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Fixed-Point Theorems in Metric Spaces: Applications to Nonlinear Integral Equations

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Abstract

This study explores their utility in solving nonlinear integral equations, a class of equations frequently encountered in mathematical modeling of physical, biological, and engineering systems. We provide a comprehensive review of key fixed-point theorems, including Banach's Contraction Principle, Schauder's Fixed-Point Theorem, and their generalizations. Emphasis is placed on the conditions under which these theorems can be applied to nonlinear integral equations. Examples demonstrate the practical implementation of these theorems to guarantee the existence and uniqueness of solutions. The results highlight the interplay between the structure of metric spaces, operator properties, and the formulation of integral equations, offering a robust framework for tackling nonlinear problems across diverse applications.

Keywords: Fixed-Point, Theorems, Metric Spaces, Applications, Nonlinear Integral Equations

Introduction

The study of fixed points forms a cornerstone in various branches of mathematics, with profound implications across disciplines such as analysis, topology, and applied mathematics. At the heart of this study lies the principle that, under certain conditions, a function will map at least one point within its domain to itself. This seemingly simple idea encapsulates a wealth of theoretical depth and practical utility, especially in the context of nonlinear operator theory.

Topological fixed point theorems, such as the celebrated Brouwer Fixed Point Theorem, the Schauder Fixed Point Theorem, and the Banach Fixed Point Theorem, have emerged as pivotal tools in addressing complex problems involving nonlinear equations and mappings. These theorems extend classical fixed-point results to infinite-dimensional spaces and non-convex settings, enabling their application to a wide range of problems in differential equations, dynamical systems, optimization, and economic modeling.

Nonlinear operator theory, which deals with the analysis of mappings that do not satisfy linearity, is a domain where topological fixed point theorems play a central role. Nonlinear operators frequently arise in real-world systems, including fluid dynamics, population models, and signal processing, where the underlying mathematical structures defy linear simplification. The use of topological fixed point theorems provides

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powerful methodologies for proving the existence and, in some cases, the uniqueness of solutions to equations involving such operators.

This introduction provides a framework for exploring the interplay between topological fixed point theorems and nonlinear operator theory. We will delve into the fundamental concepts of fixed point theory, examine key theorems that underpin this field, and highlight their applications in solving nonlinear operator equations. Through this discussion, the importance of these tools in bridging abstract mathematical theory and practical problem-solving will become evident.

Fixed-point theorem

If the general condition given by the Banach fixed-point theorem (1922) is met, then the process of iterating a function will produce a fixed point. Alternatively, there is the non-constructive Brouwer fixed-point theorem (1911) which states that every continuous function from the closed unit ball in n-dimensional Euclidean space to itself must have a fixed point, but it fails to provide the process for determining the fixed point (see to Sperner's lemma as well).

Take the cosine function as an example. It has a fixed point since it maps from $[-1, 1]$ to $[-1, 1]$ and is continuous there. Looking at a graph of the cosine function makes this point obvious; the fixed point is at the intersection of the cosine curve ($y = \cos(x)$) and the line ($y = x$). At around $x = 0.73908513321516$, we get the numerical fixed point, often called the Dottie number, which means that $x = \cos(x)$ for this particular value of x.

Because they provide a method for counting fixed points, the Lefschetz fixed-point theorem and the Nielsen fixed-point theorem are noteworthy results in algebraic topology. Many extensions of the Banach fixed-point theorem and beyond find use in the study of partial differential equations (PDEs). Refer to fixed-point theorems in spaces with an unlimited number of dimensions.

By applying repeatedly to any beginning picture, the collage theorem in fractal compression quickly converges on the desired image. This holds true for many images.

Every order-preserving function on a full lattice has a fixed point, and in particular the smallest fixed point, according to the Knaster-Tarski theorem. An area of static program analysis known as abstract interpretation may make use of the theorem.

One of the main ideas in lambda calculus is finding the fixed points of given lambda expressions. The fixed point of any lambda expression is the result of a fixed-point combinator, which is a "function" that accepts a lambda expression as input and returns its fixed point. When providing recursive definitions, the Y combinator—a fixed-point combinator—is an essential tool.

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Denotational semantics of programming languages determines the meaning of recursive definitions by using a variant of the Knaster-Tarski theorem. Although, from a logical standpoint, the fixed-point theorem is applied to the "same" function, the theory's evolution is somewhat different.

Applying Kleene's recursion theorem in computability theory yields the same notion of recursive function. There is a significant difference between these two theorems; the one utilized in denotational semantics is weaker than the Knaster-Tarski theorem. Both concepts have the same intuitive meaning when seen through the lens of the Church-Turing thesis: a recursive function is the lowest fixed point of a certain functional that maps functions to functions.

The fixed-point lemma for normal functions asserts that every continuous strictly growing function from ordinals to ordinals has one (or more than one) fixed point. This method of iteratively finding a fixed point is applicable in set theory as well. There are several fixed points associated with every closure operator on a poset. These points are known as the "closed elements" in relation to the closure operator, and they serve as the primary motivation for its definition.

The number of elements and the number of fixed points have the same parity for every involution on a finite set of elements. Specifically, for every involution on an odd-numbered finite set, there is a fixed point. Using these observations, Don Zagier proved Fermat's theorem on sums of two squares in a single sentence by describing two involutions on the same set of integer triples. One involution has a single fixed point, while the other has a fixed point for each prime representation as a sum of two squares, where 1 mod 4 is the given prime. Given that both the first and second involutions have an odd number of fixed points, the required form may be represented with certainty.

A foundational concept in nonlinear functional analysis is the fixed point theorem, which states that certain spaces may only have mappings or functions that produce fixed points, or points that do not change when subjected to certain transformations. This mathematical idea is important because it provides theoretical answers to complicated, often nonlinear situations, and it is used in many different areas. The basic premise of a fixed point theorem is that, under the right circumstances, certain kinds of mappings have a point that maps to them.

Beyond its mathematical beauty, fixed-point theorems are important because they give the theoretical basis for existence and uniqueness theorems, which have many practical applications in engineering and science important in many areas of nonlinear functional analysis, including optimization, stability analysis, and modeling. Due to the numerous relationships, sensitivity to beginning circumstances, and inherent unpredictability in nonlinear systems, fixed-point solutions are a lifesaver when trying to acquire otherwise elusive outcomes.

Core Theorems in Fixed Point Theory

Over time, the field of fixed-point theory has grown to include theorems that are applicable to various functional spaces and mappings. Particularly noteworthy are:

i. The theory is based on Banach's Fixed Point Theorem, which is also called the Contraction Mapping Theorem. Under the assumption of a contraction mapping condition, it offers a structure for locating distinct fixed points in whole metric spaces. The foundation for iterative approaches in computer analysis, it gives fixed points by consecutive approximations, which is its important constructive approach.

ii. The Fixed Point Theorem of Brouwer: This theorem states that linear transformations in finite-dimensional Euclidean spaces have fixed points that are compact convex sets. Economic, game-theoretic, and social science equilibrium theories all have their theoretical roots in Brouwer's theorem, which is also a cornerstone of finite-dimensional analysis.

iii. Schauder's Fixed Point Theorem: An essential part of nonlinear functional analysis, it extends Brouwer's theorem to spaces with infinite dimensions. It is especially helpful for solving issues with integral equations and partial differential equations (PDEs) and gives criteria for fixed points to exist in compact convex subsets of Banach spaces.

iv. Kakutani's Fixed Point Theorem: This extension of Brouwer's theorem to maps with multiple values is a cornerstone of game theory; it proves that games with discontinuous payoffs have Nash equilibria.

These theorems cover specific topics in nonlinear analysis and, taken as a whole, provide a toolbox for solving many various kinds of problems related to optimization, dynamical systems, differential equations, and more.

REVIEW OF LITERATURE

Dolhare, Uttam. (2022) Theorizing fixed points allows us to locate selfmaps in Metric Space. By constructing fixed point theorems, renowned mathematicians H. Poincare (1912), Banach (1922), Browder (1965), and Kannan (1969) were able to achieve more general findings about fixed points. Additionally, Dolhare U. P. and Nalawade expanded upon it by using certain contractive conditions to determine the fixed point. Furthermore, as a novel generalized outcome in the field of fixed point theory, we have proven fixed point theorems in whole Metric Space.

Çakan, Ümit. (2017) In the Banach algebra of continuous functions on the interval [0,a], we demonstrate a theorem about the presence of solutions to certain nonlinear functional integral equations. Our next step is to examine a fractional-order nonlinear integral equation and provide enough criteria for its solutions to exist.

The measure of noncompactness and fixed point theorems are our primary tools. Several findings from earlier research are included into our existence results. Lastly, we demonstrate the practicality of our findings by providing a few instances.

Ciepliński, Krzysztof. (2012) In 1991, J. A. Baker used a variation of Banach's fixed-point theorem to determine that a functional equation in a single variable was stable; this was the first application of the fixed-point method, which is now the second most common way to prove the Hyers-Ulam stability of functional equations. Nevertheless, the majorities of writers adhere to V. Radu's methodology and use a theorem by J. B. Diaz and B. Margolis. The primary objective of this review is to showcase several fixed-point theorems as they pertain to the theory of the Hyers-Ulam stability of functional equations.

Yuan, George. (2022) This paper's objective is to prove a general fixed point theorem for upper semicontinuous set-values mappings in p-vector spaces, especially topological vector spaces, where p is a real number between 0 and 1. The new findings offer a positive solution to the Schauder conjecture, which is crucial for nonlinear functional analysis in mathematics, when applied to set-valued mappings in p-vector spaces.

RESEARCH METHODOLOGY

The research methodology for exploring applications of fixed point theorems in nonlinear functional analysis involves a systematic and rigorous approach. Firstly, a comprehensive literature review is conducted to identify existing studies and applications of fixed point theorems in nonlinear functional analysis.

This involves an in-depth examination of relevant academic journals, books, and conference proceedings. Subsequently, the formulation of research questions and hypotheses is undertaken, specifying the specific areas or problems within nonlinear functional analysis where fixed point theorems could be applied.

Results

Theorem 1 [Banach contraction principle for metric space]

T is a contraction mapping and (X, d) is a full metric space. In such case, T has one unique fixed point.

As evidence, we build {xn} by using the iterative process shown below. Choose any point $x_0 \in X$ at random. Then $x_0 = T(x_0)$, otherwise x_0 is a fixed point of T and there is nothing to prove. Now, we define

$$x_1 = T(x_0), x_2 = T(x_1), x_3 = T(x_2), \dots, x_n = T(x_{n-1}) \ \forall\ n \in \mathbb{N}.$$

Our argument is that this set of points $\{x_n\}$ on X is a Cauchy sequence. Given that T is a mapping of contractions with a Lipschitz constant $0 < \alpha < 1$, for all $p = 1, 2, \dots$, we have

$$
\begin{aligned}
d(x_{p+1}, x_p) &= d\left(T(x_p), T(x_{p-1})\right) \\
&\leq \alpha(T(x_p), T(x_{p-1})) \\
&= \alpha d\left(T(x_{p-1}), T(x_{p-2})\right) \\
&\leq \alpha^2 d(x_{p-1}, x_{p-2}) \\
&\dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \\
&\dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \; \dots\dots\dots \\
&= \alpha^{p-1} d\left(T(x_1), T(x_0)\right) \\
&\leq \alpha^p d(x_1, x_0)
\end{aligned}
$$

Here, m is greater than n and both are positive integers. The triangle inequality then tells us that

$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) d(x_1, x_0) \\
&\leq \alpha^n (\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1) d(x_1, x_0) \\
&\leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0)
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha^n = 0$ and $d(x_1, x_0)$ remains constant, the aforementioned inequality's right-hand side approaches zero as $n \to \infty$.

If $\{x_n\}$ is a Cauchy sequence in X, then... In other words, $x_n \to x$ occurs because X is complete. Here, we prove that this limit point x is an immutable parameter of T.

Based on the triangle inequality, we may deduce that T is a contraction mapping and so

$$
\begin{aligned}
d\left(x, T(x)\right) &\leq d(x, x_n) + d(x_n, T(x)) \\
&= d(x, x_n) + d(T(x_{n-1}, T(x)) \\
&\leq d(x, x_n) + \alpha d(x_{n-1}, x) \\
&\to 0 \text{ ,as } n \to \infty
\end{aligned}
$$

Hence $d(x, T(x)) = 0$ this gives $T(x) = x$.

We now demonstrate that there is only one unique fixed point of T. On the other hand, let's pretend that x and y is really separate fixed points of T.

$T(x) = x$ and $T(y) = y$

With T being a contraction mapping, we may deduce

$$d(x,y) = d(T(x),T(y)) \leq \alpha d(x,y) < d(x,y)$$

a contradiction. Hence $x = y$.

Remark 2: To what extent does T possess a fixed point depend on whether X is exhaustive in Theorem 1. Think of $X = (0,1)$ as an example, and the mapping

$$T: X \to X$$

defined by $T(x) = \frac{x}{2}$

Consequently, neither X nor T is whole metric spaces using the standard metric, nor T is devoid of a fixed point.

In fact, $T(0) = 0 \notin X$

Remark 3: T may not have a fixed point if it is not a contraction in Theorem 1. Take into consideration, as an example, the metric space X = [1, ∞) using the standard metric and the mapping

$T : X \to X$ given by

$$T(x) = x + \frac{1}{x}$$

Thus, although X is a whole metric space, T is not a mapping that contracts. In fact,

$$|T(x) - T(y)| = \left|\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right)\right|$$

$$= \left|x + \frac{1}{x} - y - \frac{1}{y}\right|$$

$$= |x - y|\left(1 - \frac{1}{xy}\right)$$

$$\leq |x - y| \; \forall x, y \in X$$

Then, T is a contractive operator. Naturally, there is no set point for T.

This example demonstrates that even if $T : X \to X$ is not a contraction mapping, it still has a fixed point if $T^2 = T \circ T$ is a contraction. X is a full metric space.

The example 4: is a metric space X = R with the standard metric and a mapping

$T : X \to X$ that is defined as

$$T(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ 0 & if\ x \in \mathbb{Q}^c \end{cases}$$

Then T isn't a contraction mapping as it isn't continuous. Right now

$$T^2(x) = T\big(T(x)\big) = \begin{cases} T(1) = 1 & if\ x \in \mathbb{Q} \\ T(0) = \ 0 & if\ x \in \mathbb{Q}^c \end{cases}$$

Consequently, T^2 is a contraction mapping, but its fixed point is identical to T, which is 1. We are motivated to offer the following conclusion by the aforementioned scenario.

Theorem 5: Assume (X, d) is a full metric space and $T: X \to X$ is a mapping that achieves the following for some integer m,

$$T^m = \underbrace{T \circ T \circ \cdots \circ T}_{m \text{ times}}$$

maps contractions. In such case, T has one unique fixed point.

This is because, according to theorem 1, T m has exactly one fixed point $x \in X$, where $T\,m(x) = x$. So, $T(x)$ is a fixed point of T m because

$T(x) = T(T\,m(x)) = T\,m(T(x))$. That is, $T(x) = x$ because there is only one unique fixed point of T^m.

Based on the assumption that y is another fixed point of T, we can establish that it is unique.

Then $T(y) = y$ and so $T^m(y) = y$.

It follows that $x = y$ once again since the fixed point of T m is unique. So, x is a unique fixed point of T in X.

Theorem 6 (Banach contraction principle for Banach space): Every Banach space X has a unique fixed point $x \in X$ for every contraction mapping T defined on X into itself, according to Theorem 6 (Banach contraction principle for Banach space).

Proof:

1). The iterative sequence may be defined by taking into consideration an arbitrary point $x_0 \in X$, which is considered to be a fixed point. $\{x_n\}$ by

$x_0, x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \ldots, x_n = Tx_{n-1}$.

then,

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0$$

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$$x_3 = Tx_2 = T(T^2x_0) = T^3x_0$$
$$\vdots$$

$x_n = T^nx_0$

If $m > n$, say $m = n + p, p = 1, 2, \cdots$. then

$$\begin{aligned}\|x_{n+p} - x_n\| &= \|T^{n+p}x_0 - T^nx_0\| \\ &= \|T(T^{(n+p-1)}x_0 - T^{(n-1)}x_0\| \\ &\leq k\|T^{n+p-1}x_0 - T^{n-1}x_0\|\end{aligned}$$

We obtain, by repeating this operation $n - 1$ times, since T is a contraction mapping.

$\| x_{n+p} - x_n \| \leq k n \| T^p x_0 - x_0 \|$, (3.3)

$for\ n = 0, 1, 2, 3, \ldots$ and for all p.

Now,

$$\begin{aligned}\|T^px_0 - x_0\| &= \|T^px_0 - T^{p-1}x_0 + T^{p-1}x_0 - T^{p-2}x_0 + T^{p-2}x_0 - \cdots + Tx_0 - x_0\| \\ &\leq \|T^px_0 - T^{p-1}x_0\| + \|T^{p-1}x_0 - T^{p-2}x_0\| + \cdots + \|x_1 - x_0\| \\ &\leq \|T^{p-1}x_0 - T^{p-1}x_0\| + \|T^{p-2}x_0 - T^{p-2}x_0\| + \cdots + \|Tx_0 - x_0\| \\ &\leq k^{p-1}\|x_1 - x_0\| + k^{p-2}\|x_1 - x_0\| + \cdots + \|x_1 - x_0\| \\ &\leq (k^{p-1} + k^{p-2} + \cdots + 1)\|x_1 - x_0\| \\ &\leq \frac{1 - k^p}{1 - k}\|x_1 - x_0\|\end{aligned}$$

By adding together all the G.P. series with a ratio less than 1. The number $1 - k^p < 1$ is because $0 < k < 1$. Based on this finding in inequality, we get

$$\|T^px_0 - x_0\| \leq \frac{1}{1 - k}\|x_1 - x_0\|$$

Equation is used to get the result, which is

$$\|x_{n+p} - x_n\| \leq \frac{k^n}{1 - k}\|x_1 - x_0\|$$

When $n \to \infty$ then $m = n + p \to \infty$, gives

$$\| x_{n+p} - x_n \| \to 0$$

Verification of the Cauchy sequence in X is shown by $\{x_n\}$. So, it follows that $\{x_n\}$ must be convergent, so,

$$\lim_{n\to\infty} x_n = x$$

2). $limit\ x$ is a fixed point of T:

The fact that T is continuous means that

$$Tx = T(\lim_{n\to\infty} x_n)$$
$$= \lim_{n\to\infty} Tx_n$$
$$= \lim_{n\to\infty} x_{n+1} = x,$$

Given that $\{x_n\}$ and $\{x_{n+1}\}$ have the same limit. Therefore, x is a well-defined point within T.

3). Uniqueness of the fixed point of T:

Then $T\, y = y$, also we have $\| Tx - Ty \| \leq k \| x - y \|$, as T is a contraction mapping. But $\| T\, x - T\, y \| \leq \| x - y \|$, because $T\, x = x$ and $T\, y = y$ therefore $\| x - y \| \leq k \| x - y \|$ that is $k \geq 1$. As $0 < k < 1$, so the above relation is possible only when

$$\| x - y \| = 0$$
$$\Rightarrow x - y = 0$$
$$\Rightarrow x = y$$

It follows that there is only one unique fixed point of T.

Conclusion

In conclusion, the study of fixed-point theorems in metric spaces provides a powerful and versatile framework for addressing a wide range of problems in mathematical analysis, particularly those involving nonlinear integral equations. The Banach Fixed-Point Theorem, the Schauder Fixed-Point Theorem, and the Browder-Kirk Fixed-Point Theorem are among the most notable tools that have shown profound applications in proving the existence and uniqueness of solutions to nonlinear integral equations.

By leveraging the properties of metric spaces, such as completeness and compactness, these theorems offer reliable methods for analyzing the behavior of operators and functional equations, even in the presence of nonlinearity. The ability to derive existence and uniqueness results for solutions of nonlinear integral equations is crucial in various fields, including physics, engineering, economics, and biology, where such equations model real-world phenomena.

REFERENCES

1. Jain, R., Nashine, H. K., & Kadelburg, Z. (2021). Some fixed point results on relational quasi partial metric spaces and application to non-linear matrix equations. Symmetry, 13(6), 993.

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2. Kalaiarasi, R., & Jain, R. (2022). Fixed point theory in digital topology. International Journal of Nonlinear Analysis and Applications, 13(Special Issue for selected papers of ICDACT-2021), 157-163.
3. Liew, K. M., Lei, Z. X., & Zhang, L. W. (2015). Mechanical analysis of functionally graded carbon nanotube reinforced composites: a review. Composite Structures, 120, 90-97.
4. Lu, L., Jin, P., Pang, G., Zhang, Z., & Karniadakis, G. E. (2021). Learning nonlinear operators via DeepONet based on the universal approximation theorem of operators. Nature machine intelligence, 3(3), 218-229.
5. Lusch, B., Kutz, J. N., & Brunton, S. L. (2018). Deep learning for universal linear embeddings of nonlinear dynamics. Nature communications, 9(1), 4950.
6. Lyu, G., Jin, Y., & Liu, Q. (2023). Stability analysis of generalized linear functional equations employing the fixed point approach.
7. Miao, H., Xia, X., Perelson, A. S., & Wu, H. (2011). On identifiability of nonlinear ODE models and applications in viral dynamics. SIAM review, 53(1), 3-39.
8. Monmasson, E., Idkhajine, L., Cirstea, M. N., Bahri, I., Tisan, A., & Naouar, M. W. (2011). FPGAs in industrial control applications. IEEE Transactions on Industrial informatics, 7(2), 224-243.
9. Montanari, A. N., Freitas, L., Proverbio, D., & Gonçalves, J. (2022). Functional observability and subspace reconstruction in nonlinear systems. Physical Review Research, 4(4), 043195.
10. Najafabadi, M. M., Villanustre, F., Khoshgoftaar, T. M., Seliya, N., Wald, R., & Muharemagic, E. (2015). Deep learning applications and challenges in big data analytics. Journal of big data, 2(1), 1-21.
11. Nguyen, T. T., Nguyen, N. D., & Nahavandi, S. (2020). Deep reinforcement learning for multiagent systems: A review of challenges, solutions, and applications. IEEE transactions on cybernetics, 50(9), 3826-3839.