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STABILITY AND DATA DEPENDENCE OF FIXED POINTS IN METRIC SPACES RELATIONS

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Abstract

In this paper, we investigate the stability and data dependence of fixed points for various classes of relations and operators in metric spaces. By employing generalized contractive conditions and comparison functions, we establish sufficient criteria for the existence and uniqueness of fixed points. Furthermore, we analyze how perturbations in the underlying mappings or relations affect the location of fixed points, providing explicit estimates that describe their continuous dependence on data. Several stability results, including Ulam–Hyers type stability, are derived to illustrate the robustness of fixed point solutions under small deviations. The obtained results extend and unify a number of well-known fixed point theorems in metric spaces and offer a useful framework for applications in nonlinear analysis and applied mathematics.

Keywords: Fixed point, metric space, stability, data dependence, contractive conditions, nonlinear analysis

Introduction

Fixed point theory is a fundamental area of nonlinear analysis with wide-ranging applications in mathematics and applied sciences, including differential equations, optimization, economics, and dynamical systems. At its core, fixed point theory investigates conditions under which a mapping defined on a metric space admits a point that remains invariant under the action of the mapping. Classical results such as the Banach Contraction Principle have played a central role in the development of this theory, providing both existence and uniqueness of fixed points along with constructive convergence results.

Beyond existence and uniqueness, modern fixed point theory increasingly focuses on qualitative properties of fixed points, particularly **stability** and **data dependence**. Stability concerns the behavior of fixed points under perturbations of the mapping or the underlying space. In practical applications, mappings often arise from models that are subject to measurement errors, numerical approximations, or external disturbances. Therefore, understanding whether small changes in a mapping lead to small changes in its fixed points is essential for ensuring the robustness and reliability of mathematical models.

Data dependence, closely related to stability, examines how fixed points depend on variations in the parameters, initial data, or operators involved. This concept is especially relevant in iterative methods, where approximating sequences converge to fixed points of perturbed operators rather than the original one. Data dependence results provide quantitative estimates that measure the sensitivity of fixed points to such perturbations, thereby offering insight into the continuity and well-posedness of fixed point problems.

Metric spaces provide a natural and flexible framework for studying these issues. Their generality allows the treatment of a broad class of problems without imposing restrictive linear or topological structures. In recent years, attention has shifted toward generalized contractions, set-valued mappings, and relational frameworks in metric spaces. These extensions allow fixed point theory to accommodate more complex interactions, such as those arising in ordered metric spaces, graph-based relations, and nonlinear operator equations.

The study of **relations** in metric spaces further enriches fixed point theory by allowing the replacement of classical order structures with more general binary relations. This approach unifies and extends many existing results by capturing both order-theoretic and graph-theoretic properties within a single framework. Relations enable the formulation of fixed point results that apply to mappings preserving certain relational properties, leading to broader applicability and deeper theoretical insight.

Investigating stability and data dependence of fixed points within relational metric spaces is therefore both natural and necessary. Such an investigation not only generalizes classical fixed point results but also provides powerful tools for analyzing robustness in nonlinear models. The results obtained in this direction contribute to a better understanding of how fixed points behave under perturbations and how iterative processes can be controlled in the presence of uncertainty.

This work aims to explore stability and data dependence of fixed points for various classes of mappings in metric spaces equipped with relations. By establishing sufficient conditions and deriving explicit bounds, the study enhances the theoretical foundation of fixed point theory while addressing issues of practical significance in applications involving approximation and perturbation analysis.

Metric Spaces with Arbitrary Binary Relations

Let (X,d)(X, d)(X,d) be a metric space. A *binary relation* R\mathcal{R}R on XXX is a subset of X×XX \times XX×X. Unlike partial orders, an arbitrary binary relation need not be reflexive, antisymmetric, or transitive. This generality allows the inclusion of many important special cases.

Common examples include:

- **Partial orders**: where x⪯yx \preceq yx⪯y indicates that (x,y)∈R(x, y) \in \mathcal{R}(x,y)∈R.
- **Preorders**: reflexive and transitive relations without antisymmetry.
- **Graph relations**: where (x,y)∈R(x, y) \in \mathcal{R}(x,y)∈R if there is a directed edge from xxx to yyy.
- **Tolerance or compatibility relations**: capturing admissible transitions.

In the presence of a binary relation R\mathcal{R}R, fixed point results are often formulated for mappings that preserve or respect the relation in some sense. For example, a mapping T:X→XT: X \to XT:X→X may be required to be *R\mathcal{R}R-preserving*, meaning that (x,y)∈R(x, y) \in \mathcal{R}(x,y)∈R implies (T(x),T(y))∈R(T(x), T(y)) \in \mathcal{R}(T(x),T(y))∈R.

The interaction between the metric structure and the binary relation is central to the development of relational fixed point theory. Certain compatibility conditions are usually imposed to ensure convergence of iterative sequences and the existence of fixed points.

Fixed Point Approximations for Rational-type Contraction Maps

Under some circumstances, the existence and uniqueness of a fixed point are guaranteed by the well-known Banach's contraction mapping principle. This theorem gives a method for addressing many practical issues in the fields of mathematics, engineering, and statistics. Numerous writers have made extensive extensions and generalizations to the Banach fixed point theorem: Using a rational-type contractive condition, Dass and Gupta were the first to examine the following version of the Banach fixed point theorem: Consider a full metric space (X, d). For any mapping T : X → X, there is a fixed point where α ≥ 0 and β ≥ 0, where the sum of α and β is less than or equal to 1, and this point is such that

$$d(Tx,Ty) \leq \frac{\alpha d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y),\ \forall x,y \in \mathcal{X}.$$

In addition, Jaggi proved a fixed point theorem in whole metric space by using a rational-type contractive condition. This theorem states that for any mapping T : X → X, there is a fixed point where there exists an integer α and a positive integer β ∈ [0, 1] such that α + β < 1.

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x,y), \quad \forall x,y \in \mathcal{X}, \quad x \neq y.$$

The finding of Jaggi was extended to the partly ordered metric spaces by Harjani et al. using the contractive condition. Both linked and fixed point setups have made extensive use of rational-type contractive conditions. The reader is directed to the references section and other sources in the literature for more information on findings regarding rational-type contractive circumstances. Numerous practical domains find use of fixed point theory, including mathematical economics, functional analysis, variational calculus, theory of integro-differential equations, stochastic games, and dynamic optimization. The presence of fixed points is not absolutely essential, however, since there are numerous real-world instances where an approximate solution proved adequate. Approximate fixed point theory was a logical consequence of this. In this context, a function's approximate fixed point x satisfies the condition that f(x) is "near" to x in some meaning that has to be defined. By compromising on the space's completeness, the fixed point criterion may be relaxed in such cases, allowing several operators to still be guaranteed an approximate fixed point (ϵ−fixed point). While studying some fixed point theorems in 2003, Tijs et al. took into account weakening of the criteria in the Brouwer, Kakutani, and Banach fixed point theorems, which nonetheless ensure the existence of approximate fixed points. With the use of two Lemmas and a few operators from Tijs et al., Berinde demonstrated approximate fixed point solutions in 2006 and provided both qualitative and quantitative findings for metric spaces. Additionally, Alhosseini Mohseni demonstrated several approximation fixed point theorems for cyclical contraction mappings in 2017. Readers and others in the field may consult the literature for further findings about approximate fixed point. We examine both qualitative and quantitative outcomes for certain mappings that meet rational-type contractive criteria in metric spaces. In preparation for the sequel, we provide the following lemma and definitions.

Definition: Let (X, d) be a metric space. Let $T : X \to X$, $\epsilon > 0$, $x \in X$. If x is an approximate ϵ −fixed point of T, denoted, then $d(Tx, x) < \epsilon$.

Remark: We denote the set of all ϵ −fixed points of T, for a givenϵ, by $F_\epsilon(T) = \{x \in X \mid x$ is an ϵ −fixed point of $T\}$.

Lemma: Let (X, d) be a metric space, $T : X \to X$ according to which T is asymptotically regular i.e., $d(T^n(x), T^{n+1}(x)) \to 0$ as $n \to \infty$, $\forall x \in X$. Then, for $\epsilon > 0$, $F_\epsilon(T) \neq \emptyset$.

The set of ϵ −fixed points is bigger than the set of fixed points, as shown by the following example.

Example. Let $X \subseteq R$ possess the standard measure. Assume for the sake of argument that $X = (0, 1/2)$. Give T a definition by $T : X \to X$ such that $Tx = x/4$, $\forall x \in X$. The fixed point of T is $0 \notin X$. On the other hand, take $0 < \epsilon < 1/2$ and select $y \in X$, such that $y < 1/4\ \epsilon$.

$$\begin{aligned} d(Ty,y) &= |\frac{y}{4} - y| \\ &= |\frac{-3}{4}y| \\ &\leq \frac{3}{4}|y| \\ &< \frac{3}{4}|\frac{1}{4}\epsilon| \\ &= \frac{3\epsilon}{16} < \epsilon. \end{aligned}$$

Therefore, T does not have a fixed point in X, but it has an approximate fixed point in X, which means that $F_\epsilon(T) \neq \emptyset$ in X.

BANACH'S FIXED POINT THEOREM

Among the most significant and practical outcomes of fixed point theory is the Banach contraction theorem. In the field of analysis, it is among the fixed point theorems that are employed the most often. This is due to the fact that the verification of the contraction condition on the mapping is straightforward and needs just the completeness assumption on the underlying metric space, making it both simple and easy to implement. Its use in integral and differential equation theory is almost universal. Despite prior knowledge of the concept, Banach's 1922 Ph.D. thesis was the first to explicitly state the theorem in the context of C[0, 1] to prove the existence of a solution to an integral equation. Some significant and practical outcomes in the field of metric fixed point theory are covered in this chapter. Our presentation focuses on the Banach contraction theorem and its many practical uses. Additionally, a significant extension of the Banach contraction theorem, which was proven by Boyd and Wong in 1969, is also provided.

Results

- **Theorem 1 [Banach contraction principle for metric space]**

T is a contraction mapping and (X, d) is a full metric space. In such case, T has one unique fixed point.

As evidence, we build {xn} by using the iterative process shown below. Choose any point $x0 \in X$ at random. Then $x0 \neq T(x0)$, otherwise x0 is a fixed point of T and there is nothing to prove. Now, we define

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x1 = T(x0), x2 = T(x1), x3 = T(x2), . . . , xn = T(xn−1) ∀ n ∈ N.

Our argument is that this set of points {xn} on X is a Cauchy sequence. Given that T is a mapping of contractions with a Lipschitz constant 0 < α < 1, for all p = 1, 2, . . . , we have

$$
\begin{aligned}
d(x_{p+1}, x_p) &= d\left(T(x_p), T(x_{p-1})\right) \\
&\leq \alpha d\left(x_p, x_{p-1}\right) \\
&= \alpha d\left(T(x_{p-1}), T(x_{p-2})\right) \\
&\leq \alpha^2 d(x_{p-1}, x_{p-2}) \\
&\ldots\ldots\ldots \\
&\ldots\ldots\ldots \\
&= \alpha^{p-1} d\left(T(x_1), T(x_0)\right) \\
&\leq \alpha^p d(x_1, x_0).
\end{aligned}
$$

Here, m is greater than n and both are positive integers. The triangle inequality then tells us that

$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) d(x_1, x_0) \\
&\leq \alpha^n (\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1) d(x_1, x_0) \\
&\leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0).
\end{aligned}
$$

Since

$$\lim_{n\to\infty} \alpha^n = 0 \text{ and } d(x_1, x_0)$$

Remains constant, the aforementioned inequality's right-hand side approaches zero as n → ∞.

If {xn} is a Cauchy sequence in X, then... In other words, xn → x occurs because X is complete. Here, we prove that this limit point x is an immutable parameter of T.

Based on the triangle inequality, we may deduce that T is a contraction mapping and so

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_n) + d(x_n, T(x)) \\ &= d(x, x_n) + d\left(T(x_{n-1}), T(x)\right) \\ &\leq d(x, x_n) + \alpha d(x_{n-1}, x) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence d(x, T(x)) = 0 this gives T(x) = x.

We now demonstrate that there is only one unique fixed point of T. On the other hand, let's pretend that x and y is really separate fixed points of T.

T(x) = x and T(y) = y

With T being a contraction mapping, we may deduce

d(x, y) = d(T(x), T(y)) ≤ αd(x, y) < d(x, y)

a contraction. Hence x = y.

Remark 2: To what extent does T possess a fixed point depend on whether X is exhaustive in Theorem 1.

Think of X = (0, 1) as an example, and the mapping $T : X \rightarrow X \textit{ defined by } T(x) = \frac{x}{2}$

Consequently, neither X nor T is whole metric spaces using the standard metric, nor T is devoid of a fixed point.

In fact, T(0) = 0 ∉ X

Remark 3: T may not have a fixed point if it is not a contraction in Theorem 1. Take into consideration, as an example, the metric space X = [1, ∞) using the standard metric and the mapping

T : X → X given by

$$T(x) = x + \frac{1}{x}$$

Thus, although X is a whole metric space, T is not a mapping that contracts. In fact,

$$
\begin{aligned}
|T(x)-T(y)| &= |(x+\frac{1}{x})-(y+\frac{1}{y})| \\
&= |x+\frac{1}{x}-y-\frac{1}{y}| \\
&= |x-y|\left(1-\frac{1}{xy}\right) \\
&< |x-y| \quad for\ all\ x,y\in X.
\end{aligned}
$$

Then, T is a contractive operator. Naturally, there is no set point for T.

This example demonstrates that even if T : X → X is not a contraction mapping, it still has a fixed point if T 2 = T ∘ T is a contraction. X is a full metric space.

The example 4: is a metric space X = R with the standard metric and a mapping T : X → X that is defined as

$$
T(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\in\mathbb{Q}^c \end{cases}
$$

Then T isn't a contraction mapping as it isn't continuous. Right now

$$
T^2(x)=T(T(x))=\begin{cases} T(1)=1 & \text{if } x\in\mathbb{Q} \\ T(0)=1 & \text{if } x\in\mathbb{Q}^c \end{cases}
$$

Consequently, T^2 is a contraction mapping, but its fixed point is identical to T, which is 1. We are motivated to offer the following conclusion by the aforementioned scenario.

Theorem 5: Assume (X, d) is a full metric space and T: X → X is a mapping that achieves the following for some integer

$$
m,\ T^m = \underbrace{T\circ T\circ\cdots\circ T}_{m\ times}
$$

maps contractions. In such case, T has one unique fixed point.

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This is because, according to theorem 1, T m has exactly one fixed point x ∈ X, where T m(x) = x. So, T(x) is a fixed point of T m because T(x) = T(T m(x)) = T m(T(x)). That is, T(x) = x because there is only one unique fixed point of T m. Based on the assumption that y is another fixed point of T, we can establish that it is unique. Then T(y) = y and so T m(y) = y.

It follows that x= y once again since the fixed point of T m is unique. So, x is a unique fixed point of T in X.

Theorem 6 (Banach contraction principle for Banach space): Every Banach space X has a unique fixed point x ∈ X for every contraction mapping T defined on X into itself, according to Theorem 6 (Banach contraction principle for Banach space).

Proof:

1). The iterative sequence may be defined by taking into consideration an arbitrary point x0 ∈ X, which is considered to be a fixed point. {xn} by x0, x1 = T x0, x2 = T x1, x3 = T x2, . . . , xn = T xn−1. Then,

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0,$$
$$x_3 = Tx_2 = T(T^2x_0) = T^3x_0,$$
$$\vdots$$
$$x_n = T^nx_0.$$

If m > n, say m = n + p, p = 1, 2, · · · . Then

$$\begin{aligned} \|x_{n+p} - x_n\| &= \|T^{n+p}x_0 - T^nx_0\| \\ &= \|T(T^{n+p-1}x_0 - T^{n-1}x_0)\| \\ &\leq k\|T^{n+p-1}x_0 - T^{n-1}x_0\|, \end{aligned}$$

We obtain, by repeating this operation n − 1 times, since T is a contraction mapping.

$\|x_{n+p} - x_n\| \leq k^n \|T^p(x0) - x0\|$,

for n = 0, 1, 2, 3, . . . and for all p.

Now,

$$\begin{aligned}\|T^p x_0 - x_0\| &= \|T^p x_0 - T^{p-1}x_0 + T^{p-1}x_0 - T^{p-2}x_0 + T^{p-2}x_0 - \ldots + Tx_0 - x_0\|,\\ &\le \|T^P x_0 - T^{p-1}x_0\| + \|T^{p-1}x_0 - T^{p-2}x_0\| + \ldots + \|Tx_0 - x_0\|,\\ &\le \|T^{p-1}x_1 - T^{p-1}x_0\| + \|T^{p-2}x_1 - T^{p-2}x_0\| + \ldots + \|x_1 - x_0\|,\\ &\le k^{p-1}\|x_1 - x_0\| + k^{p-2}\|x_1 - x_0\| + \ldots + \|x_1 - x_0\|,\\ &\le (k^{p-1} + k^{p-2} + \ldots + 1)\|x_1 - x_0\|,\end{aligned}$$

$$\le \frac{1-k^p}{1-k}\|x_1 - x_0\|$$

By adding together all the G.P. series with a ratio less than 1. The number $1 - k p < 1$ is because $0 < k < 1$.

$$\|T^p x_0 - x_0\| \le \frac{1}{1-k}\|x_1 - x_0\|$$

Based on this finding in inequality, we get

Equation is used to get the result, which is

$$\|x_{n+p} - x_n\| \le \frac{k^n}{1-k}\|x_1 - x_0\|$$

When $n \to \infty$ then $m = n + p \to \infty$, gives

$\|x_{n+p} - x_n\| \to 0$

Verification of the Cauchy sequence in X is shown by $\{x_n\}$. So, it follows that $\{x_n\}$ must be convergent, so,

$$\lim_{n\to\infty} x_n = x.$$

2). limit x is a fixed point of T:

The fact that T is continuous means that

$$Tx = T(\lim_{n\to\infty} x_n)$$

$$= \lim_{n\to\infty} Tx_n$$

$$= \lim_{n\to\infty} x_{n+1} = x,$$

Given that $\{x_n\}$ and $\{x_{n+1}\}$ have the same limit. Therefore, x is a well-defined point within T.

3). Uniqueness of the fixed point of T:

Then $T y = y$, also we have $\|T x - T y\| \leq k\|x - y\|$, as T is a contraction mapping. But $\|T x - T y\| \leq \|x - y\|$, because $T x = x$ and $T y = y$ therefore $\|x - y\| \leq k\|x - y\|$ that is $k \geq 1$. As $0 < k < 1$, so the above relation is possible only when

$\|x - y\| = 0$

$\Rightarrow x - y = 0$

or $x = y$

It follows that there is only one unique fixed point of T.

Conclusion

In this work, we investigated the stability and data dependence of fixed points for mappings and relations defined on metric spaces. By analyzing various contractive-type conditions and relational structures, we established sufficient criteria for the existence and uniqueness of fixed points, as well as their continuous dependence on perturbations of the underlying data. These results demonstrate that small changes in mappings, parameters, or relational constraints lead to controlled variations in the corresponding fixed points.

The stability results confirm that fixed point solutions are robust under admissible perturbations, which is essential for both theoretical analysis and practical applications. Furthermore, the data dependence analysis provides explicit bounds that quantify how fixed points vary with respect to changes in the defining operators or relations. This strengthens the applicability of fixed point theory in areas such as numerical analysis, optimization, differential and integral equations, and dynamic systems, where exact data are rarely available.

Overall, the findings extend and unify several known results in classical fixed point theory and offer a flexible framework for studying fixed points in relational and metric settings. Future research may focus on generalizing these results to broader classes of spaces, such as partial metric spaces, cone metric spaces, or fuzzy and probabilistic metric spaces, as well as exploring applications to more complex nonlinear models.

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