Circuit QED Introduction

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Outline

Finding the Relevant Variables

- Want to describe a superconducting circuit in a quantized form.
- Quantum mechanics is a microscopic theory $\Rightarrow$ multibody problem/density operators
- Such a system has only a few degrees of freedom
- Potential degrees of freedom
  - Single Particle excitations
  - Bulk/Volume Plasmons

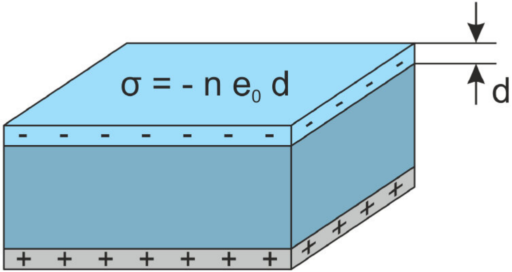
Single Particle excitations

- Feature of superconductivity is electrons build bound states $\Rightarrow$ excitation gap
- Operate the circuit at lower energies

Bulk Plasmons[1]

The Jellium Modell

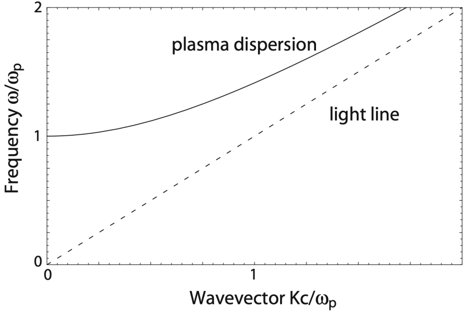
- Current: $\vec{j} = -en\vec{v}$
- Newton/long range coulomb: $\dot{\vec{v}} = -\frac{e}{m}\vec{E}$
- Maxwell and continuity:
  $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{j} = -\dot{\rho}$
- Oscillations in ρ:
  $\ddot{\rho} = -\frac{e^2 n}{m\epsilon_0}\rho = -\omega_p^2\rho$



[1] Floess, Dominik and Giessen, Harald ;Reports on Progress in Physics, Vol. 81, 2018

Bulk Plasmons[2]

- jelly modell $\Rightarrow$ constant dispersion model
- No Bulk Plasmons can be excited below ω_P
- Aluminium $\omega_p \approx 3.6 \cdot 10^{15} Hz \Leftrightarrow E \approx 15$eV
- Visible light 1.6eV $\sim$ 3.3eV $\Rightarrow$ Al is reflective
- We operate below that



[2] Stefan Alexander Maier, Plasmonics: Fundamentals and Applications

London Wavelength

Starting with the Maxwell's equations and going to a low-frequency limit

$$\nabla^2 \vec{E} = \frac{1}{\lambda_L^2}\vec{E} \Rightarrow \vec{E} = \vec{E}_0 \exp\left\{-\frac{x}{\lambda_L}\right\}$$

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n e^2}} = \frac{c}{\omega_p}$$

For Aluminium $\lambda_L \approx 14nm$ which is negletable

The Relevant Variables

To conclude:

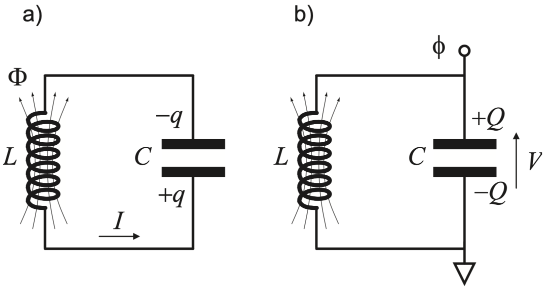
- Jellium modell breaks down in large/small wavevector limit:
  - larger wavevectors lead to an increasing frequency
  - lower wavevectors lead to possible modes due to finite size, e.g. capacitor
- Energy to create a plasmon to high and degrees of freedom are forced into ground state due to coulomb force.
- No single particle excitations
- Only incompressible fluid of electrons sloshing back and forth charging up the capacitor.

The Relevant Variables

So the relevant variables are

- Voltage
- Current
- Charge on the capacitor plates
- Magnetic Flux through the inductor

Quantization of the LC Circuit



Choosing a):

$$E_{\text{Coil}} = \frac{1}{2}LI^2 \qquad E_{\text{Cap}} = \frac{1}{2C}q^2.$$

Constructing the Lagrangian

One term a variable appears quadratic and in the other term the same variable appears quadratic in its first-time derivative. Using $I = \dot{q}$:

$$\mathcal{L} = T(\dot{q}) - V(q) = \frac{L}{2}\dot{q}^2 - \frac{1}{2C}q^2.$$

- Euler-lagrange eq.: $\ddot{q} = -\Omega^2 q \quad \Big| \; \Omega = \frac{1}{\sqrt{LC}}$
- Conjugate momentum: $\Phi = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} = LI$

Hamiltonian

- Hamiltonian:

$$H = \Phi \dot{q} - \mathcal{L} = \frac{\Phi^2}{2L} + \frac{q^2}{2C}$$

- Classical hamiltonian:

$$H_{\mathsf{Spring}} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \Rightarrow m = L,\ \omega = \Omega$$

- Equations of motion:

$$\dot{q} = \frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I \qquad \dot{\Phi} = -\frac{\partial H}{\partial q} = -\frac{q}{C} = V$$

Canonical Quantization

1. Commutator

$$\left[\hat{\Phi}, \hat{q}\right] = -i\hbar$$

2. Ladder operator

$$\hat{a} = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} + \frac{i}{\sqrt{2L\hbar\Omega}}\hat{\Phi} \Rightarrow \hat{a}^{\dagger} = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} - \frac{i}{\sqrt{2L\hbar\Omega}}\hat{\Phi}$$

3. Hamiltonian

$$H = \frac{\hbar\Omega}{2}\left(\hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a}\right) = \hbar\Omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

Intermezzo Classical Mechanics

Quantum	Classical
$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{q}^2}{2C}$	$H = \frac{\Phi^2}{2L} + \frac{q^2}{2C}$
$\hat{a} = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} + \frac{i}{\sqrt{2l\hbar\Omega}}\hat{\Phi}$	$\alpha = \frac{1}{\sqrt{2C\hbar\Omega}}q + \frac{i}{\sqrt{2l\hbar\Omega}}\Phi$
$H = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$	$H = \hbar\Omega\alpha^*\alpha$
$\left[\hat{a}, \hat{a}^\dagger\right] = 1$	$\left[\alpha, \alpha^*\right] = 0$
$\hat{a}(t) = \hat{a}(0)e^{-i\Omega t}$	$\alpha(t) = \alpha(0)e^{-i\Omega t}$
$\frac{d\langle\hat{q}\rangle}{dt} = \frac{\langle\hat{\Phi}\rangle}{L}, \frac{d\langle\hat{\Phi}\rangle}{dt} = -\frac{1}{C}\langle\hat{q}\rangle$	$\frac{dq}{dt} = \frac{\Phi}{L}, \quad \frac{d\Phi}{dt} = -\frac{1}{C}q$
$\frac{d\hat{f}}{dt} = \frac{\partial\hat{f}}{\partial t} + \frac{i}{\hbar}\left[\hat{f}, H\right]$	$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$

Exchanging Position and Momentum

Due to josephson junctions it is more convenient to work with magnetic flux as position. Using linear relationships between the flux and the charge yields

$$\varphi = \int^t d\tau V(\tau) \Rightarrow \dot{\varphi} = V \qquad CV = Q \quad \Big| \; Q = -q$$

$$E_{\mathsf{Cap}} = \frac{1}{2} C \dot{\varphi}^2 \qquad E_{\mathsf{Coil}} = \frac{1}{2L} \varphi^2$$

$$\mathcal{L} = T(\dot{q}) - V(q) = \frac{1}{2} C \dot{\varphi}^2 - \frac{1}{2L} \varphi^2$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = C \dot{\varphi}$$

Exchanging Position and Momentum

- Hamiltonian

$$H = Q\dot{\varphi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\varphi^2}{2L}$$

- Hamiltonian equations of motion

$$\dot{\varphi} = \frac{\partial H}{\partial Q} = \frac{Q}{C}$$

$$\dot{Q} = -\frac{\partial H}{\partial \varphi} = -\frac{\varphi}{L}$$

Exchanging Position and Momentum

- Commutator

$$\left[\hat{Q}, \hat{\varphi}\right] = -i\hbar \quad \Big| \text{Notice } \hat{Q} = -\hat{q},\ \hat{\Phi} = \hat{\varphi}$$

- Ladder operator

$$\hat{a} = \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi} + \frac{i}{\sqrt{2C\hbar\Omega}}\hat{Q} \Rightarrow \hat{a}^\dagger = \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi} - \frac{i}{\sqrt{2C\hbar\Omega}}\hat{Q}$$

- Inverse

$$\hat{\varphi} = \sqrt{\frac{L\hbar\Omega}{2}}\left(\hat{a} + \hat{a}^\dagger\right) = \Phi_{ZPF}\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\hat{Q} = -i\sqrt{\frac{C\hbar\Omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) = -iQ_{ZPF}\left(\hat{a} - \hat{a}^\dagger\right)$$

Interpretation of the New Quanta

Interpretation of excitation's as photons regardless of the spatial seperation. Let's get some intuiton for the circuit. Starting with the fluctuations

$$Q_{ZPF} = \sqrt{\frac{C\hbar\Omega}{2}} = \sqrt{\frac{\hbar}{2Z}} \qquad \Phi_{ZPF} = \sqrt{\frac{L\hbar\Omega}{2}} = \sqrt{\frac{\hbar Z}{2}} \quad \Big| \; Z = \sqrt{\frac{L}{C}}$$

$$\Delta Q = \sqrt{\langle 0|\hat{Q}^2|0\rangle - \langle 0|\hat{Q}|0\rangle^2} = Q_{ZPF}$$

$$\Delta \varphi = \sqrt{\langle 0|\hat{\varphi}^2|0\rangle - \langle 0|\hat{\varphi}|0\rangle^2} = \Phi_{ZPF}$$

$$Q_{ZPF}\Phi_{ZPF} = \frac{\hbar}{2}$$

Interpretation of the New Quanta

Let's now look at an absolute size of these fluctuations

$$R_Q = \frac{h}{(2e)^2} \approx 6,453.20\text{Ohms}$$

defining

$$z = \frac{Z}{R_Q}$$

to obtain

$$Q_{ZPF} = (2e)\sqrt{\frac{1}{4\pi z}} \qquad \Phi_{ZPF} = \frac{h}{2e}\frac{z}{4\pi} = \Phi_0 \frac{z}{4\pi}.$$

Interpretation of the New Quanta

$$\Phi_0 = 2.06783367 \frac{\mu V}{GHz}$$

The Voltage Operator is

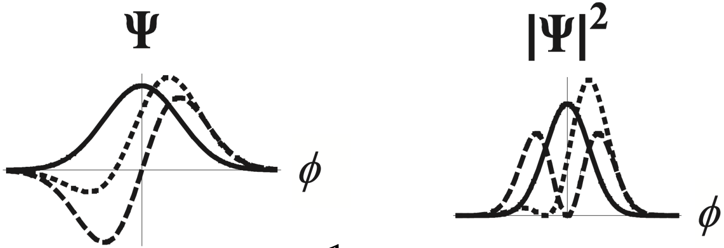
$$\hat{V} = \frac{1}{C}\hat{Q} = -i\sqrt{\frac{h\Omega}{2C}}\left(\hat{a} - \hat{a}^\dagger\right) = -iV_{ZPF}\left(\hat{a} - \hat{a}^\dagger\right)$$

so that

$$V_{ZPF} = \Omega\Phi_{\text{ZPF}} = \Omega\Phi_0\sqrt{\frac{z}{4\pi}}$$

Oscillator driven at 10 GHz and impedance $Z = 100$ Ohm fluctuations $\approx 1/3\mu V$ and correspondingly current fluctuations of $3nA$.

Interpretation of the New Quanta



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|\Psi\rangle_{QBit} = \left(\alpha\,|0\rangle + \beta\,|1\rangle\right) \otimes |0\rangle_{Field} \xrightarrow{\text{Decay}}$$

$$|\Psi\rangle_{FlyingQBit} = |0\rangle_{Atom} \otimes \left(\alpha\,|0\rangle + \beta\,|1\rangle\right)$$