Circuit QED: The LC Circuit

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1 Finding the Relevant Variables

We want to describe a superconducting circuit in a quantized form. To make our thoughts concrete, we start with a simple electrical circuit, the LC circuit. To start the discussion, we consider which variables are useful and needed to describe the state of such a circuit. One may think, that since quantum mechanics is a microscopic theory one has to use multi-body tools or density operators, but it turns out that only a few degrees of freedom can be used to describe the system very precisely. The following discussion follows strictly [1]. What kind of states or excitations may exist in such a system? Probably:

- Single Particle excitations
- Bulk Plasmons.

One of the essential features of (ordinary) superconductivity is, that electrons of opposite spin pair up (form a bound state) and a substantial excitation gap is created to break such pairs up. The energies with which our LC circuit is operating are smaller than that, so that no single electron excitations can occur.

When we have our material, the electrons act approximately as a gas which can have density fluctuations (, e.g. due to a driving Field). An example is illustrated in Figure 1. When such a density fluctuation occurs we can assign a mean velocity to the electron gas which results in a current

$$\vec{j} = -en\vec{v}. \tag{1}$$

The mean electron velocity obeys Newton laws so that

$$\dot{\vec{v}} = -\frac{e}{m}\vec{E}. \tag{2}$$

Taking the divergence and using

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{j} = -\dot{\rho}$$

in (1) and (2) yields

$$\ddot{\rho} = -\frac{e^2 n}{m\epsilon_0}\rho = -\omega_p^2 \rho. \tag{3}$$

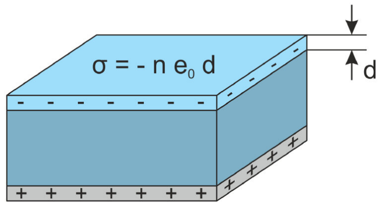

Figure 1: Illustration of a volume plasmon. The electrons in the metal are collectively driven at the plasma frequency ω_p. They are displaced by the distance d relative to positively charged ions. This results in a surface charge density $\sigma = ned$.[2]

Where we defined the Plasma frequency ω_p. The here presented model is the so-called jellium model. It gives rise to a dispersion relation independent of the wave vector. A more detailed approach, can be found in [3] which yields a more suitable dispersion relation given in Figure 2. The important result is that no Volume (Bulk) Plasmons can be excited by frequencies lower than ω_p due to the long-range coulomb force. For materials like Aluminium the plasma frequency lays at $\omega_p \approx 3.6 \cdot 10^{15} Hz \Leftrightarrow E \approx 15\text{eV}$. For comparison, the Energy of visible Photons lies around 1.6eV $\sim$ 3.3eV and our LC-circuit operates at frequencies below that. This also explains the reflectiveness of such metals at visible frequencies. Essentially the Electrons are so dense and so agile that they screen out any electrical fields almost perfectly over short distances. One can find this result directly by solving Maxwell's equations approximately in the metal in the low-frequency limit. Equation (1) and (2) lead to the first London equation

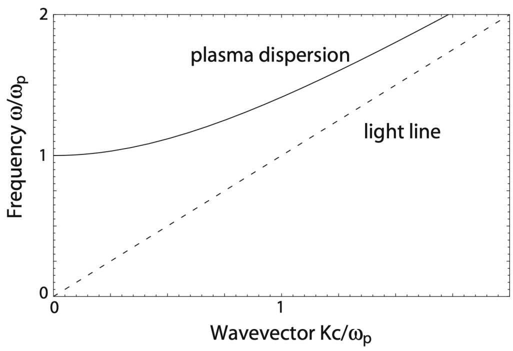

Figure 2: The dispersion relation of the free electron gas in a metal. Electromagnetic wave propagation is only allowed for $\omega > \omega_p$.[3]

$$\dot{\vec{j}} = \frac{e^2 n}{m}\vec{E} = \frac{1}{\mu_0 \lambda_L^2}\vec{E} \quad \Big| \lambda_L = \sqrt{\frac{m}{\mu_0 n e^2}} = \frac{c}{\omega_p}. \tag{4}$$

Equipped with this equation we start with the Curl Equation of the electric field

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

taking the curl again and using the curl equation of the magnetic field without the time derivative of the displacement current, since we are operating at low frequencies yields

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) \tag{5}$$

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{j}}{\partial t} \quad \Big| \text{using (4)} \tag{6}$$

$$\nabla \times \nabla \times \vec{E} = -\frac{1}{\lambda_L^2}\vec{E} \quad \Big| \nabla \times \nabla \times * = \nabla(\nabla \cdot *) - \nabla^2 * \tag{7}$$

$$\nabla \frac{\rho}{\epsilon_0} - \nabla^2 \vec{E} = -\frac{1}{\lambda_L^2}\vec{E} \quad \Big| \text{Approximating } \left|\nabla \frac{\rho}{\epsilon_0}\right| \ll 1 \tag{8}$$

$$\nabla^2 \vec{E} = \frac{1}{\lambda_L^2}\vec{E} \Rightarrow \vec{E} = \vec{E}_0 \exp\left\{\pm \frac{x}{\lambda_L}\right\}. \tag{9}$$

A divergent field is unphysical so an exponentially decaying field arises in this scenario. For Aluminium $\lambda_L \approx 14nm$. This is neglectable for our purposes. To conclude the energy to create a volume plasmon is too high and the degrees of freedom are forced into the ground state.

As we have seen the jellium model breaks down for large wave vectors. Conversely, for small wave vectors, there is a cutoff associated with the finite size of the sample. Capacitance between the finite size of Lumps of materials in the circuit can arise. The capacity matrix is then modified such that there exist collective charge oscillation modes in the microwave range, e. g. our capacitor in the LC circuit.

Applying this to our LC circuit: No degrees of freedom for single particle fluctuations due to superconductivity and no density fluctuations due to the Coulomb force except in the modelled capacitor. All left is a rigid collective motion of incompressible electron fluid sloshing back and forth. Therefore the only 4 variables in our circuit are:

- Voltage
- Current
- Charge on the capacitor plates
- Magnetic Flux trrough the Inductor

2 Quantization of the LC Circuit

Let's construct the lagrangian. Then go to the hamiltonian formulation of the classical problem and canonical quantize our system. How do we formulate our lagrangian? We have no position or impulse in the system. So we are going to abstract a bit and search for terms concerning the energy of the coil and the capacitor where in one term a variable appears quadratic and in the other term the same variable appears quadratic in its first-time derivative.

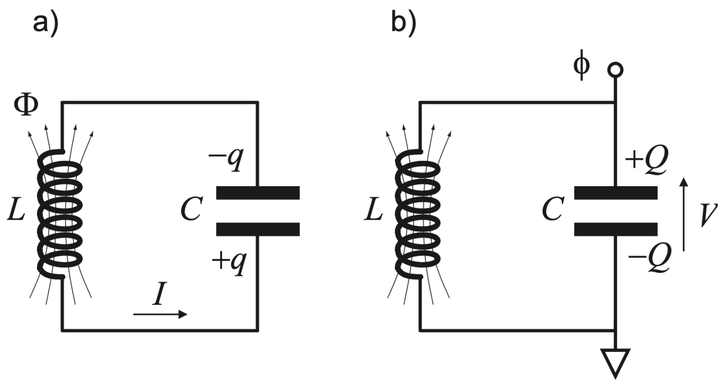

Figure 3: Simple LC electrical oscillator analogous to a mass and spring mechanical oscillator. In panel a) the position coordinate of the mass is taken to be q, the charge accumulated on the capacitor by the current I flowing through the inductor, and the flux Φ through the inductor is the momentum. b) the position coordinate is now taken to be φ, the time integral of the voltage V across the capacitor (i.e., the node flux) and the conjugate momentum is Q, the charge on the capacitor.

The energy of a coil and a capacitor can be written as

$$E_{\text{Coil}} = \frac{1}{2}LI^2 \qquad E_{\text{Cap}} = \frac{1}{2C}q^2. \tag{10}$$

Using the convention $\dot{q} = I$ the Lagrangian can be written by

$$\mathcal{L} = T(\dot{q}) - V(q) = \frac{L}{2}\dot{q}^2 - \frac{1}{2C}q^2. \tag{11}$$

The Euler-Lagrange equations yield

$$\ddot{q} = -\Omega^2 q \quad \Big| \, \Omega = \frac{1}{\sqrt{LC}}, \tag{12}$$

so that they correctly predict the charge oscillations. Going to hamiltonian formulism via the legendre transformation, the conjugate momentum is

$$\Phi = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} = LI \tag{13}$$

the magnetic flux through the coil. When performing the quantization one should remember that we have chosen q as the position coordinate and Φ as the momentum. Continuing

$$H = \Phi\dot{q} - \mathcal{L} = \frac{\Phi^2}{2L} + \frac{q^2}{2C} \tag{14}$$

we arrive at the hamiltonian. Hamiltonian equations of motion reproduces some known laws regarding coils and capacitors:

$$\dot{q} = \frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I \qquad \dot{\Phi} = -\frac{\partial H}{\partial q} = -\frac{q}{C} = V \tag{15}$$

Comparing this to the classical harmonic oszillator

$$H_{\text{Spring}} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{16}$$

one recognizes that $m = L$ and $\omega = \Omega$. We now lift Φ and q to operators $\hat{\Phi}$ and $\hat{q}$ which obey the canonical kommutation relation

$$\left[\hat{\Phi}, \hat{q}\right] = -i\hbar. \tag{17}$$

We now can define a complex operator

$$\hat{a} = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} + \frac{i}{\sqrt{2L\hbar\Omega}}\hat{\Phi} \Rightarrow \hat{a}^\dagger = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} - \frac{i}{\sqrt{2L\hbar\Omega}}\hat{\Phi} \tag{18}$$

with the inversion by taking a scale of the real or imaginary part

$$\hat{q} = \sqrt{\frac{C\hbar\Omega}{2}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{\Phi} = -i\sqrt{\frac{L\hbar\Omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \tag{19}$$

and the commutator

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{20}$$

The Hamiltonian then reads

$$H = \frac{\hbar\Omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{21}$$

2.1 Intermezzo Classical Mechanics

In the following we want to point out the similarities and differences between the classical and quantum description of the harmonic oscillator:

Some important aspects of the comparison in table 1 between the classical and the quantum world is that a zero point energy plops out due to the different commutation relations and that the analogous to the commutator is a rescaled poisson bracket

$$\{*, *\} \to \frac{i}{\hbar}[*, *] \tag{22}$$

Quantum	Classical
$\hat{H}=\frac{\hat{\Phi}^2}{2L}+\frac{\hat{q}^2}{2C}$	$H=\frac{\Phi^2}{2L}+\frac{q^2}{2C}$
$\hat{a}=\frac{1}{\sqrt{2C\hbar\Omega}}\hat{q}+\frac{i}{\sqrt{2l\hbar\Omega}}\hat{\Phi}$	$\alpha=\frac{1}{\sqrt{2C\hbar\Omega}}q+\frac{i}{\sqrt{2l\hbar\Omega}}\Phi$
$H=\hbar\Omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)$	$H=\hbar\Omega\alpha^*\alpha$
$\left[\hat{a},\hat{a}^\dagger\right]=1$	$[\alpha,\alpha^*]=0$
$\hat{a}(t)=\hat{a}(0)e^{-i\Omega t}$	$\alpha(t)=\alpha(0)e^{-i\Omega t}$
$\frac{d\langle\hat{q}\rangle}{dt}=\frac{\langle\hat{\Phi}\rangle}{L},\quad\frac{d\langle\hat{\Phi}\rangle}{dt}=-\frac{1}{C}\langle\hat{q}\rangle$	$\frac{dq}{dt}=\frac{\Phi}{L},\quad\frac{d\Phi}{dt}=-\frac{1}{C}q$
$\frac{d\hat{f}}{dt}=\frac{\partial\hat{f}}{\partial t}+\frac{i}{\hbar}[f,H]$	$\frac{df}{dt}=\frac{\partial f}{\partial t}+\{f,H\}$

Table 1: Comparison between quantum and classical descriptions of harmonic oscillator.

2.2 Exchanging Position and Momentum

At first sight, this quantization seems very natural from our intuitive view of the capacitance storing the potential energy and the inductor storing the kinetic energy. When working with Josephson junctions it will be more convenient to work with the magnetic flux as the position coordinate. To achieve that we use linear relations (15) which connect the magnetic flux and the charge

$$\varphi=\int^t d\tau V(\tau)\Rightarrow\dot{\varphi}=V\qquad CV=Q\;\Big|\;Q=-q. \tag{23}$$

The so introduced convention of the charge sign and the new label for the magnetic flux will become convenient as soon as we want to quantize the System (notice $\varphi=\Phi$). See Figure 3 for the setup. So that we can rewrite the potential energy of a capacitor and also use eq. (13) to do so for the potential energy of the coil

$$E_{\text{Cap}}=\frac{1}{2}C\dot{\varphi}^2\qquad E_{\text{Coil}}=\frac{1}{2L}\varphi^2. \tag{24}$$

We now reversed the dependency of the dotted variables and the undotted ones, so that we can write the lagrangian

$$\mathcal{L}=T(\dot{q})-V(q)=\frac{1}{2}C\dot{\varphi}^2-\frac{1}{2L}\varphi^2. \tag{25}$$

The conjugate momemtun now reads

$$Q=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}=C\dot{\varphi}. \tag{26}$$

We reversed the roles of charge and magnetic flux. The magnetic flux is now the position and the charge in the conjugate momentum. The hamiltonian and hamilton equations of motion are then

$$H=Q\dot{\varphi}-\mathcal{L}=\frac{Q^2}{2C}+\frac{\varphi^2}{2L} \tag{27}$$

$$\dot{\varphi}=\frac{\partial H}{\partial Q}=\frac{Q}{C} \tag{28}$$

$$\dot{Q}=-\frac{\partial H}{\partial\varphi}=-\frac{\varphi}{L} \tag{29}$$

Which is äquivalent to the previous hamilton equation of motion. Let's impose the canonical commutation relations

$$\left[\hat{Q}, \hat{\varphi}\right] = -i\hbar \tag{30}$$

Notice that since we have chosen our sign convention for the charge differently this relation doesn't conflict with the commutation relation (17) and $\hat{\Phi} = \hat{\varphi}$ and $\hat{Q} = -\hat{q}$. We now proceed with the usual procedure. We introduce a new operator

$$\hat{a} = \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi} + \frac{i}{\sqrt{2C\hbar\Omega}}\hat{Q} \Rightarrow \hat{a}^\dagger = \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi} - \frac{i}{\sqrt{2C\hbar\Omega}}\hat{Q} \tag{31}$$

with the commutator

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \tag{32}$$

and the Inverse

$$\hat{\varphi} = \sqrt{\frac{L\hbar\Omega}{2}}\left(\hat{a} + \hat{a}^\dagger\right) = \Phi_{ZPF}\left(\hat{a} + \hat{a}^\dagger\right) \tag{33}$$

$$\hat{Q} = -i\sqrt{\frac{C\hbar\Omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) = -iQ_{ZPF}\left(\hat{a} - \hat{a}^\dagger\right). \tag{34}$$

The Hamiltonian then reads

$$H = \frac{\hbar\Omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{35}$$

In the following section, we will work with this interpretation of position and momentum.

2.3 Interpretation of the New Quanta?

By the excitation and the superposition of the newly introduced states, one creates a collective electron motion through the wire and generates an electric field between the plates of the capacitor and a magnetic field inside the coil. Even though there is a spatial separation of the two fields one wants to think of its excitations as photons.

First, let's look at the fluctuations that can arise in the circuit. We can rewrite the introduced constants in eqs. (33) and (34) with the impedance

$$Q_{ZPF} = \sqrt{\frac{C\hbar\Omega}{2}} = \sqrt{\frac{\hbar}{2Z}} \qquad \Phi_{ZPF} = \sqrt{\frac{L\hbar\Omega}{2}} = \sqrt{\frac{\hbar Z}{2}} \quad \Big| \, Z = \sqrt{\frac{L}{C}} \tag{36}$$

and the variance regarding the charge and flux operator is

$$\Delta Q = \sqrt{\langle 0|\hat{Q}^2|0\rangle - \langle 0|\hat{Q}|0\rangle^2} = Q_{ZPF} \tag{37}$$

$$\Delta\varphi = \sqrt{\langle 0|\hat{\varphi}^2|0\rangle - \langle 0|\hat{\varphi}|0\rangle^2} = \Phi_{ZPF}. \tag{38}$$

Therefore the heisenberg uncertanty relation for the groundstate is fullfilled with

$$Q_{ZPF}\Phi_{ZPF} = \frac{\hbar}{2} \tag{39}$$

We now want to get a feeling for the absolute size of these fluctuations. Therefore we introduce the superconducting resistance quantum

$$R_Q = \frac{h}{(2e)^2} \approx 6,453.20\text{Ohms} \tag{40}$$

defining

$$z = \frac{Z}{R_Q} \tag{41}$$

to obtain

$$Q_{ZPF} = (2e)\sqrt{\frac{1}{4\pi z}} \qquad \Phi_{ZPF} = \frac{h}{2e}\frac{z}{4\pi} = \Phi_0\frac{z}{4\pi}. \tag{42}$$

Where the superconducting flux current is give by

$$\Phi_0 = 2.06783367\frac{\mu V}{GHz} \tag{43}$$

Eqquiped with that we now look at the voltage and current fluctuations. The Voltage Operator is

$$\hat{V} = \frac{1}{C}\hat{Q} = -i\sqrt{\frac{h\Omega}{2C}}\left(\hat{a} - \hat{a}^\dagger\right) = -iV_{ZPF}\left(\hat{a} - \hat{a}^\dagger\right) \tag{44}$$

so that

$$V_{ZPF} = \Omega\Phi_{\text{ZPF}} = \Omega\Phi_0\sqrt{\frac{z}{4\pi}} \tag{45}$$

This tells us that for example an oscillator driven at 10 GHz and an impedance $Z = 100$ Ohm has fluctuations at the scale of $1/3\mu V$ and correspondingly current fluctuations of $3nA$. Since the hamiltonian was equal to the one of a harmonic oscillator we inherit all characteristics of it. So to say to create a state with a non-zero expectation value for the magnetic flux one has to create a superposition of states

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{46}$$

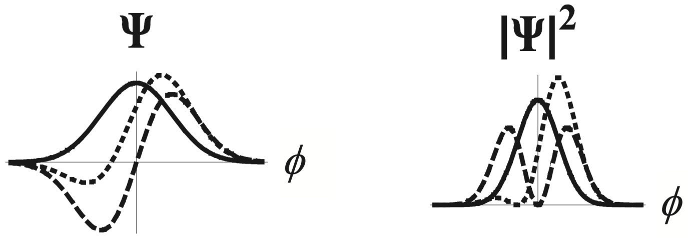

Figure 4: LC oscillator wave function amplitude (left panel) and probability density (right panel) plotted vs. the coordinate φ. Solid: ground state, $|0\rangle$; Long-Dashed: first excited state, $|1\rangle$; Short-dashed: linear combination of the ground and first excited states, $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

Such one and two-number photon states cannot be created by simply driving the LC circuit, rather than some laser excitation shenanigans. This procedure essentially creates a qubit. Considering the full Hilbert space with the surrounding quantized electromagnetic field the state decays into the ground state and creates a so-called flying qubit which is an essential step towards communicating quantum information via photons.

$$|\Psi\rangle_{QBit} = \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes |0\rangle_{Field} \xrightarrow{\text{Decay}} |\Psi\rangle_{FlyingQBit} = |0\rangle_{Atom} \otimes \left(\alpha|0\rangle + \beta|1\rangle\right) \tag{47}$$

Note that phase coefficients of the field photon are inherited from the qubit. Experiments regarding the expectation value of the operators $\hat{Q}$ and $\hat{\varphi}$ and the flying qubits have been made in the experiment of Houck et al. [4] and are in agreement with the presented theory here.

References

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