partial Q_1(b_2)}{\partial b_2}(1-\delta)b_1} > 0. \tag{31}$$

This means that, for all $b_1 \geq b_L$, B_2 is increasing in b_1.

Next, we define $b_1 = b_H$ such that for all $0 \leq b_1 \leq b_H$, we have $\left[\frac{\partial Q_1(b_2)}{\partial b_2}(b_2 - (1-\delta)b_1) + Q_1(b_2)\right] > 0$. This means that we can characterize the shape of B_2 function as having two regions where B_2 is increasing in b_1. The first region, $b_1 \geq b_L$, is such that the government defaults in period two with probability one, i.e. $B_2 \geq \overline{Z}$. The second region, $0 \leq b_1 \leq b_H$, is associated with $B_2 < \overline{Z}$.

Then it follows that $b_1 \in \mathcal{B}_{\mathcal{M}}$ if and only if $b_H(a_1) \geq b_1 > b_L(a_1)$.

□

Proof of Lemma 2 (Thresholds of Debt and Reserve)

Proof. Recall that for any a_1, $b_L(a_1)$ is associated with $B_2 = \overline{Z}$. So we can use the budget constraint in period one to write

$$b_L(a_1) = \frac{z_1 + Q_1(\overline{Z})\,\overline{Z} + a_1}{\kappa_1 + (1-\delta)Q_1(\overline{Z})}. \tag{32}$$

Then,

$$\frac{\partial b_L(a_1)}{\partial a_1} = \frac{1}{\kappa_1 + (1-\delta)Q_1(\overline{Z})} > 0. \tag{33}$$

Next, we define

$$h(b_1, a_1) = \max_{\{b_2 | F(b_2) < 1\}} Q_1(b_2)(b_2 - (1-\delta)b_1) + a_1 - \kappa_1 b_1 \tag{34}$$

and

$$\overline{b}_2(b_1, a_1) = \operatorname*{argmax}_{\{b_2 | F(b_2) < 1\}} Q_1(b_2)(b_2 - (1-\delta)b_1) + a_1 - \kappa_1 b_1. \tag{35}$$

Then we define $b_H(a_1)$ such that

$$b_H(a_1) = \frac{z_1 + Q_1(\overline{b}_2)\overline{b}_2 + a_1}{\kappa_1 + (1-\delta)Q_1(\overline{b}_2)}. \tag{36}$$

or

$$z_1 + Q_1(\overline{b}_2)(\overline{b}_2 - (1-\delta)b_H) + a_1 = \kappa_1 b_H. \tag{37}$$

Taking total derivative with respect to a_1:

$$\frac{\partial Q_1}{\partial b_2}\left[\frac{\partial \overline{b}_2}{\partial b_1}\frac{\partial b_H}{\partial a_1} + \frac{\partial \overline{b}_2}{\partial a_1}\right]\left[\overline{b}_2 - (1-\delta)b_H\right] + Q_1(\overline{b}_2)\left[\frac{\partial \overline{b}_2}{\partial b_1}\frac{\partial b_H}{\partial a_1} + \frac{\partial \overline{b}_2}{\partial a_1} - (1-\delta)\frac{\partial b_H}{\partial a_1}\right] + 1 = \kappa_1 \frac{\partial b_H}{\partial a_1}. \tag{38}$$

Rearranging terms:

$$\frac{\partial \bar{b}_2}{\partial a_1}\left[\frac{\partial Q_1}{\partial b_2}\left(\bar{b}_2-(1-\delta)b_H\right)+Q_1(\bar{b}_2)\right]+\frac{\partial b_H}{\partial a_1}\frac{\partial \bar{b}_2}{\partial a_1}\left[\frac{\partial Q_1}{\partial b_2}\left(\bar{b}_2-(1-\delta)b_H\right)+Q_1(\bar{b}_2)\right]+1 = \frac{\partial b_H}{\partial a_1}\left[\kappa_1+Q_1(\bar{b}_2)(1-\delta)\right]. \tag{39}$$

Using the definition of $\bar{b}_2$, we know that $\left[\frac{\partial Q_1}{\partial b_2}\left(\bar{b}_2-(1-\delta)b_H\right)+Q_1(\bar{b}_2)\right]=0$:

$$\frac{\partial b_H}{\partial a_1}\left[\kappa_1+Q_1(\bar{b}_2)(1-\delta)\right]=1, \tag{40}$$

$$\frac{\partial b_H}{\partial a_1}=\frac{1}{\kappa_1+Q_1(\bar{b}_2)(1-\delta)} \tag{41}$$

$$=\frac{1}{\frac{1}{\beta}+(1-\delta)(Q_1(\bar{b}_2)-1)} \tag{42}$$

$$>0. \tag{43}$$

Therefore, $b_L(\hat{a}_1)>b_L(\tilde{a}_1)$ and $b_H(\hat{a}_1)>b_H(\tilde{a}_1)$ for all $\hat{a}_1>\tilde{a}_1\geq 0$. □

Proof of Proposition 1

Proof. Suppose $\bar{b}_2(b_H(\bar{a}_1),\bar{a}_1)=\overline{Z}$. Then we know from the expressions of $b_L(a_1)$ and $b_H(a_1)$ that $b_L(\bar{a}_1)=b_H(\bar{a}_1)$. Also, by definition of $\bar{b}_2$, we have

$$\frac{\partial Q_1}{\partial b_2}(b_2-(1-\delta)b_1)+Q_1(b_2)=0 \tag{44}$$

$$\frac{\frac{\partial Q_1}{\partial b_2}b_2+Q_1(b_2)}{\frac{\partial Q_1}{\partial b_2}(1-\delta)}=b_1. \tag{45}$$

This means

$$\frac{\left.\frac{\partial Q_1}{\partial b_2}\right|_{b_2=\overline{Z}}\overline{Z}+Q_1(\overline{Z})}{\left.\frac{\partial Q_1}{\partial b_2}\right|_{b_2=\overline{Z}}(1-\delta)}=\frac{z_1+Q_1(\,\overline{Z}\,)\,\overline{Z}\,+\bar{a}_1}{\kappa_1+(1-\delta)Q_1(\,\overline{Z}\,)}, \tag{46}$$

$$\bar{a}_1=\frac{\left.\frac{\partial Q_1}{\partial b_2}\right|_{b_2=\overline{Z}}\overline{Z}+Q_1(\overline{Z})}{\left.\frac{\partial Q_1}{\partial b_2}\right|_{b_2=\overline{Z}}(1-\delta)}\left[\kappa_1+(1-\delta)Q_1(\overline{Z})\right]-z_1-Q_1(\overline{Z})\overline{Z}. \tag{47}$$

□

Proof of Proposition 3 (Elasticity of the Debt Laffer Curve)

Proof. For any b_1, we can write

$$\frac{\partial \mathcal{L}(b_1, b_2)}{\partial b_2} = Q_1(b_2) + b_2 \frac{dQ(b_2)}{db_2} - (1-\delta) b_1 \frac{dQ_1(b_2)}{db_2}. \tag{48}$$

Also, note that $\frac{dQ_1(b_2)}{db_2} < 0$ and b_1 only enters the third term in the above equation. Therefore, if $\hat{b}_1 > \tilde{b}_1$, then

$$\frac{\partial \mathcal{L}(\hat{b}_1, b_2)}{\partial b_2} > \frac{\partial \mathcal{L}(\tilde{b}_1, b_2)}{\partial b_2}. \tag{49}$$

Lastly, we also know that $\mathcal{L}(\hat{b}_1, b_2) < \mathcal{L}(\tilde{b}_1, b_2)$ for all b_2. Then, the lemma follows. □

Proof of Proposition 2

Proof. Define $a^\star$ such that

$$z_0 + Q_0(b_L(a^\star), a^\star) b_L(a^\star) = \beta a^\star. \tag{50}$$

Using definition of Q_0 and b_L:

$$z_0 + \frac{\left[\beta\pi(\kappa_1 + (1-\delta)Q_1(B_2^\star)) + \beta(1-\pi)(\kappa_1 + (1-\delta)Q_1(\overline{Z}))\right]\left[z_1 + Q_1(\overline{Z})\overline{Z} + a^\star\right]}{\kappa_1 + (1-\delta)Q_1(\overline{Z})} - \beta a^\star = 0, \tag{51}$$

where $B_2^\star$ is the low-debt equilibrium. Defining $X = \frac{\kappa_1 + (1-\delta)Q_1(B_2^\star)}{\kappa_1 + (1-\delta)Q_1(\overline{Z})} > 1$ and rearranging terms, we obtain

$$\frac{z_0}{\beta} + \left[\pi X + (1-\pi)\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right] = [1-X]\pi a^\star. \tag{52}$$

That is,

$$a^\star = \frac{z_0}{\beta\pi(1-X)} + \left[\frac{X}{1-X} + \frac{1-\pi}{\pi(1-X)}\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right] \tag{53}$$

Since we know that $a^\star > 0$, we then have

$$\frac{z_0}{\beta\pi(1-X)} + \left[\frac{X}{1-X} + \frac{1-\pi}{\pi(1-X)}\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right] > 0 \tag{54}$$

$$\frac{z_0}{\beta\pi} + \left[X + \frac{1-\pi}{\pi}\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right] > 0 \tag{55}$$

Because $X > 1$, we have

$$\frac{z_0}{\beta\pi} + \left[X + \frac{1-\pi}{\pi}\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right] > \frac{z_0}{\beta\pi} + \left[1 + \frac{1-\pi}{\pi}\right]\left[z_1 + Q_1(\overline{Z})\overline{Z}\right]. \tag{56}$$

So as long as the condition $\frac{z_0}{\beta\pi} + \left[1 + \frac{1-\pi}{\pi}\right][z_1 + \phi\mathbb{E}[z_2]] > 0$ is satisfied, we have a sufficient condition for unique equilibrium. □

Proof of Proposition 3

Proof. First, notice that for any $a_1 \geq 0$ such that the economy has multiple equilibria in period zero, the bad equilibrium associated with $b_1 > b_L(a_1)$ also has multiple equilibria in period one. Denote $b_2^L(a_1)$ and $b_2^H(a_1) \geq \overline{Z}$ the equilibrium debt b_2 associated with the pair (b_1, a_1). Then we have

$$b_1 = \frac{z_1 + Q_1(b_2^L(a_1))b_2^L(a_1) + a_1}{\kappa_1 + (1-\delta)Q_1(b_2^L(a_1))} = \frac{z_1 + Q_1(b_2^H(a_1))b_2^H(a_1) + a_1}{\kappa_1 + (1-\delta)Q_1(b_2^H(a_1))} = \frac{z_1 + \beta\phi\mathbb{E}[z_2] + a_1}{\kappa_1 + (1-\delta)Q_1(b_2^H(a_1))}. \tag{57}$$

The last equation follows because $b_2^H(a_1) \geq \overline{Z}$. Next, to characterize $b_2^L(a_1)$, we compare two economies with different level of reserve $\hat{a}_1 > \tilde{a}_1 \geq 0$. Since both economies have the same z_0, we have

$$Q_0(\hat{b}_1)\hat{b}_1 - \beta\hat{a}_1 = Q_0(\tilde{b}_1)\tilde{b}_1 - \beta\tilde{a}_1, \tag{58}$$

$$\begin{aligned}
&\left[\beta\pi(\kappa_1 + (1-\delta)Q_1(b_2^L(\hat{a}_1))) + \beta(1-\pi)(\kappa_1 + (1-\delta)Q_1(b_2^H(\hat{a}_1)))\right]\hat{b}_1 - \beta\hat{a}_1 \\
= &\left[\beta\pi(\kappa_1 + (1-\delta)Q_1(b_2^L(\tilde{a}_1))) + \beta(1-\pi)(\kappa_1 + (1-\delta)Q_1(b_2^H(\tilde{a}_1)))\right]\tilde{b}_1 - \beta\tilde{a}_1,
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\beta\pi\left[z_1 + Q_1(b_2^L(\hat{a}_1))b_2^L(\hat{a}_1)\right] + \beta\pi\hat{a}_1 + \beta(1-\pi)\left[z_1 + \beta\phi\mathbb{E}[z_2]\right] + \beta(1-\pi)\hat{a}_1 - \beta\hat{a}_1 \\
= &\beta\pi\left[z_1 + Q_1(b_2^L(\tilde{a}_1))b_2^L(\tilde{a}_1)\right] + \beta\pi\tilde{a}_1 + \beta(1-\pi)\left[z_1 + \beta\phi\mathbb{E}[z_2]\right] + \beta(1-\pi)\tilde{a}_1 - \beta\tilde{a}_1,
\end{aligned} \tag{60}$$

$$\beta\pi\left[z_1 + Q_1(b_2^L(\hat{a}_1))b_2^L(\hat{a}_1)\right] = \beta\pi\left[z_1 + Q_1(b_2^L(\tilde{a}_1))b_2^L(\tilde{a}_1)\right]. \tag{61}$$

This means that $b_2^L(a_1) = \underline{b}_2$ is independent of a_1.

Next, for a given (b_1, a_1), we define $\mathcal{L}_{drop}(b_1, a_1) = Q_1(\overline{Z})(\overline{Z} - (1-\delta)b_1) + a_1 - \kappa_1 b_1$. This is the total revenue for a government that issue $b_2 = \overline{Z}$ in period one with a portfolio (b_1, a_1) that satisfies equilibrium in period zero.

Because (b_1, a_1) is part of the equilibrium, we have

$$b_1 = \frac{z_1 + Q_1(\underline{b}_2)\underline{b}_2 + a_1}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}. \tag{62}$$

Substituting this in:

$$\begin{aligned} \mathcal{L}_{drop}(b_1, a_1) = Q_1(\overline{Z})\overline{Z} - Q_1(\overline{Z})(1-\delta)\left[\frac{z_1 + Q_1(\underline{b}_2)\underline{b}_2}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}\right] - Q_1(\overline{Z})(1-\delta)\left[\frac{a_1}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}\right] \\ +a_1 - \kappa_1\left[\frac{z_1 + Q_1(\underline{b}_2)\underline{b}_2}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}\right] - \kappa_1\left[\frac{a_1}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}\right]. \end{aligned} \tag{63}$$

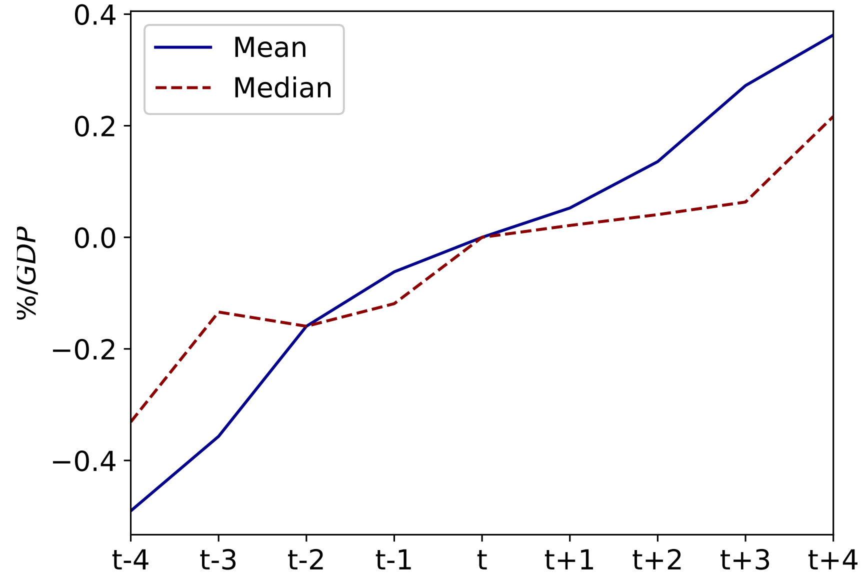
Taking derivative with respect to a_1:

$$\begin{aligned} \frac{\partial \mathcal{L}_{drop}}{\partial a_1} &= 1 - \frac{Q_1(\overline{Z})(1-\delta)}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)} - \frac{\kappa_1}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)} &(64) \\ &= 1 - \left[\frac{\kappa_1 + (1-\delta)Q_1(\overline{Z})}{\kappa_1 + (1-\delta)Q_1(\underline{b}_2)}\right] &(65) \\ &> 0. &(66) \end{aligned}$$

The last inequality follows because $\underline{b}_2 < \overline{Z}$. This means that, for a given z_1, there exists $a_1 > 0$ such that $\mathcal{L}_{drop}(b_1, a_1) > z_1$. This implies that there is no multiplicity in period one, which translates to no multiplicity in period zero as well. □

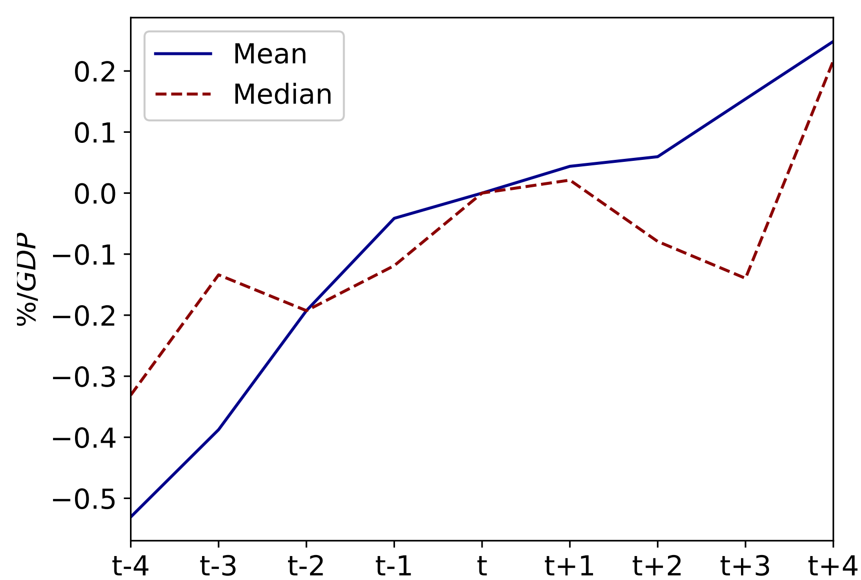
B Additional Plots

Figure 15: Mean and Median Reserve to GDP (Full Sample)



Source: Bloomberg.

Figure 16: Mean and Median Reserve to GDP (Floating Regime)



Note: We take the data on exchange rate regime from the online appendix of Catão and Mano (2017).