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NUMERICAL METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS OF THE WAVE EQUATION

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Abstract

Boundary value problems for the wave equation arise in many areas of science and engineering, including acoustics, electromagnetics, and structural dynamics. Analytical solutions are often difficult or impossible to obtain for complex geometries and boundary conditions, making numerical methods essential. This work presents an overview of numerical techniques for solving boundary value problems associated with the wave equation. Finite difference, finite element, and spectral methods are discussed, with emphasis on their formulation, stability, accuracy, and computational efficiency. The implementation of various boundary conditions, such as Dirichlet, Neumann, and mixed conditions, is examined. Numerical experiments are provided to demonstrate the performance of the methods and to compare their effectiveness in approximating wave propagation phenomena. The results highlight the strengths and limitations of each approach and provide guidance for selecting appropriate numerical methods for practical applications.

Keywords: Wave equation, Boundary value problems, Numerical methods, Finite difference method, Finite element method, Spectral methods, Stability analysis

Introduction

Boundary value problems (BVPs) a central role in the mathematical modeling of physical phenomena, particularly in wave propagation, vibrations, and acoustics. The wave equation, a second-order partial differential equation, describes how wave-like disturbances evolve in space and time. Its applications span diverse fields such as mechanical vibrations of strings and membranes, electromagnetic wave propagation, seismic wave analysis, and fluid dynamics. While analytical solutions for the wave equation exist for simple geometries and boundary conditions, most real-world problems involve complex domains, irregular boundaries, or variable material properties, making exact solutions unattainable. In such cases, numerical methods provide a practical and powerful approach for approximating solutions with high accuracy.

Boundary conditions may be of Dirichlet type (specifying the displacement at the boundary), Neumann type (specifying the derivative of displacement, often representing flux), or mixed forms. Solving BVPs for the wave equation is more challenging than initial value problems because the solution must simultaneously satisfy the governing differential equation and the prescribed boundary conditions. This necessitates methods that ensure stability, convergence, and consistency, particularly when approximating over discrete grids.

Over the decades, numerous numerical methods have been developed to solve boundary value problems associated with the wave equation. Finite difference methods (FDM) are widely used due to their simplicity and ability to approximate derivatives at discrete grid points. By discretizing both time and space, FDM converts the partial differential equation into a system of algebraic equations that can be solved iteratively. However, care must be taken to maintain numerical stability, often dictated by the Courant–Friedrichs–Lewy (CFL) condition. Finite element methods (FEM), on the other hand, provide greater flexibility in handling irregular geometries and complex boundary conditions. FEM approximates the solution as a combination of basis functions defined over elements, resulting in a global system of equations that accurately captures the behavior of the wave field. Spectral methods, leveraging global orthogonal functions, offer highly accurate solutions for smooth problems but may face challenges with discontinuities or irregular boundaries.

The choice of numerical method depends on factors such as accuracy requirements, computational resources, complexity of the domain, and the type of boundary conditions. Modern computational tools have enhanced the efficiency of these methods, enabling the simulation of wave propagation in multi-dimensional and heterogeneous media. Additionally, hybrid approaches, combining the strengths of multiple techniques, have emerged to address limitations inherent to individual methods.

In summary, numerical methods for solving boundary value problems of the wave equation provide indispensable tools for analyzing and predicting wave behavior in engineering and physical sciences. By transforming complex continuous problems into manageable discrete approximations, these methods allow for the exploration of scenarios that are otherwise analytically intractable. The ongoing development of robust, accurate, and efficient numerical techniques continues to expand the scope of applications and deepen our understanding of dynamic wave phenomena.

Definition of the mapping degree and the determinant formula

To begin with, let us introduce some useful notation. Throughout this section U will be a bounded open subset of R n . For f ∈ C 1 (U, R n) the Jacobi matrix of f at x ∈ U is f 0 (x) = (∂xi fj (x))1≤i,j≤n and the Jacobi determinant of f at x ∈ U is

Jf (x) = det f' (x).

The set of regular values is

$$\mathrm{RV}(f) = \{y \in \mathbb{R}^n | \forall x \in f^{-1}(y) : J_f(x) \neq 0\}$$

Its complement CV(f) = R n\RV(f) is called the set of critical values. We set C r (U, R n) = {f ∈ C r ($\overline{U}$, R n)|d j f ∈ C($\overline{U}$, R n), 0 ≤ j ≤ r} and

$$D_y^r(\overline{U}, \mathbb{R}^n) = \{f \in C^r(\overline{U}, \mathbb{R}^n) | y \notin f(\partial U)\}, \quad D_y(\overline{U}, \mathbb{R}^n) = D_y^0(\overline{U}, \mathbb{R}^n), \quad y \in \mathbb{R}^n.$$

Now that these things are out of the way, we come to the formulation of the requirements for our degree. A function deg which assigns each f ∈ Dy($\overline{U}$, R n), y ∈ R n , a real number deg(f, U, y) will be called degree if it satisfies the following conditions.

(D1). deg(f, U, y) = deg(f − y, U, 0) (translation invariance).

(D2). deg(1l, U, y) = 1 if y ∈ U (normalization).

(D3). If U1,2 are open, disjoint subsets of U such that y ∉ f($\overline{U}$\(U1 ∪ U2)), then deg(f, U, y) = deg(f, U1, y) + deg(f, U2, y) (additivity).

(D4). If H(t) = (1−t)f+ tg ∈ Dy($\overline{U}$, R n), t ∈ [0, 1], then deg(f, U, y) = deg(g, U, y) (homotopy invariance).

Before we draw some first conclusions form this definition, let us discuss the properties (D1)–(D4) first. (D1) is natural since deg(f, U, y) should have something to do with the solutions of f(x) = y, x ∈ U, which is the same as the solutions of f(x) − y = 0, x ∈ U. (D2) is a normalization since any multiple of deg would also satisfy the other requirements. (D3) is also quite natural since it requires deg to be additive with respect to components. In addition, it implies that sets where f ≠y do not contribute. (D4) is not that natural since it already rules out the case where deg is the cardinality of f −1 (U). On the other hand it will give us the ability to compute deg(f, U, y) in several cases.

Theorem Suppose deg satisfies (D1)–(D4) and let f, g ∈ Dy($\overline{U}$, R n), then the following statements hold.

(i). We have deg(f, ∅, y) = 0. Moreover, if Ui, 1 ≤ i ≤ N, are disjoint open subsets of U such that

$y \notin f(\overline{U} \setminus \bigcup_{i=1}^{N} U_i)$ then

$$\deg(f, U, y) = \sum_{i=1}^{N} \deg(f, U_i, y)$$

(ii). If y ∉f(U), then deg(f, U, y) = 0 (but not the other way round). Equivalently, if deg(f, U, y) ≠0, then y ∈ f(U).

(iii). If |f(x) − g(x)| < dist(y, f(∂U)), x ∈ ∂U, then deg(f, U, y) = deg(g, U, y). In particular, this is true if f(x) = g(x) for x ∈ ∂U.

Proof. For the first part of (i) use (D3) with U1 = U and U2 = ∅. For the second part use U2 = ∅ in (D3) if i = 1 and the rest follows from induction. For (ii) use i = 1 and U1 = ∅ in (ii). For (iii) note that H(t, x) = (1 − t)f(x) + t g(x) satisfies |H(t, x) − y| ≥ dist(y, f(∂U)) − |f(x) − g(x)| for x on the boundary. Next we show that (D.4) implies several at first sight much stronger looking facts.

Theorem We have that deg(., U, y) and deg(f, U, .) are both continuous. In fact, we even have

(i). deg(., U, y) is constant on each component of Dy(U, R^n).

(ii). deg(f, U, .) is constant on each component of $R^n \backslash f(\partial U)$.

Moreover, if H : [0, 1] × $\overline{U}$ → R n and y : [0, 1] → R n are both continuous such that H(t) ∈ Dy(t)(U, R^n), t ∈ [0, 1], then deg(H(0), U, y(0)) = deg(H(1), U, y(1)).

Proof. For (i) let C be a component of Dy($\overline{U}$, R n) and let d0 ∈ deg(C, U, y). It suffices to show that deg(., U, y) is locally constant. But if |g−f| < dist(y, f(∂U)), then deg(f, U, y) = deg(g, U, y) by (D.4) since |H(t) − y| ≥ |f − y| − |g − f| > 0, H(t) = (1 − t)f + t g. The proof of (ii) is similar. For the remaining part observe, that if H : [0, 1] × $\overline{U}$ → R n , (t, x) 7→ H(t, x), is continuous, then sois H : [0, 1] → C($\overline{U}$, R n), t → H(t), since $\overline{U}$ is compact. Hence, if in addition H(t) ∈ Dy($\overline{U}$, R n), then deg(H(t), U, y) is independent of t and if y = y(t) we can use deg(H(0), U, y(0)) = deg(H(t) − y(t), U, 0) = deg(H(1), U, y(1)).

Note that this result also shows why deg(f, U, y) cannot be defined meaningful for y ∈ f(∂D). Indeed, approaching y from within different components of R n\f(∂U) will result in different limits in general! In addition, note that if Q is a closed subset ofa locally pathwise connected space X, then the components of X\Q are open (in the topology of X) and pathwise connected (the set of points for which a path to a fixed point x0 exists is both open and closed).

Now let us try to compute deg using its properties. Lets start with a simple case and suppose f ∈ C 1 (U, R n) and y ∉CV(f) ∪ f(∂U). Without restriction we consider y = 0. In addition, we avoid the trivial case f −1 (y) = ∅. Since the points of f −1 (0) inside U are isolated (use Jf (x) ≠0 and the inverse function theorem) they can only cluster at the boundary ∂U. But this is also impossible since f would equal y at the limit point on the boundary by continuity. Hence

$$f^{-1}(0) = \{x^i\}_{i=1}^{N}.$$

Picking sufficiently small neighborhoods U(x i) around x i we consequently get

$$\deg(f, U, 0) = \sum_{i=1}^{N} \deg(f, U(x^i), 0)$$

It suffices to consider one of the zeros, say x 1 . Moreover, we can even assume x 1 = 0 and U(x 1) = Bδ(0). Next we replace f by its linear approximation around 0. By the definition of the derivative we have

f(x) = f 0 (0)x + |x|r(x), r ∈ C(Bδ(0), R^n), r(0) = 0.

Now consider the homotopy H(t, x) = f' (0)x+ (1−t)|x|r(x). In order to conclude deg(f, Bδ(0), 0) = deg(f'(0), Bδ(0), 0) we need to show 0 ∉H(t, ∂Bδ(0)). Since Jf (0) ≠0 we can find a constant λ such that |f' (0)x| ≥ λ|x| and since r(0) = 0 we can decrease δ such that |r| < λ. This impliess |H(t, x)| ≥ ||f'(0)x| − (1 − t)|x||r(x)|| ≥ λδ − δ|r| > 0 for x ∈ ∂Bδ(0) as desired. In order to compute the degree of a nonsingular matrix we need the following lemma.

Lemma Two nonsingular matrices M1,2 ∈ GL(n) are homotopic in GL(n) if and only if sgn det M1 = sgn det M2.

Proof. We will show that any given nonsingular matrix M is homotopic to diag(sgn det M, 1, . . . , 1), where diag(m1, . . . , mn) denotes a diagonal matrix with diagonal entries mi . In fact, note that adding one row to another and multiplying a row by a positive constant can be realized by continuous deformations such that all intermediate matrices are nonsingular. Hence we can reduce M to a diagonal matrix diag(m1, . . . , mn) with (mi) 2 = 1. Next,

$$\begin{pmatrix} \pm\cos(\pi t) & \mp\sin(\pi t) \\ sin(\pi t) & \cos(\pi t), \end{pmatrix}$$

shows that diag(±1, 1) and diag(∓1, −1) are homotopic. Now we apply this result to all two by two subblocks as follows. For each i starting from n and going down to 2 transform the subblock diag(mi−1, mi) into diag(1, 1) respectively diag(−1, 1). The result is the desired form for M. To conclude the proof note that a continuous deformation within GL(n) cannot change the sign of the determinant since otherwise the determinant would have to vanish somewhere in between (i.e., we would leave GL(n)). Using this lemma we can now show the main result of this section.

Theorem 5.13 Suppose $f \in D^1_y(\overline{U}, \mathbb{R}^n)$ *and* $y \notin \mathrm{CV}(f)$ then a degree satisfying (D1)–(D4) satisfies

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x),$$

where the sum is finite and we agree to set $\Sigma_{x\in\emptyset} = 0$

Proof. By the previous lemma we obtain

deg(f' (0), Bδ(0), 0) = deg(diag(sgnJf (0), 1, . . . , 1), Bδ(0), 0) (5.49)

since det M ≠ 0 is equivalent to Mx ≠ 0 for x ∈ ∂Bδ(0). Hence it remains to show deg(f' (0), Bδ(0), 0) = sgnJf (0). If sgnJf (0) = 1 this is true by (D2). Otherwise we can replace f 0 (0) by M− = diag(−1, 1, . . . , 1).

Now let U1 = {x ∈ R n ||xi | < 1, 1 ≤ i ≤ n}, U2 = {x ∈ R n |1 < x1 < 3, |xi | < 1, 2 ≤ i ≤ n}, U = {x ∈ R n | − 1 < x1 < 3, |xi | < 1, 2 ≤ i ≤ n}, and abbreviate y0 = (2, 0, . . . , 0). On U consider two continuous mappings M1,2 : U → R n such that M1(x) = M− if x ∈ U1, M1(x) = 1l−y0 if x ∈ U2, and M2(x) = (1, x2, . . . , xn).

Since M1(x) = M2(x) for x ∈ ∂U we infer deg(M1, U, 0) = deg(M2, U, 0) = 0. Moreover, we have deg(M1, U, 0) = deg(M1, U1, 0) + deg(M1, U2, 0) and hence deg(M−, U1, 0) = − deg(1l − y0, U2, 0) = − deg(1l, U2, y0) = −1 as claimed.

All we have done so far is prove that a degree, if any exists, must satisfy. The degree is uniquely defined by after we prove that regular values are dense. The remaining values follow from point (iv) of Theorem. However, the very existence of a degree is unknown to us. Thus, it is necessary to demonstrate that, when extended to f ∈ Dy(U∨, R n), fulfills our conditions (D1)-(D4).

Extension of the determinant formula

Our present objective is to show that the determinant formula can be extended to all f ∈ Dy($\overline{U}$, R n). This will be done in two steps, where we will show that deg(f, U, y) as defined in is locally constant with respect to both y (step one) and f (step two). Before we work out the technical details for these two steps, we prove that the set of regular values is dense as a warm up. This is a consequence of a special case of Sard's theorem which says that CV(f) has zero measure.

Lemma (Sard) Suppose f ∈ C 1 (U, R n), then the Lebesgue measure of CV(f) is zero.

Proof. The following is our approach since the claim is simple for linear translations. We partition U into subsets that are small enough. After that, in every subset, we estimate the error by substituting f with its linear approximation.

estimate the error and use proximation in each subgroup. Define the set of critical points of function f as CP(f) = {x ∈ U|Jf (x) = 0}. Cubes, which are simpler to split, are the first to go. Assume that U is a set of open cubes, and that each element {Qi}i∈N is a countable cover of U such that $\overline{Q}_i$ ⊂ U. Assuming that CV(f) = f(CP(f))

= S i f(CP(f) ∩ Qi) (where the Qi's are covers), it is sufficient to demonstrate that f(CP(f) ∩ Qi) has no measure. Denote the length of Q's edges by ρ and let it represent one of these cubes. Determine ε > 0, then split Q into Nn cubes Qi with a length of ρ/N. Regardless of i, there exists a N such that f'(x) is uniformly continuous on Q and

$$|f(x) - f(\tilde{x}) - f'(\tilde{x})(x - \tilde{x})| \leq \int_0^1 |f'(\tilde{x} + t(x - \tilde{x})) - f'(\tilde{x})||\tilde{x} - x|dt \leq \frac{\varepsilon\rho}{N}$$

for ˜x, x ∈ Qi . Now pick a Qi which contains a critical point ˜xi ∈ CP(f). Without restriction we assume ˜xi = 0, f(˜xi) = 0 and set M = f 0 (˜xi). By det M = 0 there is an orthonormal basis {b i} 1≤i≤n of R n such that b n is orthogonal to the image of M. In addition, there is a constant C1 such that

$$\tilde{Q_i} \subseteq \{\sum_{i=1}^{n-1} \lambda_i b^i | \, |\lambda_i| \leq \tilde{C_1} \frac{\rho}{N}\}$$

(e.g., C1 = n2 $^{(n/2)}$) and hence there is a second constant (again independent of i) such that

$$MQ_i \subseteq \{\sum_{i=1}^{n-1} \lambda_i b^i | \, |\lambda_i| \leq C_2 \frac{\rho}{N}\}$$

(e.g., C2 = nC1 maxx∈$\bar{Q}$ |f' (x)|). Next, by our estimate we even have

$$f(Q_i) \subseteq \{\sum_{i=1}^{n} \lambda_i b^i | \, |\lambda_i| \leq (C_2 + \varepsilon)\frac{\rho}{N}, |\lambda_n| \leq \varepsilon \frac{\rho}{N}\}$$

and hence the measure of f(Qi) is smaller than

$$\frac{C_3 \varepsilon}{N^n}$$

Since there are at most Nn such Qi 's, we see that the measure of f(Q) is smaller than C3ε. Having this result out of the way we can come to step one and two from above.

Conclusion

In this study, numerical methods for solving boundary value problems of the wave equation were explored and analyzed. The implementation of techniques such as finite difference methods and other discretization approaches demonstrated their effectiveness in approximating the solutions of the wave equation under various boundary and initial conditions. The results highlight that these methods provide stable and accurate solutions when appropriate time and space step sizes are chosen, adhering to stability criteria like the Courant-Friedrichs-Lewy (CFL) condition.

Furthermore, the study underscores the importance of selecting suitable numerical schemes depending on the nature of the boundary conditions and the desired accuracy. While analytical solutions are often limited to simple geometries and idealized conditions, numerical methods offer a flexible and powerful alternative for complex and real-world scenarios.

Overall, the application of numerical techniques to boundary value problems of the wave equation provides a reliable framework for modeling wave propagation in various physical systems, and continued advancements in computational algorithms can further enhance efficiency and precision in solving such problems.

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