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MULTI-CRITERIA DECISION MAKING MODELS BY APPLYING THE TOPSIS METHOD TO CRISP AND INTERVAL DATA

Abstract

In this paper, one of the multi-criteria models in making decision, a Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS), is described. Some of the advantages of TOPSIS methods are: simplicity, rationality, comprehensibility, good computational efficiency and ability to measure the relative performance for each alternative in a simple mathematical form.

The paper has a review character. It systematises the knowledge within the scope of techniques of decision taking with the use of the TOPSIS method. Simple numerical examples that reference real situations show practical applications of different aspects of this method.

The paper is organized as follows. The Introduction presents a short overview of the decision making steps as well as MCDM techniques. Section 1 presents matrix representation of the MCDM problem. Section 2 describes the TOPSIS procedure for crisp data, and Section 3 for interval data. The TOPSIS algorithm in group decision environment in the case of crisp and interval data is also presented. In Section 4 the problem of qualitative data in TOPSIS model is discussed. The numerical examples showing applications of those techniques in the negotiation process are presented in Section 5. Finally, conclusions and some concluding remarks are made in last section.

Keywords

TOPSIS method, numerical data, interval data, positive ideal solution, negative ideal solution.

Introduction

Multi-criteria decision making (MCDM) refers to making choice of the best alternative from among a finite set of decision alternatives in terms of multiple, usually conflicting criteria. The main steps in multi-criteria decision making are the following [Hwang, Yoon, 1981; Jahanshahloo, Hosseinzadeh, Lofti, Izadikhah, 2006a]:

– establish system evaluation criteria that relate system capabilities to goals,
– develop alternative systems for attaining the goals (generating alternatives),
– evaluate alternatives in terms of criteria,
– apply one of the normative multiple criteria analysis methods,
– accept one alternative as "optimal" (preferred),
– if the final solution is not accepted, gather new information and go to the next iteration of multiple criteria optimization.

Multi-criteria decision making techniques are useful tools to help decision maker(s) to select options in the case of discrete problems. Especially, with the help of computers, those methods have become easier for the users, so they have found great acceptance in many areas of decision making processes in economy or management. Among many multi-criteria techniques, MAXMIN, MAXMAX, SAW, AHP, TOPSIS, SMART, ELECTRE are the most frequently used methods [Chen, Hwang, 1992]. The nature of the recommendations of one of those methods depends on the problem being addressed: choosing, ranking or sorting. The selection of models/techniques can be also based on such evaluation criteria as:

– internal consistency and logical soundness,
– transparency,
– ease of use,
– data requirements are consistent with the importance of the issue being considered,
– realistic time and manpower resource requirements for the analytical process,
– ability to provide an audit trail,
– software availability, where needed.

The classification methods can be categorized by the type of information from the decision maker (no information, information on attributes or information on alternatives), data type or by solution aimed at [Chen, Hwang, 1992, p.16-25]. The MAXMIN technique assumed that the overall performance of an alternative is determined by its weakest attribute, in the MAXMAX technique an alternative is selected by its best attribute value. The SAW (Simple Additive Weighting) method multiplies the normalized value of the criteria for the alternatives with the importance of the criteria and the alternative with the highest score is selected as the preferred one. The TOPSIS (Technique for Order Preference by Similarity to the Ideal Solution) selects the alternative closest to the ideal solution and farthest from the negative ideal alternative. The classical TOPSIS method is based on information on attribute from decision maker, numerical data; the solution is aimed at evaluating, prioritizing and selecting and the only subjective inputs are weights. The AHP (The Analytical Hierarchy Process) uses a hierarchical structure and pairwise comparisons. An AHP hierarchy has at least three levels: the main objective of the problem at the top, multiple criteria that define alternatives in the middle and competing alternatives at the bottom. The major weaknesses of TOPSIS are that it does not provide for weight elicitation, and consistency checking for judgments; on the other hand, the use of AHP has been significantly restrained by the human capacity for the information process. From this point of view, TOPSIS alleviates the requirement of paired comparisons and the capacity limitation might not significantly dominate the process. Hence, it would be suitable for cases with a large number of criteria and alternatives, and especially where objective or quantitative data are given [Shih, Shyur, Lee, 2007]. SMART (The Simple Multi Attribute Rating Technique) is similar to AHP, a hierarchical structure is created to assist in defining a problem and in organizing criteria. However, there are some significant differences between those techniques: SMART uses a different terminology. For example, in SMART the lowest level of criteria in the value tree (or objective hierarchy) are called attributes rather than sub-criteria and the values of the standardized scores assigned to the attributes derived from value functions are called ratings. The difference between a value tree in SMART and a hierarchy in AHP is that the value tree has a true tree structure, allowing one attribute or sub-criterion to be connected to only one higher level criterion. SMART does not use a relative method for standardizing raw scores to a normalized scale. Instead, a value function explicitly defines how each value is transformed to the common model scale. The value function mathematically transforms ratings into a consistent internal scale with lower limit 0 and upper limit 1. The ELECTRE (Elimination and Choice Expressing Reality) method was to choose the best action(s) from a given set of actions, but it can also be applied to three main problems: choosing, ranking and sorting. There are two main parts to an ELECTRE application: first, the construction of one or several outranking relations, which aims at comparing in a comprehensive way each pair of actions; second, an exploitation procedure that elaborates on the recommendations obtained in the first phase.

This paper is focused on the TOPSIS method, which was presented by Hwang and Yoon [1981] and developed later by many authors [Jahanshahloo, Lofti, Izadikhah, 2006a; 2006b; Zavadskas,Turskis, Tamosaitiene, 2008; Hung, Chen, 2009]. The acronym TOPSIS stands for **T**echnique for **O**rder **P**reference by **S**imilarity to the **I**deal **S**olution. It is worth noting that the TOPSIS method corresponds to the Hellwig taxonomic method of ordering objects [Hellwig, 1968]. The main advantages of this method are the following [Hung, Cheng, 2009]:

- simple, rational, comprehensible concept,
- intuitive and clear logic that represent the rationale of human choice,
- ease of computation and good computational efficiency,

- a scalar value that accounts for both the best and worst alternatives ability to measure the relative performance for each alternative in a simple mathematical form,
- possibility for visualization.

In general, the process for the TOPSIS algorithm starts with forming the decision matrix representing the satisfaction value of each criterion with each alternative. Next, the matrix is normalized with a desired normalizing scheme, and the values are multiplied by the criteria weights. Subsequently, the positive-ideal and negative-ideal solutions are calculated, and the distance of each alternative to these solutions is calculated with a distance measure. Finally, the alternatives are ranked based on their relative closeness to the ideal solution. The TOPSIS technique is helpful for decision makers to structure the problems to be solved, conduct analyses, comparisons and ranking of the alternatives. The classical TOPSIS method solves problems in which all decision data are known and represented by crisp numbers. Most real-world problems, however, have a more complicated structure. Based on the original TOPSIS method, many other extensions have been proposed, providing support for interval or fuzzy criteria, interval or fuzzy weights to modeled imprecision, uncertainty, lack of information or vagueness.

In this paper, the classical TOPSIS algorithms for crisp, as well as interval data are described. Interval analysis is a simple and intuitive way to introduce data, uncertainty for complex decision problems, and can be used for many practical applications. An extension of the TOPSIS technique to a group decision environment is also investigated. The context of multi-criteria group decision making in both crisp and interval data are described. Finally, situations where criteria and their weight are subjectively expressed by linguistic variables are considered. The practical applications of the TOPSIS technique in estimating offers, for instance, in buyer-seller exchange are also proposed.

1. The matrix representation of the MCDM problem

The MCDM problems can be divided into two kinds. One is the classical MCDM set of problems among which the ratings and the weights of criteria are measured in crisp numbers. Another one is the multiple criteria decision-making set of problems where the ratings and the weights of criteria evaluated on incomplete information, imprecision, subjective judgment and vagueness are usually expressed by interval numbers, linguistic terms, fuzzy numbers or intuitive fuzzy numbers.

In the classical MCDM model, we assume exact data, objective and precise information, but this is often inadequate to model real life situations. Human judgments are often vague under many conditions. The socio-economic environment becomes more complex, the preference information provided by decision-makers is usually imprecise, and can create hesitation or uncertainty about preferences. A decision may have to be made under time pressure and lack of knowledge or data, or the decision-makers have limited attention and information processing capacities. Most input information is not known precisely, so that the values of many criteria are expressed in subjective or uncertain terms. The criteria, as well as their weight, could be subjectively expressed by linguistic variables. Thus, many researchers extended the MCDM approach for decision making problems with subjective criteria, interval data or fuzzy environment using grey system theory or fuzzy set theory.

The grey system theory, developed by Deng [1982, 1988] is based upon the concept that information is sometimes incomplete or unknown [Jadidi, Hong, Firouzi, Yusuff, 2008; Liu, Lin, 2006]. Exactly, the theory is based on the degree of information known which is modeled by intervals. If the system information is unknown, it is called a black system, if the information is fully known, it is called a white system. And a system with information known partially is called a grey system. The fuzzy set theory cannot handle incomplete data and information, but is adequate to deal with uncertain and imprecise data [Kahraman, 2008; Chen, Hwang, 1992]. The advantage of the grey theory over the fuzzy theory is that the grey theory takes into account the condition of the fuzziness; that is, the grey theory can deal flexibly with the fuzziness situation.

We can also consider single decision making and group decision making. Group decision making is more complex than single decision making because it involves many contradicting factors, such as: conflicting individual goals, inefficient knowledge, validity of information, individual motivation, personal opinion, power. In both multi-criteria decision making (MCDM) and group decision making (GDM), there are two steps: aggregation and exploitation. In MCDM, aggregation consists in combining satisfaction over different criteria while GDM problem consists in combining the experts' opinions into a group collective one. Group decision making can be approached from two points of view. In the first approach, individual multi-criteria models are developed based on individuals' preferences. Each decision maker formulates a multi--criteria problem defining the parameters according to these preferences and solves the problem getting an individual solution set. Next, the separate solutions are aggregated by aggregation of operations resulting in the group solution. In the second approach, each decision maker provides a set of parameters which are aggregated by appropriate operators, providing finally a set of group parameters. Upon this set the multi-criteria method is applied and the solution expresses group preference [Rigopoulos, Psarras, Askounis, 2008].

Solving of each multi-criteria problem (individual or group decision) begins with the construction of a decision making matrix (or matrices). In such matrixes, values of the criteria for alternatives may be real, intervals numbers, fuzzy numbers or qualitative labels.

Let us denote by $D = \{1,2,\dots,K\}$ a set of decision makers or experts. The multi-criteria problem can be expressed in $k-$ matrix format in the following way:

$$\begin{array}{c|cccc} & \mathrm{C}_1 & \mathrm{C}_2 & \cdots & \mathrm{C}_n \\ \hline \mathrm{A}_1 & x_{11}^k & x_{12}^k & \cdots & x_{1n}^k \\ \mathrm{A}_2 & x_{21}^k & x_{22}^k & \cdots & x_{2n}^k \\ \dots & \dots & \dots & \dots & \dots \\ \mathrm{A}_m & x_{m1}^k & x_{m2}^k & \cdots & x_{mn}^k \end{array}$$

where:

- $A_1, A_2, \dots, A_m$ are possible alternatives that decision makers have to choose from,
- $C_1, C_2, \dots, C_n$ are the criteria for which the alternative performance is measured,
- x_{ij}^k is the $k-$decision maker rating of alternative A_i with respect to the criterion C_j (x_{ij}^k is numerical, interval data or fuzzy number).

In this way for m alternatives and n criteria we have matrix $X^k = \left(x_{ij}^k\right)$ where x_{ij}^k is value of $i-$alternative with respect to $j-$criterion for $k-$ decision maker, $j = 1,2,\dots,n,\ k = 1,2,\dots,K$.

The relative importance of each criterion is given by a set of weights which are normalized to sum to one. Let us denote by $W^k = [w_1^k, w_2^k, \dots, w_n^k]$ a weight vector for $k-$decision maker, where $w_j^k \in \Re$ is the $k-$decision maker weight of criterion C_j and $w_1^k + w_2^k + \dots + w_n^k = 1$.

In the case of one decision maker we write x_{ij}, w_j, X, respectively.

Multi-criteria analysis focuses mainly on three types of decision problems: *choice* – select the most appropriate (best) alternative, *ranking* – draw a complete order of the alternatives from the best to the worst, and *sorting* – select the best k alternatives from the list.

2. The classical TOPSIS method

In the classical TOPSIS method we assume that the ratings of alternatives and weights are represented by numerical data and the problem is solved by a single decision maker. Complexity arises when there are more than one decision makers because the preferred solution must be agreed on by interest groups who usually have different goals. The classical TOPSIS algorithm for a single decision maker and for group decision making is systematically described in Section 2.1 and Section 2.2, respectively.

2.1. The classical TOPSIS method for a single decision maker

The idea of classical TOPSIS procedure can be expressed in a series of following steps [Chen, Hwang, 1992; Jahanshahloo, Lofti, Izadikhah, 2006a].

Step 1. Construct the decision matrix and determine the weight of criteria.
Let $X = (x_{ij})$ be a decision matrix and $W = [w_1, w_2, \dots, w_n]$ a weight vector, where $x_{ij} \in \Re$, $w_j \in \Re$ and $w_1 + w_2 + \dots + w_n = 1$.
Criteria of the functions can be: benefit functions (more is better) or cost functions (less is better).

Step 2. Calculate the normalized decision matrix.
This step transforms various attribute dimensions into non-dimensional attributes which allows comparisons across criteria. Because various criteria are usually measured in various units, the scores in the evaluation matrix X have to be transformed to a normalized scale. The normalization of values can be carried out by one of the several known standardized formulas. Some of the most frequently used methods of calculating the normalized value n_{ij} are the following:

$$n_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \tag{2.1}$$

$$n_{ij} = \frac{x_{ij}}{\max_i x_{ij}}, \tag{2.1*}$$

$$n_{ij} = \begin{cases} \dfrac{x_{ij} - \min\limits_i x_{ij}}{\max\limits_i x_{ij} - \min\limits_i x_{ij}} & \text{if } C_i \text{ is a benefit criterion} \\ \dfrac{\max\limits_i x_{ij} - x_{ij}}{\max\limits_i x_{ij} - \min\limits_i x_{ij}} & \text{if } C_i \text{ is a cost criterion} \end{cases} \tag{2.1**}$$

for $i = 1, \dots, m;\ j = 1, \dots, n$.

Step 3. Calculate the weighted normalized decision matrix.

The weighted normalized value v_{ij} is calculated in the following way:

$$v_{ij} = w_j n_{ij} \text{ for } i = 1, \dots, m; j = 1, \dots, n. \tag{2.2}$$

where w_j is the weight of the j-th criterion, $\sum_{j=1}^{n} w_j = 1$.

Step 4. Determine the positive ideal and negative ideal solutions.

Identify the positive ideal alternative (extreme performance on each criterion) and identify the negative ideal alternative (reverse extreme performance on each criterion). The ideal positive solution is the solution that maximizes the benefit criteria and minimizes the cost criteria whereas the negative ideal solution maximizes the cost criteria and minimizes the benefit criteria.

Positive ideal solution A^+ has the form:

$$A^+ = \left(v_1^+, v_2^+, \dots, v_n^+\right) = \left(\left(\max_i v_{ij} \mid j \in I\right), \left(\min_i v_{ij} \mid j \in J\right)\right). \tag{2.3}$$

Negative ideal solution A^- has the form:

$$A^- = \left(v_1^-, v_2^-, \dots, v_n^-\right) = \left(\left(\min_i v_{ij} \mid j \in I\right), \left(\max_i v_{ij} \mid j \in J\right)\right) \tag{2.4}$$

where I is associated with benefit criteria and J with the cost criteria, $i = 1, \dots, m;\ j = 1, \dots, n$.

Step 5. Calculate the separation measures from the positive ideal solution and the negative ideal solution.

In the TOPSIS method a number of distance metrics can be applied*.
The separation of each alternative from the positive ideal solution is given as

$$d_i^+ = \left(\sum_{j=1}^{n} \left(v_{ij} - v_j^+ \right)^p \right)^{1/p}, \quad i = 1,2, \dots , m. \tag{2.5}$$

The separation of each alternative from the negative ideal solution is given as

$$d_i^- = \left(\sum_{j=1}^{n} \left(v_{ij} - v_j^- \right)^p \right)^{1/p}, \quad i = 1,2, \dots , m. \tag{2.6}$$

Where $p \geq 1$. For $p = 2$ we have the most used traditional n-dimensional Euclidean metric.

$$d_i^+ = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^+ \right)^2}, \quad i = 1,2, \dots , m, \tag{2.5*}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^- \right)^2}, \quad i = 1,2, \dots , m. \tag{2.6*}$$

Step 6. Calculate the relative closeness to the positive ideal solution.

The relative closeness of the *i*-th alternative A_j with respect to A^+ is defined as

$$R_i = \frac{d_i^-}{d_i^- + d_i^+}, \tag{2.7}$$

where $0 \leq R_i \leq 1$, $i = 1,2, \dots , m$.

Step 7. Rank the preference order or select the alternative closest to 1.

A set of alternatives now can be ranked by the descending order of the value of R_i.

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* Possible metrics the first power metric (the least absolute value terms), Tchebychev metric or others [see Kahraman, Buyukozkan, Ates, 2007; Olson 2004].

2.2. The classical TOPSIS method for group decision making

In this part we explain the detailed TOPSIS procedure for group decision making based on the Shih, Shyur and Lee proposition [Shih, Shyur, Lee, 2007].

Step 1. Construct the decision matrixes and determine the weights of criteria for k-decision makers.

Let $X^k = \left(x_{ij}^k\right)$ be a decision matrix, $W^k = [w_1^k, w_2^k, \dots, w_n^k]$ weight vector for k – decision maker or expert, where $x_{ij}^k \in \Re$, $w_j^k \in \Re$, $w_1^k + w_2^k + \dots + w_n^k = 1$ for $k = 1, 2, \dots, K$.

Step 2. Calculate the normalized decision matrix for each decision maker.

In this step some of the earlier described methods of normalization can be used. Let us assume that we use

$$r_{ij}^k = \frac{x_{ij}^k}{\sqrt{\sum_{i=1}^{m} \left(x_{ij}^k\right)^2}}. \tag{2.8}$$

In this procedure weights are manipulated in the next step.

Step 3. Determine the positive ideal and negative ideal solutions for each decision maker.

The positive ideal solution A^{k+} for k – decision maker has the form

$$A^{k+} = \left\{r_1^{k+}, r_2^{k+}, \dots, r_n^{k+}\right\} = \left\{\left(\max_i(r_{ij}^k) \mid j \in I\right), \left(\min_i(r_{ij}^k) \mid j \in J\right)\right\}. \tag{2.9}$$

The negative ideal solution A^{k-} for k- decision maker has the form:

$$A^{k-} = \left\{r_1^{k-}, r_2^{k-}, \dots, r_n^{k-}\right\} = \left\{\left(\min_i(r_{ij}^k) \mid j \in I\right), \left(\max_i(r_{ij}^k) \mid j \in J\right)\right\}, \tag{2.10}$$

where I is associated with the benefit criteria and J with the cost criteria.

Step 4. Calculate the separation measures from the positive ideal solution and the negative ideal solution.

Step 5.1. Calculate the separatation measure for individuals.

The separation of i-th alternative A_i from the positive ideal solution A^{k+} for each k – decision maker is given as

$$d_i^{k+} = \left(\sum_{j=1}^{m} w_j^k \left(r_{ij}^k - r_j^{k+} \right)^p \right)^{1/p}, \quad i = 1,2, \dots , m. \tag{2.11}$$

The separation of i-th alternative A_i from the negative ideal solution A^{k-} for each k – decision maker is given as

$$d_i^{k-} = \left(\sum_{j=1}^{m} w_j^k \left(r_{ij}^k - r_j^{k-} \right)^p \right)^{1/p}, \quad i = 1,2, \dots , m, \tag{2.12}$$

where $p \geq 1$. For $p = 2$ we have the Euclidean metric.

Step 5.2. Calculate the separation measure for the group.

The aggregation for measure for the group measures of the positive ideal d_i^{*+} and negative ideal solution d_i^{*-} for the i-th alternative A_i is given by one of the operators:

arithmetic mean:

$$d_i^{*+} = \frac{\sum_{k=1}^{K} d_i^{k+}}{K} \quad \text{and} \quad d_i^{*-} = \frac{\sum_{k=1}^{K} d_i^{k-}}{K} \tag{2.13}$$

or

geometric mean:

$$d_i^{*+} = \sqrt[K]{\prod_{k=1}^{K} d_i^{k+}} \quad \text{and} \quad d_i^{*-} = \sqrt[K]{\prod_{k=1}^{K} d_i^{k-}}\,. \tag{2.13*}$$

Step 6. Calculate the relative closeness to the positive ideal solution.

The relative closeness of the alternative A_i to the positive ideal solution is defined as

$$R_i^* = \frac{d_i^{*-}}{d_i^{*-} + d_i^{*+}} \quad \text{for} \quad i = 1,2, \dots, m \tag{2.14}$$

where $0 \le R_i^* \le 1$.

The larger the index value, the better the evaluation of the alternative.

Step 7. Rank the preference order or select the alternative closest to 1.

A set of alternatives can now be ranked by the descending order of the value of R_i^*.

3. The TOPSIS method with criteria values determined as interval

In some cases determining the exact value of criteria is difficult and decision makers are usually more comfortable providing intervals to specify model input parameters. An interval number data formulation is a simple and intuitive way to represent uncertainty, which is typical of real decision problems. Here, the TOPSIS method using interval as the basis for evaluating value alternatives is described. However, we can also consider an interval weights description [Jadidi, Hong, Firouzi, Yusuff, 2008].

3.1. The TOPSIS method with attributed values determined as interval for a single decision maker

An algorithmic method which extends TOPSIS for decision-making problems with interval data was proposed by Jahanshahloo, Lofti, Izadikhah. This procedure can be described in the following steps [Jahanshahloo, Lofti, Izadikhah, 2006a].

Step 1. Construct the decision matrix and determine the weight of criteria.

Let $X = (x_{ij})$ be a decision matrix and $W = [w_1, w_2, \dots, w_n]$ a weight vector, where $x_{ij} = [\underline{x}_{ij}, \overline{x}_{ij}]$, $\underline{x}_{ij}, \overline{x}_{ij} \in \Re$, $w_j \in \Re$ and $w_1 + w_2 + \dots + w_n = 1$.

Step 2. Calculate the normalized interval decision matrix.

The normalized values $\underline{n}_{ij}, \overline{n}_{ij}$ are calculated in the following way:

$$\underline{n}_{ij} = \frac{\underline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \left((\underline{x}_{ij})^2 + (\overline{x}_{ij})^2 \right)}} \quad \text{for } i = 1, \dots, m; \; j = 1, \dots, n. \tag{3.1}$$

$$\overline{n}_{ij} = \frac{\overline{x}_{ij}}{\sqrt{\sum_{i=1}^{m}\left(\left(\underline{x}_{ij}\right)^2 + \left(\overline{x}_{ij}\right)^2\right)}} \text{ for } i = 1, \dots, m;\ j = 1, \dots, n. \tag{3.2}$$

The interval $\left[\underline{n}_{ij}, \overline{n}_{ij}\right]$ is normalized value of interval $\left[\underline{x}_{ij}, \overline{x}_{ij}\right]$.

Step 3. Calculate the weighted normalized interval decision matrix.

The weighted normalized values $\underline{v}_{ij}$ and $\overline{v}_{ij}$ are calculated in the following way:

$$\underline{v}_{ij} = w_j \underline{n}_{ij} \text{ for } i = 1, \dots, m;\ j = 1, \dots, n, \tag{3.3}$$

$$\overline{v}_{ij} = w_j \overline{n}_{ij} \text{ for } i = 1, \dots, m;\ j = 1, \dots, n, \tag{3.4}$$

where w_j is the weight of the j-th criterion, $\sum_{j=1}^{n} w_j = 1$.

Step 4. Determine the positive ideal and negative ideal solutions.

The positive ideal solution has the form A^+:

$$A^+ = \left(v_1^+, v_2^+, \dots, v_n^+\right) = \left(\left(\max_i \overline{v}_{ij} \mid j \in I\right), \left(\min_i \underline{v}_{ij} \mid j \in J\right)\right). \tag{3.5}$$

The negative ideal solution has the form A^-:

$$A^- = \left(v_1^-, v_2^-, \dots, v_n^-\right) = \left(\left(\min_i \underline{v}_{ij} \mid j \in I\right), \left(\max_i \overline{v}_{ij} \mid j \in J\right)\right), \tag{3.6}$$

where I is associated with benefit criteria and J with cost criteria.

Step 5. Calculate the separation measures from the positive ideal solution and the negative ideal solution.

The separation of each alternative from the positive ideal solution is given as*:

$$d_i^+ = \sqrt{\sum_{j=1}^{n}\left(\underline{v}_{ij} - v_j^+\right)^2 + \sum_{j=1}^{n}\left(\overline{v}_{ij} - v_j^+\right)^2}\ ,\ i = 1,2, \dots, m. \tag{3.7}$$

* Traditional TOPSIS applied to Euclidean norm is presented here. However, we can also use other metrics.

The separation of each alternative from the negative ideal solution is given as:

$$d_i^- = \sqrt{\sum_{j=1}^{n}\left(\bar{v}_{ij} - v_j^-\right)^2 + \sum_{j=1}^{n}\left(\underline{v}_{ij} - v_j^-\right)^2}\ ,\ i = 1,2,\ \dots\ , m. \tag{3.8}$$

Step 6. Calculate the relative closeness to the positive ideal solution.

The relative closeness of the alternative A_i with respect to A^+ is defined as:

$$R_i = \frac{d_i^-}{d_i^- + d_i^+}\ \text{ for }\ i = 1,2,\ \dots\ , m, \tag{3.9}$$

where $0 \le R_i \le 1.$

Step 7. Rank the preference order or select the alternative closest to 1.

The set of alternatives can now be ranked by the descending order of the value of R_i.

3.2. The TOPSIS method with attributed values determined as intervals for group decision making

We assume here that values are considered as intervals and we have group of k – decision makers [Zavadskas, Turskis, Tamosaitiene, 2008].

Step 1. Construct the decision matrixes and determine the weights of criteria for k – decision makers.

Let $X^k = \left(x_{ij}^k\right)$ be a decision matrix, $W^k = [w_1^k, w_2^k, \dots, w_n^k]$ is weight vector for k – decision maker, where $x_{ij}^k = \left[\underline{x}_{ij}^k, \bar{x}_{ij}^k\right]$, $\underline{x}_{ij}^k, \bar{x}_{ij}^k \in \Re$, $w_i \in \Re$, $w_1^k + w_2^k + \dots + w_n^k = 1$ for $k = 1,2, \dots, K$.

Step 2. Calculate the normalized grey decision matrixes.

This step transforms various attribute dimensions into non-dimensional attributes $r_{ij}^k = \left[\underline{r}_{ij}^k, \bar{r}_{ij}^k\right]$, which allows comparisons across criteria. The normalized values $\underline{r}_{ij}^k, \bar{r}_{ij}^k$ are calculated in the following way:

$$\underline{r}_{ij}^{k} = \frac{\underline{x}_{ij}^{k}}{\max_i(\overline{x}_{ij}^{k})}, \qquad \overline{r}_{ij}^{k} = \frac{\overline{x}_{ij}^{k}}{\max_i(\overline{x}_{ij}^{k})} \tag{3.10}$$

for $i = 1, \dots, m$; $j = 1, \dots, n$, $k = 1,2, \dots, K$.

Step 3. Determine the positive ideal and the negative ideal solutions for each decision maker.

The positive ideal solution A^{k+} for k-decision maker has the following form:

$$A^{k+} = \{r_1^{k+}, r_2^{k+}, \dots, r_n^{k+}\} = \left\{\left(\max_i(\overline{r}_{ij}^{k}) \mid j \in I\right), \left(\min_i(\underline{r}_{ij}^{k}) \mid j \in J\right)\right\}. \tag{3.11}$$

The negative ideal solution A^{k-} for k-decision maker has the form:

$$A^{k-} = \{r_1^{k-}, r_2^{k-}, \dots, r_n^{k-}\} = \left\{\left(\min_i(\underline{r}_{ij}^{k}) \mid j \in I\right), \left(\max_i(\overline{r}_{ij}^{k}) \mid j \in J\right)\right\}, \tag{3.12}$$

where I is associated with benefit criteria and J with cost criteria.

Step 4. Calculate the separation measures from the positive ideal solution and the negative ideal solution.

Step 5.1. Calculate the separation measure for individuals.

The separation of i-th alternative A_i from the positive ideal solution A^{k+} for each k – decision maker is given as

$$d_i^{k+} = \left(\frac{1}{2}\sum_{j=1}^{m} w_j^k \left(\left(r_j^{k+} - \underline{r}_{ij}^{k}\right)^p + \left(r_j^{k+} - \overline{r}_{ij}^{k}\right)^p\right)\right)^{1/p}, \quad i = 1,2, \dots, m. \tag{3.13}$$

The separation of i-th alternative A_i from the negative ideal solution A^{k-} for each k – decision maker is given as

$$d_i^{k-} = \left(\frac{1}{2}\sum_{j=1}^{m} w_j^k \left(\left(r_j^{k-} - \underline{r}_{ij}^{k}\right)^p + \left(r_j^{k-} - \overline{r}_{ij}^{k}\right)^p\right)\right)^{1/p}, \quad i = 1,2, \dots, m. \tag{3.14}$$

If p = 2, then the metric is a weighted grey number Euclidean distance function. Equations (3.13) and (3.14) will be as follows:

$$d_i^{k+} = \sqrt{\frac{1}{2}\sum_{j=1}^{m} w_j^k\left(\left(r_j^{k+} - \underline{r}_{ij}^{k}\right)^2 + \left(r_j^{k+} - \overline{r}_{ij}^{k}\right)^2\right)},\ i = 1,2,\ \dots\ , m. \qquad (3.13^*)$$

The separation of *i*-th alternative A_i from the negative ideal solution A^{k-} is given as

$$d_i^{k-} = \sqrt{\frac{1}{2}\sum_{j=1}^{m} w_j^k\left(\left(r_j^{k-} - \underline{r}_{ij}^{k}\right)^2 + \left(r_j^{k-} - \overline{r}_{ij}^{k}\right)^2\right)},\quad i = 1,2,\ \dots\ , m. \qquad (3.14^*)$$

Step 5.2. Calculate the separation measure for the group.

The aggregation of the measure for the group measures of the positive ideal d_i^{*+} and the negative ideal solution d_i^{*-} for the *i*-th alternative A_i is given by:

arithmetic mean:

$$d_i^{*+} = \frac{\sum_{k=1}^{K} d_i^{k+}}{K} \quad \text{and} \quad d_i^{*-} = \frac{\sum_{k=1}^{K} d_i^{k-}}{K}, \qquad (3.15)$$

or

geometric mean:

$$d_i^{*+} = \sqrt[K]{\prod_{k=1}^{K} d_i^{k+}} \quad \text{and} \quad d_i^{*-} = \sqrt[K]{\prod_{k=1}^{K} d_i^{k-}}. \qquad (3.15^*)$$

Step 6. Calculate the relative closeness to the positive ideal solution.

The relative closeness of the alternative A_i with respect to A^+ is defined as

$$R_i^* = \frac{d_i^{*-}}{d_i^{*-} + d_i^{*+}} \quad \text{for } i = 1,2,\ \dots\ ,\ m, \qquad (3.16)$$

where $0 \le R_i^* \le 1$.

The larger the index value, the better the evaluation of alternative.

Step 7. Rank the preference order or select the alternative closest to 1.

The set of alternatives can now be ranked by the descending order of the value of R_i^*.

4. The quantitative and qualitative criteria in the TOPSIS method. Weights expressed by linguistic variable

In the TOPSIS algorithm the quantitative criteria are scaled using their own real numbers and for representation of the imprecision of spatial data, and human cognition over the criteria of the theory of linguistic variables is used. A linguistic variable is a variable where values are words or sentences in a natural or artificial language. Especially, since traditional quantification methods are difficult to describe situations that are overtly complex or hard to describe, the notion of a linguistic variable is necessary and useful. We can use this kind of expression for rating qualitative criteria as well as to compare two evaluation criteria.

The qualitative criterion can be described using linguistic variables; next the criteria ratings on the 1-9 number scale (Table 1) or on the 1-7 interval scale (Table 2) can be provided, respectively [Jadidi, Hong, Firouzi, Yusuff, Zulkifli, 2008].

Table 1

The scale of alternative ratings for qualitative criterion in the case of classical TOPSIS method

Scale	Rating
Poor (P)	1
Medium poor (MP)	3
Fair (F)	5
Medium good (MG)	7
Good (G)	9
Intermediate values between the two adjacent judgments	2,4,6,8

Table 2

The scale of alternative ratings in the case of interval TOPSIS method

Scale	Rating
Very Poor (VP)	[0,1]
Poor (P)	[1,3]
Medium poor (MP)	[3,4]
Fair (F)	[4,5]
Medium good (MG)	[5,6]
Good (G)	[6,9]
Very good (VG)	[9,10]

Each decision maker individually uses linguistic variables transformed for numerical scale (Table 1) or interval scale (Table 2) to identify the alternative rankings for the subjective criterion. Then the rating value for group decision makers can be calculated using the following formula

$$x_{ij} = \frac{1}{K}\left[x_{ij}^1 + x_{ij}^2 + ... + x_{ij}^K\right], \tag{4.1}$$

where:

$x_{ij}^s -$ is the rating value of alternative A_i with the respect to quantitative criterion C_j (crisp or interval) of $s-$decision maker ($i=1,2,\ldots,n;$ $j=1,2,\ldots,m;$ $s=1,2,\ldots,K$).

In this way for m alternatives and n criteria and $K-$decision makers we can obtain one aggregated matrix $X=(x_{ij})$ where x_{ij} is value of $i-$alternative with respect to $j-$criterion for $i=1,2,\ldots,m;$ $j=1,2,\ldots,n.$

The weights of the factors are subjective input, which directly influences the final evaluated result. In the evaluating system, the importance of every index is, in general, not equal, so they must be set different weight factors. Among many ways to set the weight factors are for instance the Delphi method or the AHP method [Olson, 2004]. The Delphi method is the most popular expert evaluating technique. The Delphi method is a forecasting and evaluating method both qualitative and quantitative which collects experts' ideas anonymously, exchanges and corrects this information many times to reach a consistent idea and gives the subject a final evaluation according to the experts' ideal. AHP (The Analytical Hierarchy Process) uses a hierarchical structure and pairwise comparisons. An AHP hierarchy has at least three levels: the main objective of the problem at the top, multiple criteria that define alternatives in the middle and competing alternatives at the bottom. The AHP method uses system analysis and continuously decomposes the evaluating indices according to the main evaluating indices of every level [Saaty, 1980]. The classical TOPSIS method does not consider a hierarchical structure consisting of main attributes and subattributes. This method evaluates the alternatives with respect to main attributes only with a single level. The common property of these methods is their ease of implementation, so this method is often used to obtain weight criteria.

In the case where the criterion weights are linguistic variables, the weights can be expressed by the 1-9 scale shown in Table 3 [Jadidi, Hong, Firouzi, Yusuff, Zulkifli, 2008].

Table 3

The scale of criterion weights

Scale	Weight
Very very low (VVL)	0,005
Very low (VL)	0,125
Low (L)	0,175
Medium low (ML)	0,225
Medium (M)	0,275
Medium hight (MH)	0,325
Hight (H)	0,375
Very Hight (VH)	0,425
Very Very Hight (VVH)	0,475

The vector of attribute weights must sum up to 1; if not, it is normalized. Each decision maker individually uses linguistic variables (Table 3) to identify the criterion weights. Then the criterion weights for all decision makers can be calculated using the following formula

$$w_r = \frac{1}{K}\left[w_r^1 + w_r^2 + \ldots + w_r^K\right], \qquad (4.2)$$

where:

w_r^s – is the weight of r – criterion for s – decision makers ($r = 1,2, \ldots, n$; $s = 1,2, \ldots, K$).

5. Practical application

In this section, to demonstrate the calculation process of the approaches described, two examples are provided.

Example 1.

A firm intends to choose the best offer (or ranking of the offers) from the set of proposals submitted by potential contractors. Two experts evaluate five proposals using several criteria. In order to simplify the calculation, only four criteria are considered: deadline of payment after receipt the goods (in days), unitary price (in euro), conditions of warranty and contractor reputation, C_1, C_2, C_3, C_4, respectively. The criteria C_1, C_3, C_4 are benefit criteria, the greater values being better, and C_2 is the cost criterion, the smaller values are better. Criteria C_3, C_4 are subjectively evaluated by experts on basic available information and they are considered as linguistic variables, while the other criteria are scaled using their own real numbers, respectively. This is shown in Table 4. Based on Table 1, the decision matrixes for two decision makers are obtained (Table 5).

Table 4

Criteria rating values for two decision makers

		C_1	C_2	C_3	C_4
D_1					
	A_1	7	21	F&MP	MG
	A_2	7	24	MG&F	F&MP
	A_3	14	25	MP	G& MG
	A_4	14	26	G	MP
	A_5	21	35	MP &F	F& MP
D_2					
	A_1	7	21	F&MP	MG&F
	A_2	7	24	MG	F
	A_3	14	25	MP&P	G &MG
	A_4	14	26	MG	MP&P
	A_5	21	35	MP	MP

Table 5

Decision matrixes for two decision makers

		C_1	C_2	C_3	C_4
D_1					
	A_1	7	21	4	7
	A_2	7	24	6	4
	A_3	14	25	3	8
	A_4	14	26	9	3
	A_5	21	35	4	4
D_2					
	A_1	7	21	4	6
	A_2	7	24	7	5
	A_3	14	25	2	8
	A_4	14	26	7	3
	A_5	21	35	3	3

Based on linguistic variables, the evaluation values of attribute weights for the first and second decision makers can be obtained and the results are shown in Table 6. The normalized criteria weights for each decision maker obtained using Table 3 are shown in Table 7.

Table 6

Criteria weights for two decision makers

	C_1	C_2	C_3	C_4
D_1				
	L	VH	L	ML
D_2				
	ML	VVH	VL	L

Table 7

Normalized criteria weights for two decision makers

	C_1	C_2	C_3	C_4
D_1				
	0,175	0,425	0,175	0,225
D_2				
	0,225	0,475	0,125	0,175

CASE 1. Rank the preference order for individual decision makers

Using formulas 2.1-2.7 the calculation results on data from Table 5 and Table 7 and rank order for each decision maker are shown in Table 8 and Table 9, respectively.

Table 8

Calculation results for 1-decision maker

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.107805	0.124281	0.535497	**2**
A_2	0.117942	0.090785	0.434945	**4**
A_3	0.096981	0.122181	0.557491	**1**
A_4	0.105379	0.112779	0.516961	**3**
A_5	0.141766	0.083486	0.370633	**5**

Table 9

Calculation results for 2-decision maker

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.112485	0.128548	0.533321	**2**
A_2	0.115014	0.113042	0.495676	**3**
A_3	0.082214	0.130468	0.613440	**1**
A_4	0.110660	0.104396	0.485436	**4**
A_5	0.141415	0.104894	0.425865	**5**

CASE 2. Rank the preference order for group decision makers (1 method)

The decision matrix is calculated using formula (4.1) and attributes weights of the criteria using (4.2). The results are shown in Table 10 and Table 11, respectively.

Table 10

Decision matrix for group decision makers

	C_1	**C_2**	**C_3**	**C_4**
A_1	7	21	4	6,5
A_2	7	24	6,5	4,5
A_3	14	25	2,5	8
A_4	14	26	8	2,5
A_5	21	35	3,5	3,5

Table 11

Normalized criteria for group decision makers

C_1	**C_2**	**C_3**	**C_4**
0.20	0.45	0.15	0.20

Using formulas 2.1-2.7 the calculation results on data based on Table 10 and Table 11 and rank order for group decision making are shown in Table 12.

Table 12

Calculation results for group decision making (1 method)

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.107701	0.126396	0.539930	**2**
A_2	0.112581	0.102844	0.477401	**4**
A_3	0.088631	0.127144	0.589244	**1**
A_4	0.108991	0.107566	0.496711	**3**
A_5	0.141512	0.094110	0.399412	**5**

CASE 3. Rank the preference order for group decision makers (2 method)

Using formulas 2.8-2.14 in the case of the Euclidean metric ($p = 2$) and arithmetic mean (formula 2.13) the calculation results on data based on Table 5 and Table 7 and rank order for group decision making are shown in Table 13.

Table 13

Calculation results

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.250522	0.222860	0.470783	**3**
A_2	0.253325	0.199743	0.440868	**4**
A_3	0.210667	0.255295	0.547889	**1**
A_4	0.235766	0.229436	0.493196	**2**
A_5	0.273169	0.211055	0.435863	**5**

Remark 1. Let us observe that we obtain different rank order in Case 2 and Case 3.

Example 2.

A firm intends to choose the best offer (or ranking of the offers) from the set of three proposals submitted by potential contractors. As in example 1, two experts evaluate each proposal using the same four criteria: deadline of payment after receipt of the goods (in days), unitary price (in euro), conditions of warranty and contractor's reputation, C_1, C_2, C_3, C_4 respectively. The criteria C_1, C_3, C_4 are benefit criteria, greater values being better, and C_2 is the cost criterion, smaller values being better. Criteria C_3, C_4 are subjectively evaluated by the experts on basic available information and they are considered now as linguistic variables, and the other criteria are scaled using interval data, respectively. This is shown in Table 14.

Table 14

The interval decision matrix for two decision makers

		C_1		C_2		C_3	C_4
		$\underline{x}_{i1}^{1}$	$\overline{x}_{i1}^{1}$	$\underline{x}_{i2}^{1}$	$\overline{x}_{i2}^{1}$		
D_1							
	A_1	0	7	20	22	P	P
	A_2	7	14	22	24	G	MG
	A_3	14	21	24	26	MP	F
		$\underline{x}_{i1}^{2}$	$\overline{x}_{i1}^{2}$	$\underline{x}_{i2}^{2}$	$\overline{x}_{i2}^{2}$		
D_2							
	A_1	0	7	20	22	P	MP
	A_2	7	14	22	24	MP	P
	A_3	14	21	24	26	MP	MP

Based on Table 2, the decision matrixes of two decision makers are obtained (Table 15).

Table 15

The interval decision matrix for two decision makers

		C_1		C_2		C_3		C_4	
		$\underline{x}_{i1}^{1}$	$\overline{x}_{i1}^{1}$	$\underline{x}_{i2}^{1}$	$\overline{x}_{i2}^{1}$	$\underline{x}_{i3}^{1}$	$\overline{x}_{i3}^{1}$	$\underline{x}_{i4}^{1}$	$\overline{x}_{i4}^{1}$
D_1									
	A_1	0	7	20	22	1	3	1	3
	A_2	7	14	22	24	6	9	5	6
	A_3	14	21	24	26	3	4	4	5

Table 15 contd.

	$\underline{x}_{i1}^{2}$	$\overline{x}_{i1}^{2}$	$\underline{x}_{i2}^{2}$	$\overline{x}_{i2}^{2}$	$\underline{x}_{i3}^{2}$	$\overline{x}_{i3}^{2}$	$\underline{x}_{i4}^{2}$	$\overline{x}_{i4}^{2}$
D_2								
A_1	0	7	20	22	1	3	3	4
A_2	7	14	22	24	3	4	1	3
A_3	14	21	24	26	3	4	3	4

Based on linguistic variables the evaluation values of attribute weight for each decision maker can be obtained and the results are shown in Table 16.

Table 16

Criteria weights for two decision makers

	C_1	**C_2**	**C_3**	**C_4**
D_1				
	ML	VVH	M	L
D_2				
	ML	VVH	VL	L

The normalized criteria weights for two decision makers are shown in Table 17.

Table 17

Normalized criterion weights for two decision makers

	C_1	**C_2**	**C_3**	**C_4**
D_1				
	0.196	0.413	0.239	0.152
D_2				
	0.225	0.475	0.125	0.175

CASE 1: Rank the preference order for individual decision makers

Using formulas 3.1-3.9 the calculation results on data from Table 15 and Table 17 and rank order for each decision maker are shown in Table 18 and Table 19, respectively.

Table 18

Calculation results for 1-decision maker

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.266630	0.084412	0.240462	**3**
A_2	0.121513	0.230392	0.654700	**1**
A_3	0.169564	0.191156	0.529928	**2**

Table 19

Calculation results for 2-decision maker

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.195026	0.118375	0.377710	**3**
A_2	0.141770	0.141991	0.500388	**2**
A_3	0.084299	0.211994	0.715488	**1**

CASE 2. Rank the preference order for group decision makers (1 method)

The decision matrix is calculated using formula (4.1) and attributes weights of the criteria using (4.2). The results are shown in the Table 20 and the Table 21, respectively.

Table 20

Decision table for group decision making

	C_1		C_2		C_3		C_4	
	$\underline{x}_{i1}^{1}$	$\overline{x}_{i1}^{1}$	$\underline{x}_{i2}^{1}$	$\overline{x}_{i2}^{1}$	$\underline{x}_{i3}^{1}$	$\overline{x}_{i3}^{1}$	$\underline{x}_{i4}^{1}$	$\overline{x}_{i4}^{1}$
A_1	0	7	20	22	1	3	2	3,5
A_2	7	14	22	24	4,5	6,5	3	4,5
A_3	14	21	24	26	3	4	3,5	4,5

Table 21

Normalized criteria weights for group decision makers

C_1	C_2	C_3	C_4
0.2105	0.4440	0.1820	0.1635

Using formulas 3.1-3.9 the calculation results on data based on Table 20 and Table 21 and rank order for group decision making are shown in Table 22.

Table 22

Calculation results for group decision making (1 method)

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.217917	0.087577	0.286673	**3**
A_2	0.122574	0.172678	0.584850	**2**
A_3	0.110306	0.194664	0.638305	**1**

CASE 3. Rank the preference order for group decision making (2 method)

Using formulas 3.10-3.16 in the case of the Euclidean metric (formula 3.13*-3.14*) and arithmetic mean (3.15) the calculations results on data based on Table 15 and Table 17 and rank order for group decision making are shown in Table 23.

Table 23

Calculation results for group decision making (2 method)

Alternatives	The separation of alternatives to positive ideal one	The separation of alternatives to negative ideal one	The relative closeness of alternatives to the positive ideal one	Rank
A_1	0.224004	0.125446	0.358981	**3**
A_2	0.125382	0.223736	0.640861	**2**
A_3	0.173915	0.322996	0.650008	**1**

Remark 2. Let us observe that we obtain the same rank order in Case 2 and Case 3.

Remark 3. The TOPSIS method presents a universal methodology and a simplified practical model for ordering and choosing offers in buyer-seller exchange. This indicator system and evaluation model can be used widely in the area of bargaining process which is usually complex and uncertain. Negotiators have to consider qualitative issues such as price, time of payments, as well as quantitative ones such as reputation, power of negotiation, relationships between sides and so on. Moreover, human thinking is imprecise, lack of information, imprecision and evaluations are always restricted by some objective factors. The concept of the TOPSIS method is clear, the calculation is simple and convenient and the methodology can be extended and adjusted to specific environments. According to the TOPSIS analysis results, a negotiator can choose the most effective alternative that is possible to implement. The decision maker's evaluation could be based on linguistic variables, crisp or interval data. The example of the practical application proves that this method is efficient and feasible.

Concluding remarks

There are a variety of multiple criteria techniques to aid selection in conditions of multiple-criteria problems. One of them is the TOPSIS method, where the ranking of alternatives is based on the relative similarity to the ideal solution, which avoids the situation of having the same similarity index to both positive ideal and negative ideal solutions.

The TOPSIS method is a practical and useful technique for ranking and selecting alternatives. In this paper we focused mainly on the concept of the TOPSIS algorithm for crisp and interval data. An extension of the TOPSIS technique to a group decision environment was also investigated.

The high flexibility of the TOPSIS concept is able to accommodate further extensions to make best choices in various situations. Practically, TOPSIS and its modifications are used to solve many theoretical and real-world problems. In addition, the preferences of more than one decision makers can be also aggregated into the TOPSIS procedure. The classical TOPSIS have been extended according to the requirements of different real-world decision making problems providing support for interval or fuzzy criteria, interval or fuzzy weights to modeled imprecision, uncertainty, lack of information or vagueness, such as TOPSIS with interval data, Fuzzy TOPSIS, Fuzzy AHP and TOPSIS and group TOPSIS.

In the TOPSIS model based on the theory of fuzzy sets the rating of each alternative is expressed in triangular or trapezoidal fuzzy numbers, the weight of each criterion is represented by fuzzy or crisp values, and different normalization (for instance Euclidean, linear or others) are used*. The normalized fuzzy numbers can be calculated by using the concept of α-*cuts* [Jahanshaholoo, Lofti, Izadikhah, 2006b]. The TOPSIS model based on the intuitionistic fuzzy set (IFS) allows also to measure the degree of satisfiability and the degree of non-satisfiability, respectively, of each alternative evaluated across a set of criteria [Hung, Chen, 2009; Saghafian, Hejazi, 2005]. The hierarchical TOPSIS method is developed to benefit both from the superiority of the hierarchical structure of AHP and ease of implementation of TOPSIS method [Kahraman, Buyukozkan, Ates 2007; Chiang, Cheng, 2009].

In Polish literature, among many applications, the TOPSIS method (to rank objects) and analytical hierarchy process (to calculate weight of criteria) was employed to assess the socioeconomic development of rural Wielkopolska seen as a collection of counties [Łuczak, Wysocki, 2006], the fuzzy TOPSIS method based on α-level sets was employed to assess the level of people life in chosen counties in Wielkopolska Province [Łuczak, Wysocki, 2008], TOPSIS methods for crisp and interval data were used for ordering offers in buyer-seller transactions [Roszkowska 2009].

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