INTRODUCTION TO MULTICRITERIA DECISION ANALYSIS (MCDA)

04 – Fuzzy Analytical Hierarchical Process (FAHP)

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SNEAK PEEK

Today we are going to explore the most famous method in MCDA (probably) – the Analytical Hierarchical Process (AHP).

I hope you had time to read the corresponding materials, and let’s dive in!

TABLE OF CONTENTS

Recap

- As is the set of alternatives ($A = \{a_1, a_2, \dots, a_m\}$).
- These alternatives can be analyzed through a set of orderable criteria X_g: $\left(<_g, X_g\right)$.
- Each criterion $g(a)$ may be used in a quantitative way to inform us about its importance
- Criteria can thus be compared - $g(a) > g(b)$.
- Comparisons can take many forms, according to each method (dominance): preference (strict or pure versus weak), indifference, or incomparability.
- In most methods, we can also gauge the intensity or degree of preference.
- Given that $g_1(a), \dots, g_n(a)$ and $g_1(b), \dots, g_n(b)$, a logic of aggregation will somehow compare both alternatives
- Other inter-criterion and technical parameters (weights, scales, constraints, etc.) can also be parts of the method.

Recap - AHP

- As is the set of alternatives ($A = \{a_1, a_2, \dots, a_m\}$).
- The ratio between criteria ($C_1$and C_2) or alternatives (A_1 and A_2) is inferred from the Saaty Scale (1 = equal value or $C_1 = C_2$; 9 = absolute difference, or $C_1$is absolutely $> C_2$).
- The reciprocals (or inverse) of any C_1, C_2 ratio is ${}^{1}/_{r}$ with r being the ratio in the Saaty Scale.
- In its simplest (but not classical) form by averaging and normalizing each row (and there are more than one way of doing these two steps).
- Alternatives and criteria are non-distinguishable in the calculations, alternatives being the bottom level in the hierarchy.
- As such alternatives are optional.
- This method allows ranking of alternatives, and obtaining ranking / weights of criteria.

FUZZY ANALYTICAL HIERARCHICAL PROCESS (FAHP)

A nice extension to AHP

What does FAHP stand for?

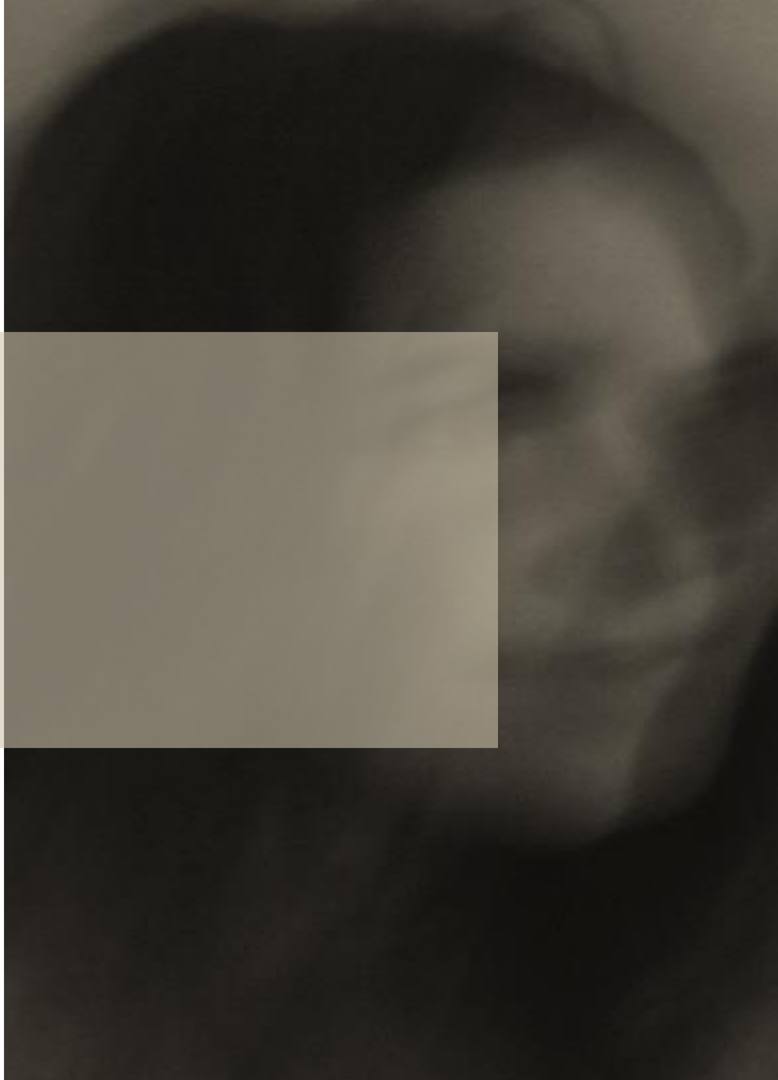
FAHP stands for ***Fuzzy*** Analytic Hierarchy Process.

Here we are going to learn the **Chang Extent Analysis Method** (as there are other FAHP approaches).

It is an extension of the traditional AHP method, but with a twist – **changing the base of calculations from real (usually discrete/continuous) numbers to fuzzy numbers**.

- Discrete (1, 2, 3, 4, ….)
- Continuous (1.567, 3.917, 9.348, …)

But what are fuzzy numbers?



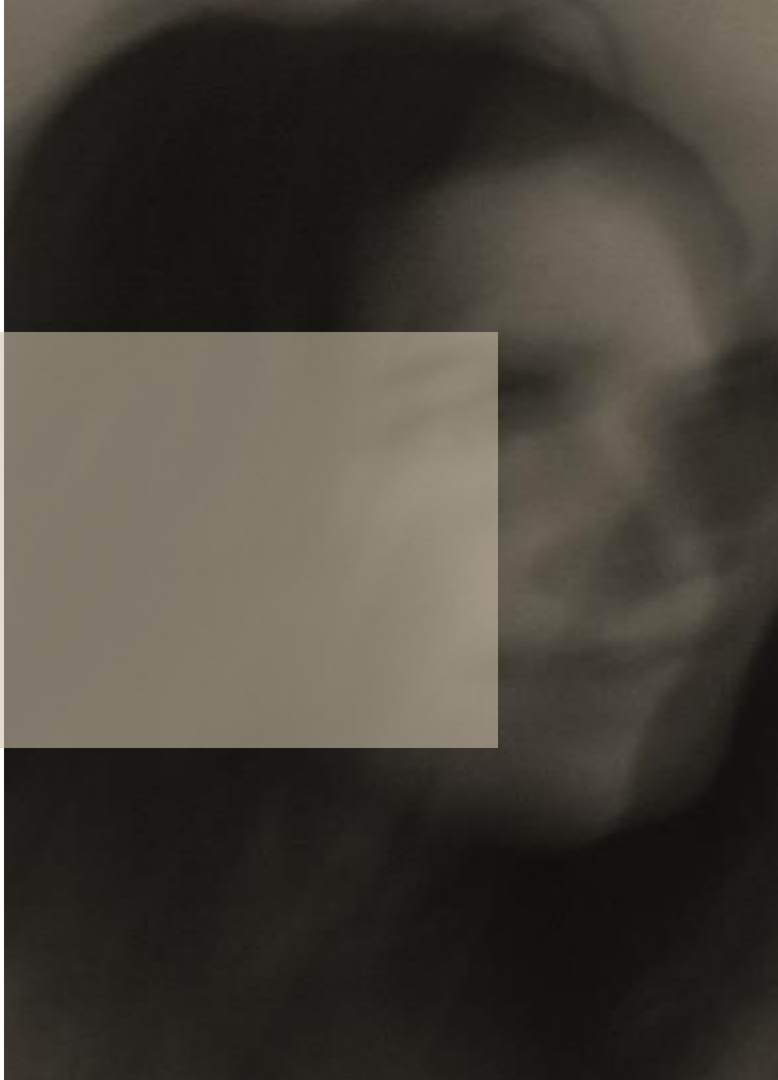
Fuzzy numbers

Fuzzy numbers are a generalization of real numbers, **but accounting for uncertainty and vagueness**.

Let's compare them to real numbers:

- A number "4" is precise in the sense that 3.9999999 is not 4 and 4.0000001 is also not 4.
- But *in practice* we could *assume* they are equal to 4 because they are within a range of *acceptability*.
- This is wrong in math (all of them are, indeed, different numbers), but from an engineering perspective it could be understood as "tolerance", "margin of error", etc.

In a way, fuzzy numbers allow numbers to be more than "one-dimensional", and to accept a range of acceptable values.



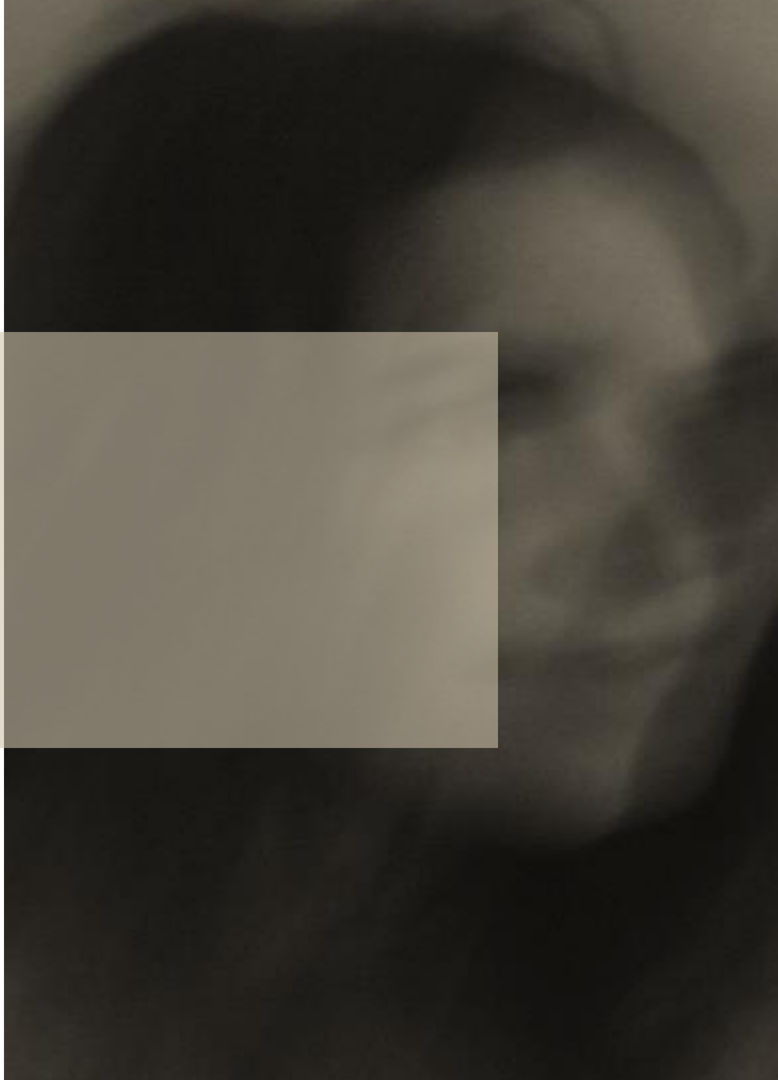
Fuzzy numbers

A fuzzy number M is an element of $F(R)$ - i.e., M is a fuzzy set over the real number R.

Definition 1

$x_o \in R$ exists such that $\mu_M(x_o) = 1$, i.e.,

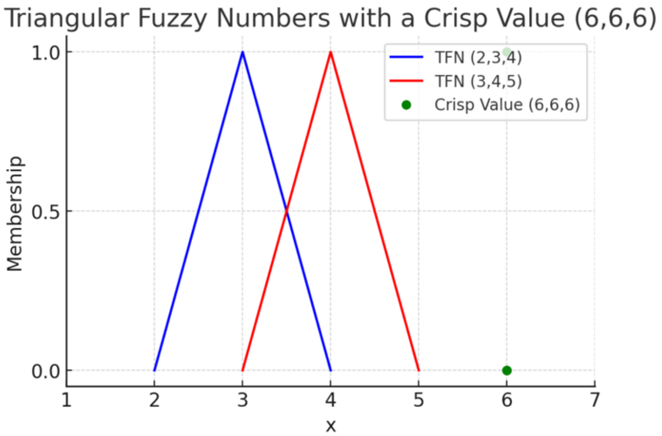
- This means that there is a specific point x_o where the **membership function** $\mu_M(x)$ reaches **its maximum value of 1**.
- In simple terms, **there is (at least) one most “true” value for the fuzzy number**.

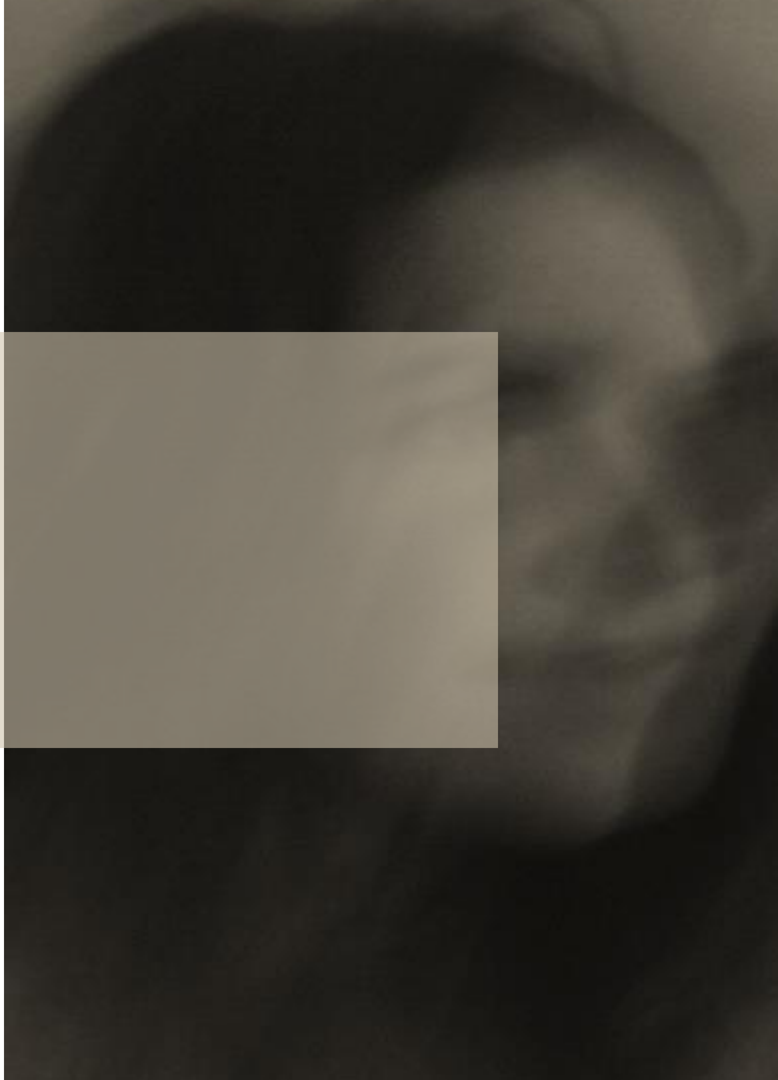


Fuzzy numbers

This means that for each real number, there is at least one point where it is optimally member of the fuzzy set.

- The real number 6 can be understood as a fuzzy number 6, 6, 6 (i.e., 6 ± 0).
- The real number 4 can be understood as a fuzzy number ranging from values under and over 4 (here, 4 ± 1).





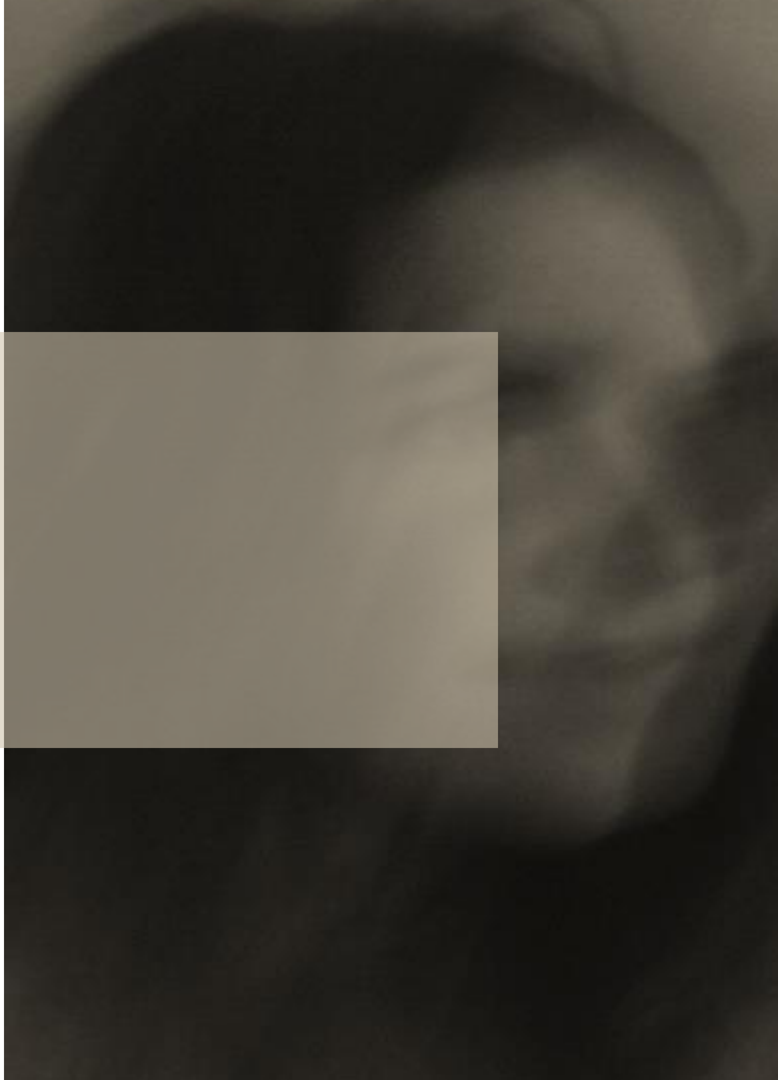
Fuzzy numbers

This simple explanation serves to introduce the notion of ***membership***, i.e, while a fuzzy number typically refers to a range, the numbers in the range vary in terms of membership to the real number to which they refer to.

Since membership varies from 0 – 1, a fuzzy number 2, 3, 4, refering to the real number 3 has the maximum membership (it reaches 1 at the membership function) at the peak of 3 (i.e., there is is the closer to the real mccoy).

This μ function (membership) is an indicator of how much a real number belongs to the fuzzy number $[0, 1]$.

- $\mu_M(x) = 1$ it means it is fully valid
- $\mu_M(x) = 0.5$ it means it is partially valid
- $\mu_M(x) = 0$ it means it is fully outside the fuzzy range



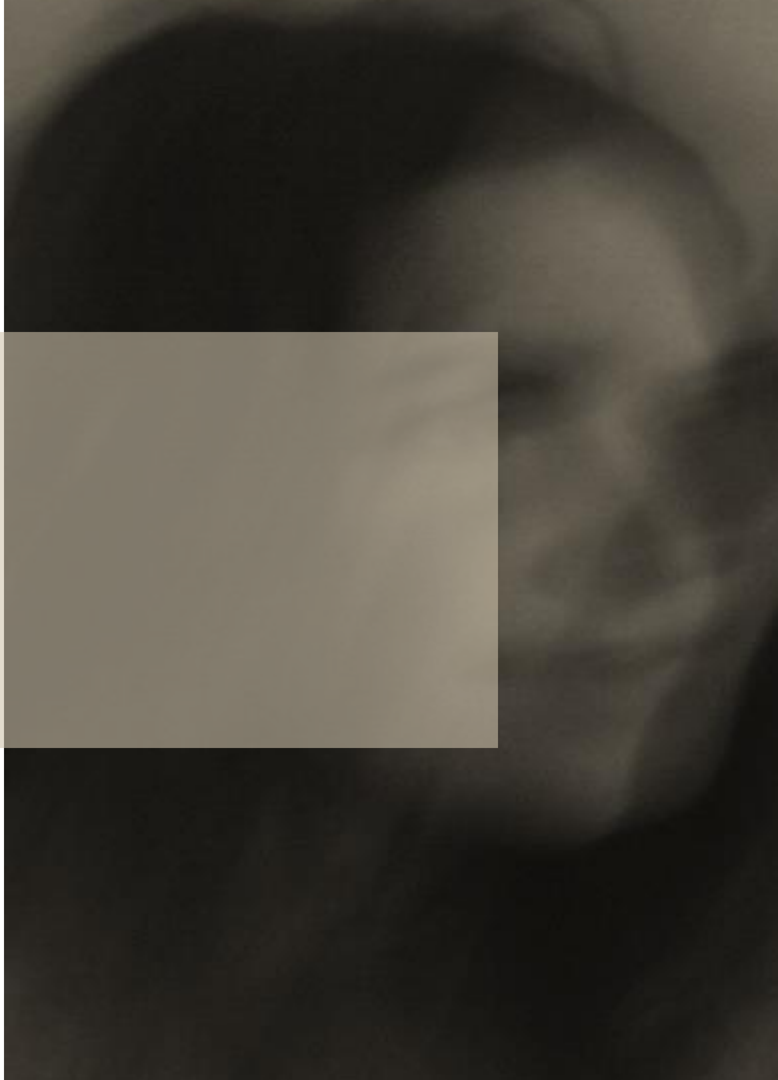
Fuzzy numbers

Let’s imagine we are programmers and we are tasked to define the range of temperatures for air conditioners.

We need to use verbal descriptors to match the actual degrees in the physical AC system.

What is “hot”?

- Given that the AC can produce temperatures from 16 to 30 how would you define hot?
- Warm could be a a fuzzy number ranging from 25º C to the maximum 30º C.
- However, here in Brazil 25º C is kind of starting to become hot but it really is hot to our perception when it gets closer to 30º C.
- Thus, all real numbers from 25-30 can be considered hot, however the nearer they are to 30 the more they truly “belong” to “hot”.



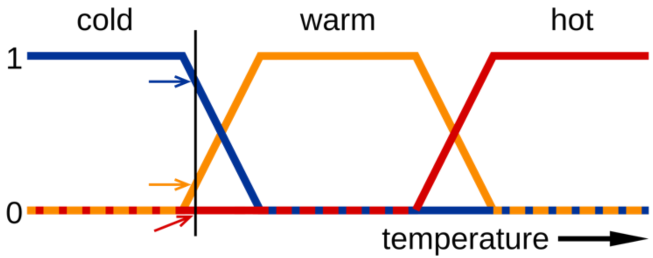
Fuzzy numbers

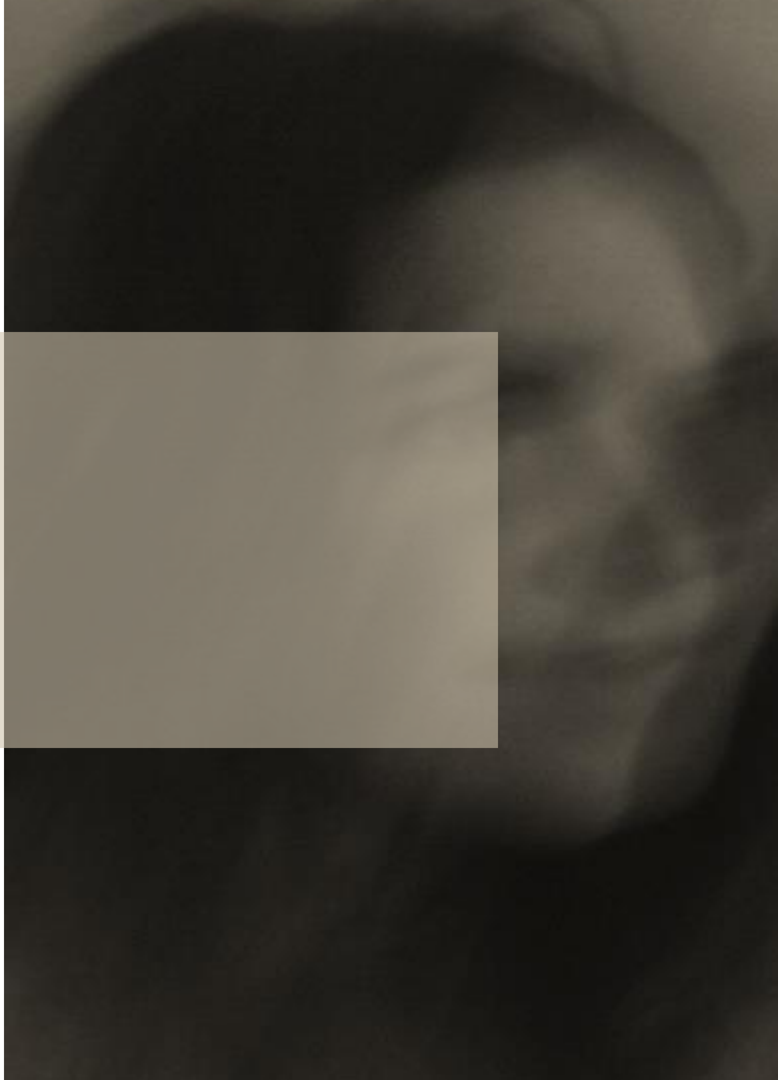
For any $\alpha \in [0\,,1]$, the α-cut A_α forms a closed interval

• The α-cut represents the set of values where the membership function is greater than or equal to A_α :

$$A_\alpha = \; [x, \mu_A(x) \geq \alpha]$$

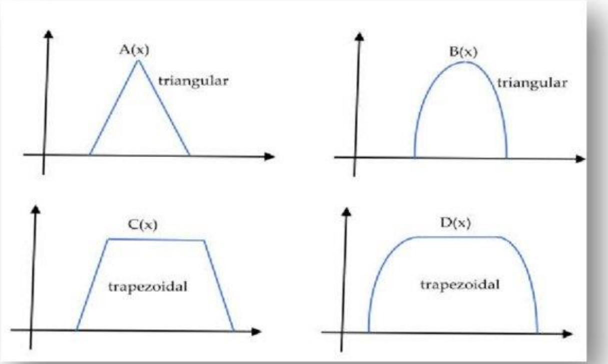
- This ensures that **fuzzy numbers behave in a structured way**, forming intervals rather than arbitrary sets.
- Eg.: in the AC case, “warm” ranges from 25-30 but it could go on forever, as long as for each real value is greater or equal to the previous one: 25 → 25.5 → 27 → 30 → 120



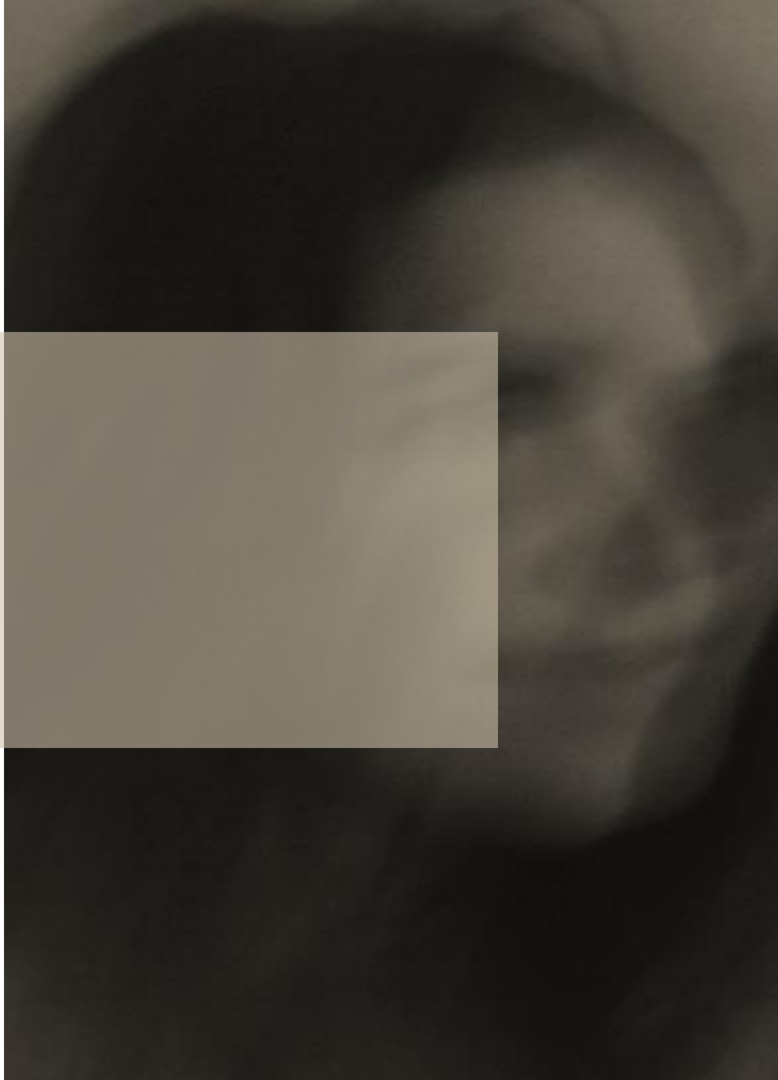


Fuzzy numbers

So far, we have seen **Triangular Fuzzy Numbers (TFNs)** and in the last slide trapezoidal ones, but fuzzy numbers can take several shapes (after all, they are functions or distributions):



We are going to use TFNs only today.



Fuzzy numbers

Definition 2

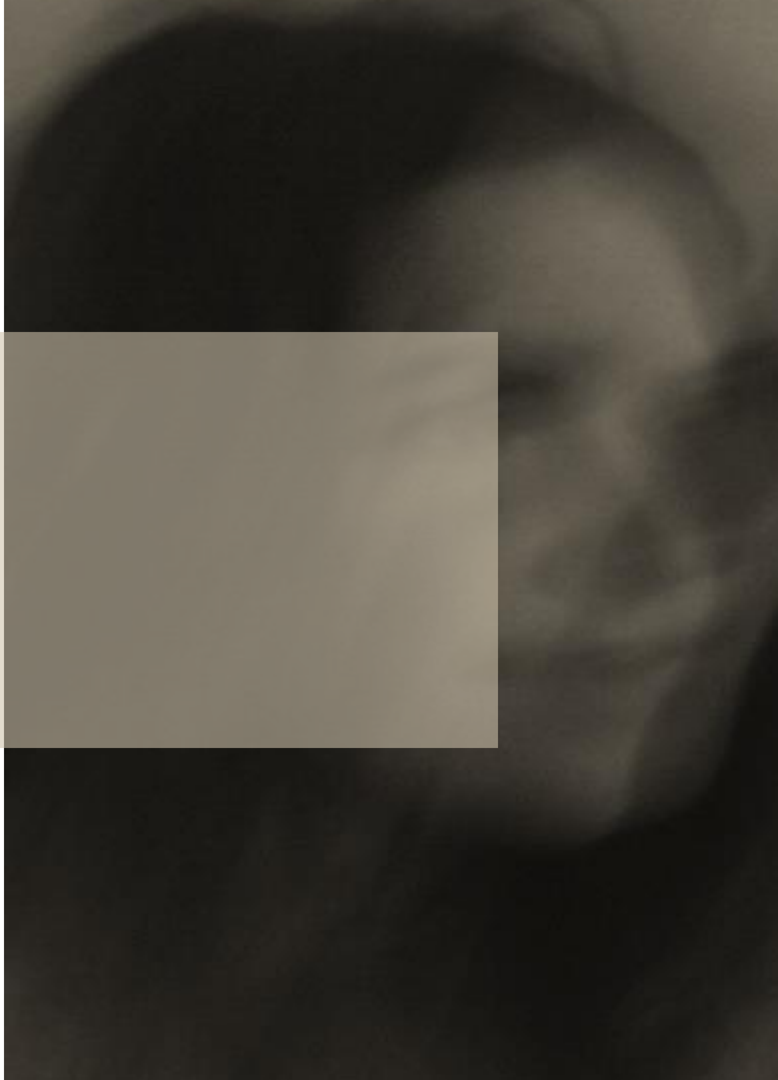
A triangular fuzzy number M is defined by three values:

$$M = (l, m, u)$$

where:

- l (**lower bound**): The smallest possible value in the fuzzy set.
- m (**mode**): The most representative or “true” value, where $\mu_M(x) = 1$.
- u (**upper bound**): The largest possible value in the fuzzy set.

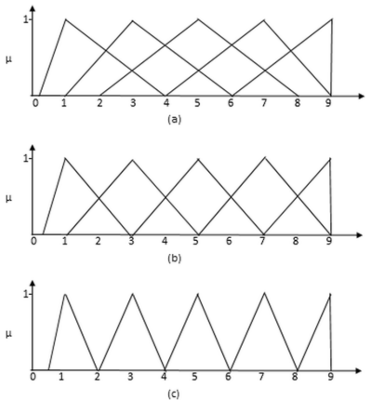
These values must be monotonic, i.e., $l \leq m \leq u$.
This makes real numbers a subcase of fuzzy numbers (as $(3, 3, 3) = 3$

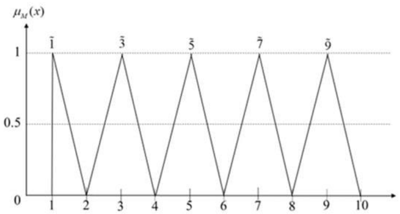


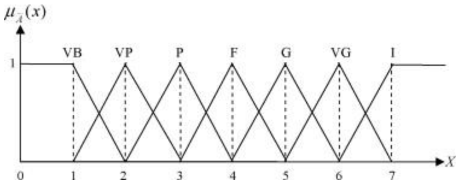
Fuzzy numbers

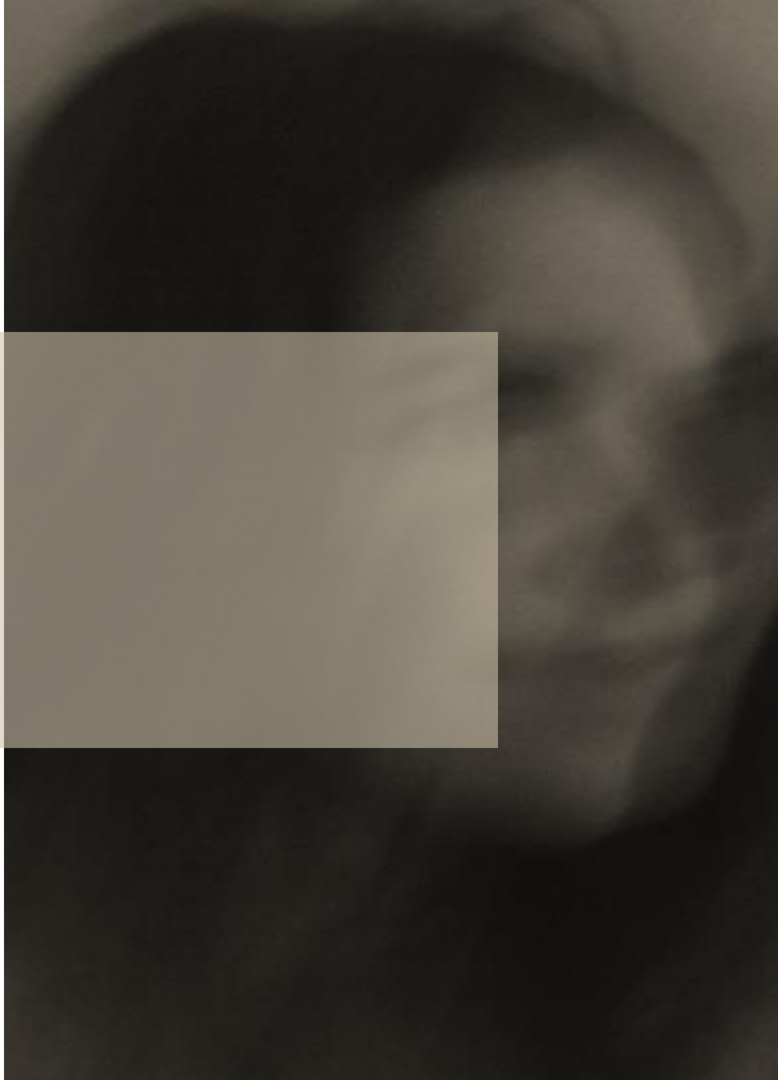
Definition 2

Triangular fuzzy numbers can take many forms







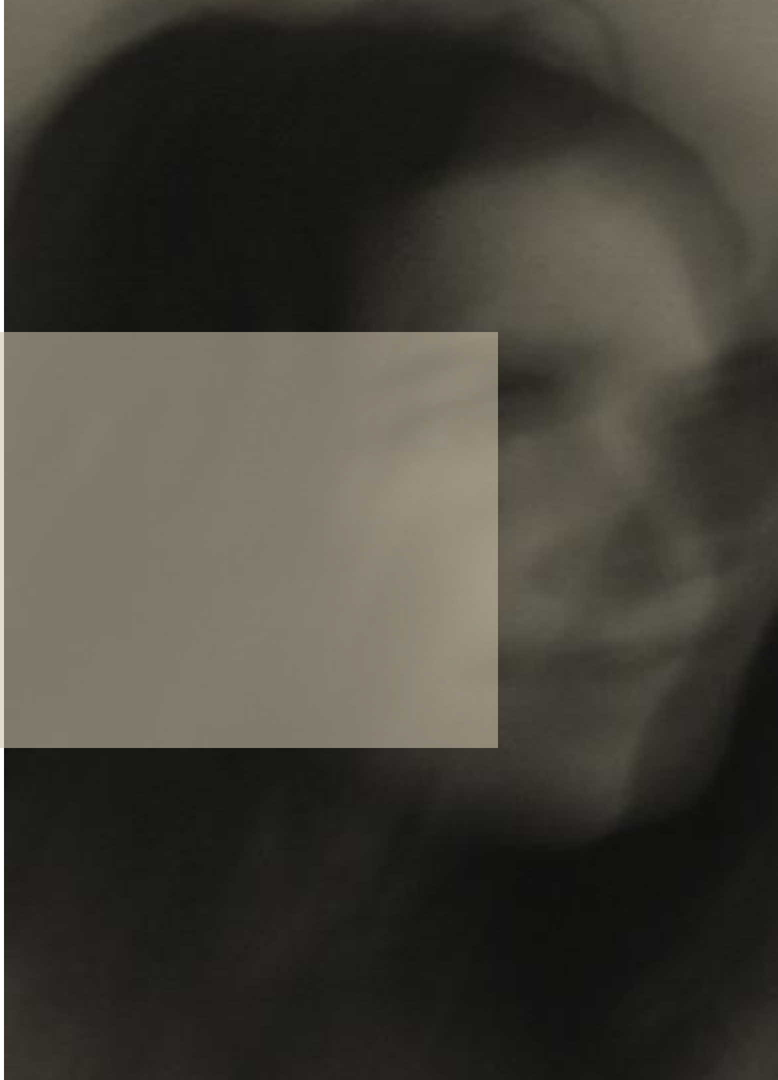


Fuzzy numbers

We will use the following Scale for the examples today (as per Chang, 1996).

However, there is some criticism for the crisp treatment of the cut offs (1,1,1 and 9,9,9 do now allow imprecisions or vagueness).

The relative importance of the two sub-elements	Fuzzy triangular
Equally important	1 1 1
intermediate value between 1 and 3	1 2 3
Slightly important	2 3 4
intermediate value between 3 and 5	3 4 5
Important	4 5 6
intermediate value between 5 and 7	5 6 7
Strongly important	6 7 8
intermediate value between 7and	7 8 9
Extremely important	9 9 9



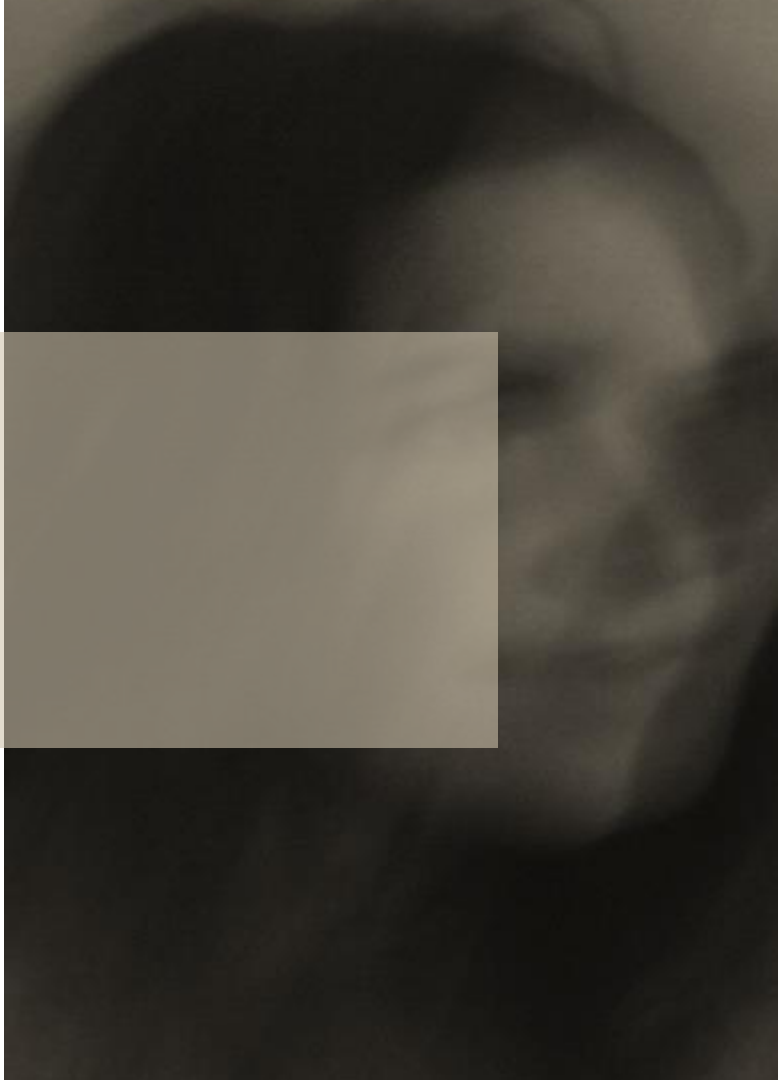
Fuzzy numbers

The **membership function** $\mu_M(x)$ follows this piecewise formula:

$$\mu_M(x) = \begin{cases} \frac{x-l}{m-l}, & x \in [l, m] \\ \frac{u-x}{u-m}, & x \in [m, u] \\ 0, & \text{otherwise} \end{cases}$$

What does this mean?

- If x is between l and m, the function increases linearly from 0 to 1 .
- If x is between m and u , the function decreases linearly from 1 to 0 .
- If x is outside $[l, u]$, the membership is 0 (completely outside the fuzzy set).



Fuzzy numbers

Consider we have two TFNs
$M_1 = (l_1, m_1, u_1), \quad M_2 = (l_2, m_2, u_2)$

We can have the following operations:

$$(l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$$

$$(l_1, m_1, u_1) \odot (l_2, m_2, u_2) \approx (l_1 l_2, m_1 m_2, u_1 u_2)$$

$$(\lambda, \lambda, \lambda) \odot (l_1, m_1, u_1) = (\lambda l_1, \lambda m_1, \lambda u_1) \text{ where } \quad \lambda > 0, \lambda \in R.$$

$$(l_1, m_1, u_1)^{-1} \approx (1/u_1, 1/m_1, 1/l_1)$$

Notice that the in the reciprocals we **invert the order** from $l_1 < m_1 < u_1$ to $1/u_1 < 1/m_1 < 1/l_1$,
otherwise it would violate the monotonicity rule in TFNs.

Fuzzy AHP

According to Chang's method, the next step is to find the **Fuzzy Synthetic Extent Value (S_i)**.

This S_i **is not the final weight or rank of the alternative**, but rather an intermediate value that helps in the computation.

$$S_i = \sum_{j=1}^{m} M_j^i \times \left(\sum_{i=1}^{n} \sum_{j=1}^{m} M_j^i \right)^{-1}$$

In a way, it is similar to what we did in the AHP procedure (summing rows and normalizing them) but now we are doing this using TFNs.

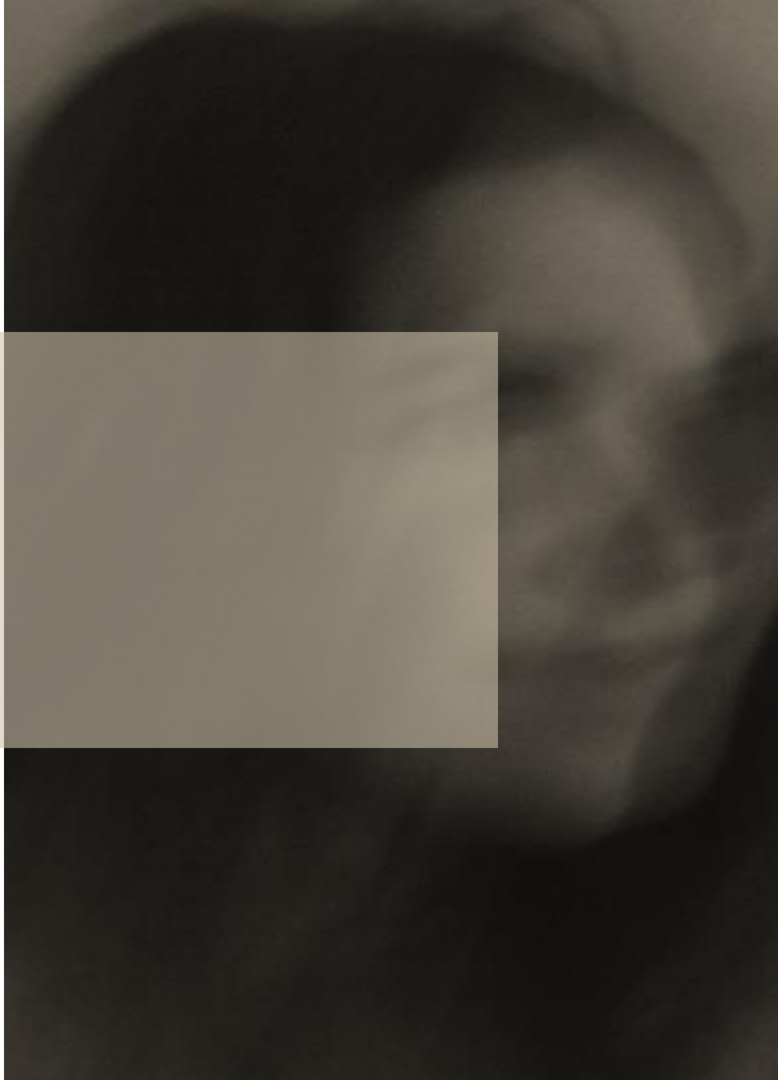
In this formula $M_j^i = (l_j^i, m_j^i, u_j^i)$ is a TFN that aggregates the evaluation of each criterion G_j for each alternative X_i.

Fuzzy AHP

Now, given any two TFNs, we need to compare them, right?

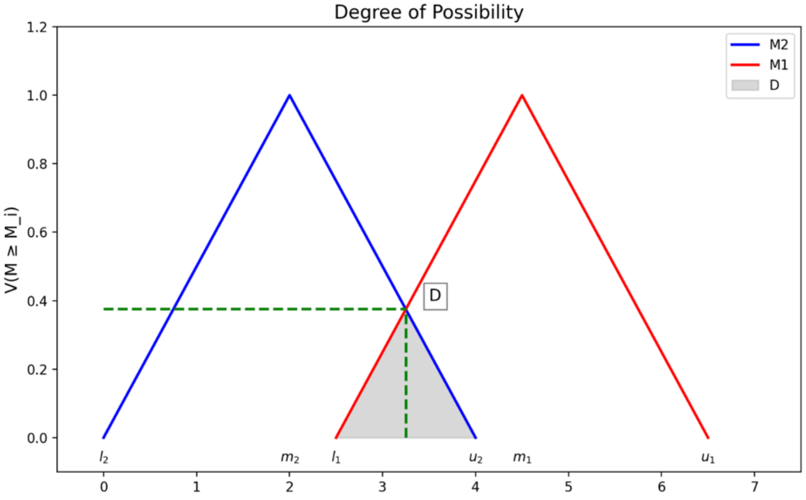
The problem is we cannot directly compare TFNs, because they can overlap entirely, partially or one cover the other.

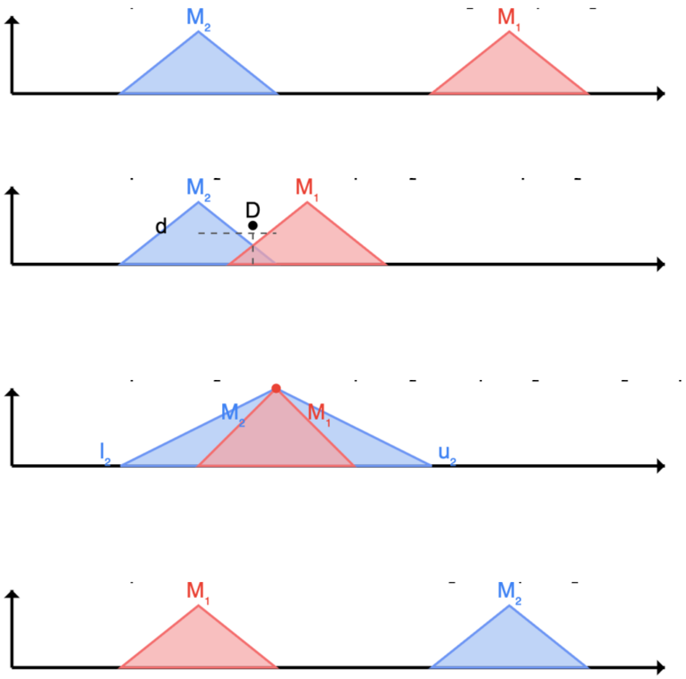
We can infer relationships between two TFNs by using the **Degree of Possibility** – i.e., the likelihood of a TFN being greater than another.



Fuzzy AHP

I hope these two figures will help you understand the intuition behind the degree of possibility (formulae on the next slide).





Fuzzy AHP

We can indirectly estimate the **degree of possibility** (i.e., how likely is a TFN be greater than another).

$V(M_1 \geq M_2) = \sup[\min(\mu_{M_1}(x), \mu_{M_2}(y))]$ where $\mu_M(x)$ is the **membership function**.

For two fuzzy numbers $M_1 = (l_1, m_1, u_1), \quad M_2 = (l_2, m_2, u_2)$ the degree of possibility is:

$$V(M_1 \geq M_2) = 1 \quad \text{iff } m_1 \geq m_2$$

(M_1 is clearly greater)

$$V(M_2 \geq M_1) = hgt(M_1 \cap M_2)$$

(Partial overlap)

$$V(M_2 \geq M_1) = \frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)}$$

This formula gives us a degree of possibility of $M_1 \geq M_2$ (for instance, 0, 0.25, 0.56, up to 1)

Fuzzy AHP

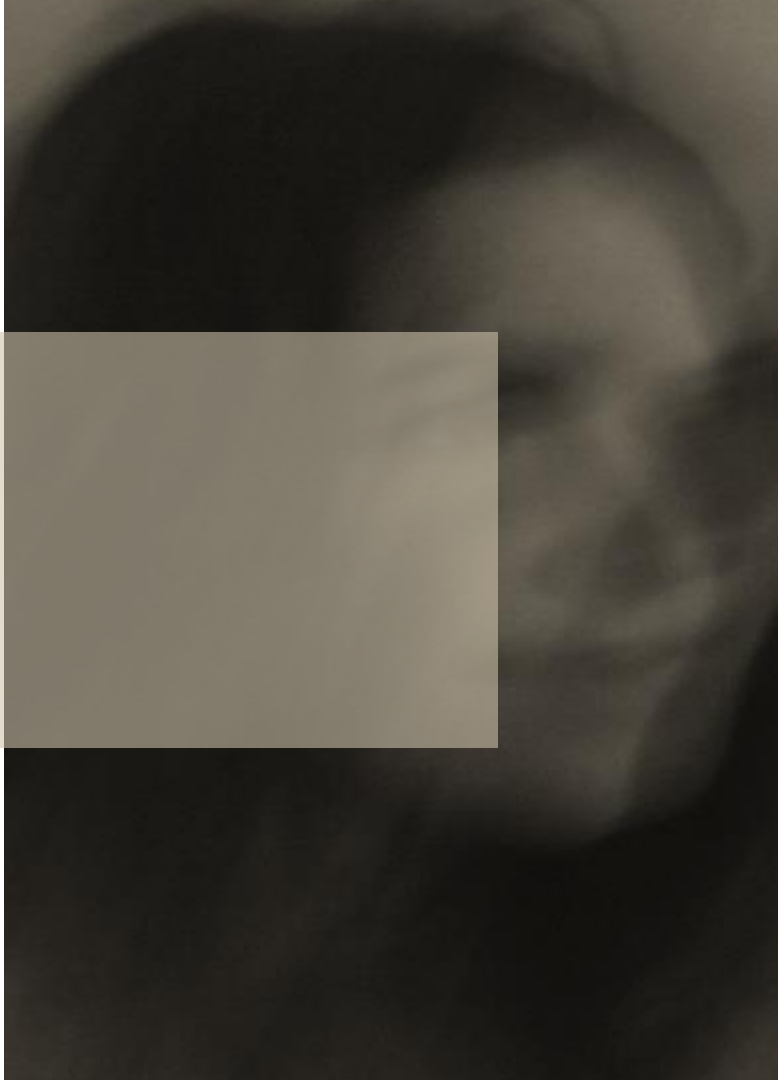
Case 1:

If $m_1 \geq m_2 \rightarrow V(M_1 \geq M_2)$, this means M_1 is always greater than M_2.

- Example:

$M_1 = (4, 6, 8)$, $M_2 = (2, 5, 7)$

- Since $m_1 = 6$ and $m_2 = 5$, we conclude M_1 is greater.
- Degree of Possibility = 1 (100% sure that $M_1 \geq M_2$).



Fuzzy AHP

Case 2

If $l_2 \geq u_1 \rightarrow$ compute using the formula:

$$V(M_1 \geq M_2) = \frac{(l_2 - u_1)}{(m_1 - u_1) - (m_2 - l_2)}$$

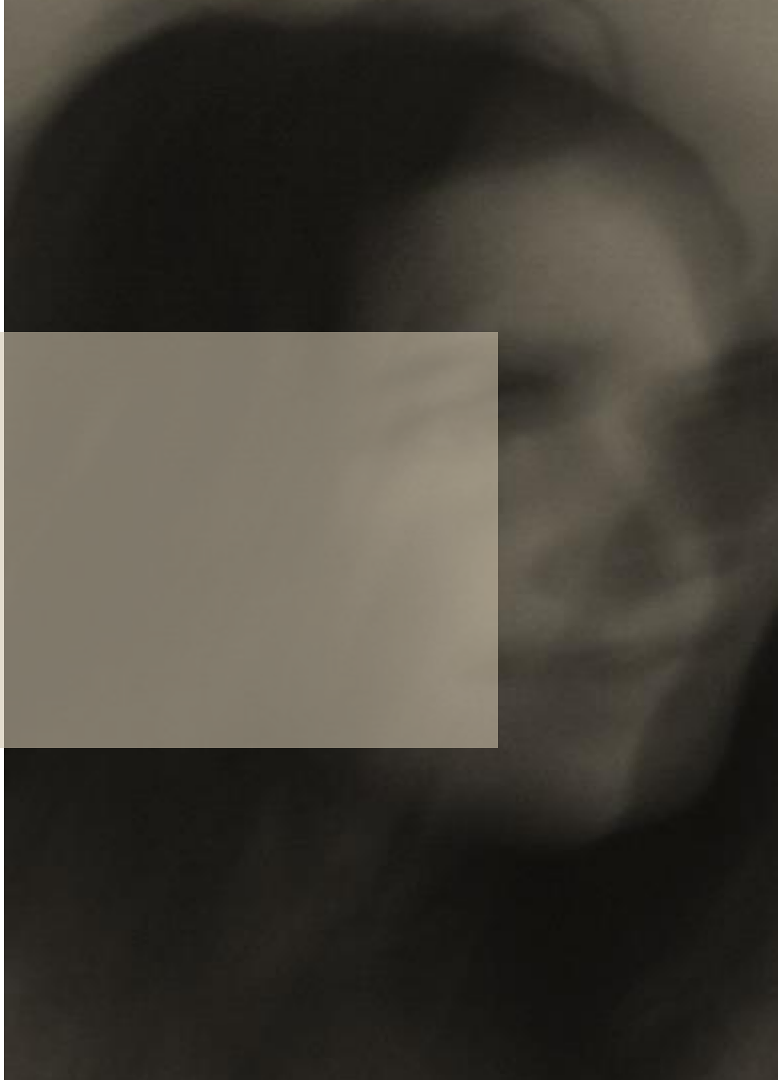
This happens when $M_1$and M_2 have partial overlap

Example: $M_1 = (3, 5, 7)$, $M_2 = (6, 8, 10)$

Here, $l_2 = 6$ and $u_1 = 7$, so there is **partial overlap**.

$$V(M_1 \geq M_2) = \frac{(6-7)}{(5-7)-(8-6)} = \frac{-1}{-2-2} = \frac{-1}{-4} = 0.25$$

Degree of Possibility = 0.25 (meaning M_1 is 25% likely to be greater than M_2).



Fuzzy AHP

Actually….

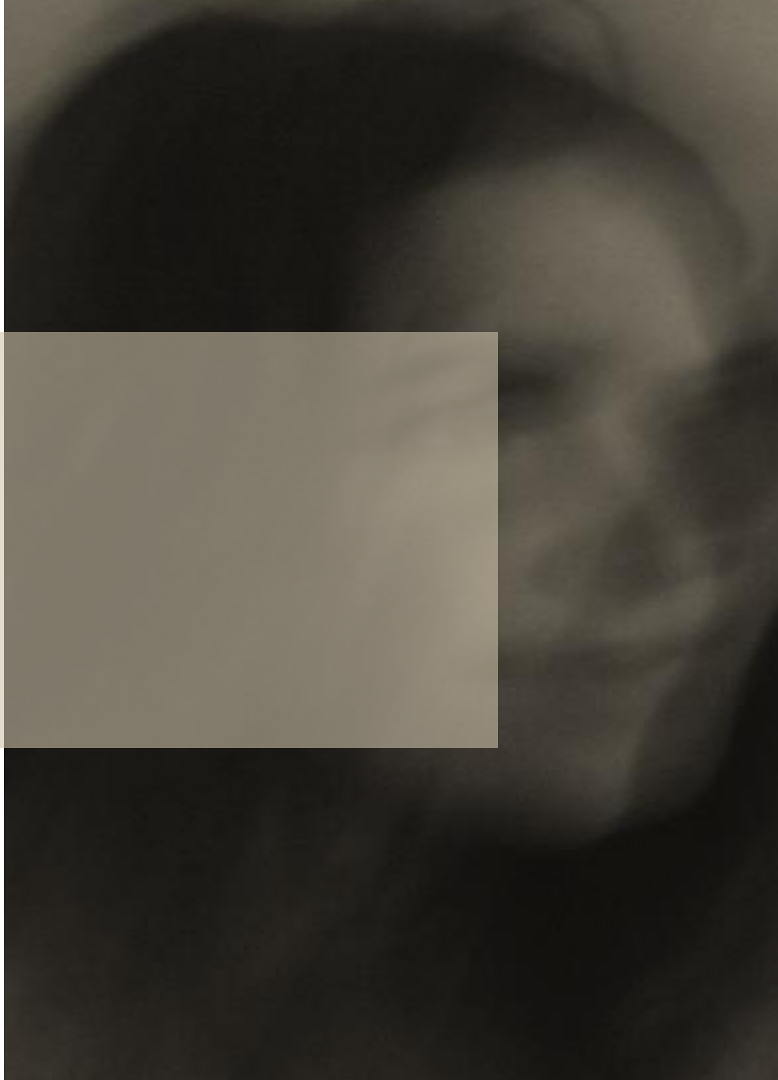
You can compute all $V(M_1 \geq M_2)$ using the formula:

$$V(M_1 \geq M_2) = \frac{(l_2 - u_1)}{(m_1 - u_1) - (m_2 - l_2)}$$

However, the results will be numbers ranging from 0, but going *over* 1.

And since the maximum value has to be 1, all you have to do is convert any $V(M_1 \geq M_2) \geq 1$ to 1.

Eg.: $V(M_1 \geq M_2) = 1.206 \rightarrow 1$



Fuzzy AHP

Ok, so far, we have seen how to compare two TFNs but let's face it – it would be a very limited model if we had only two values to model a problem.

Now we need to compare more than two TFNs:

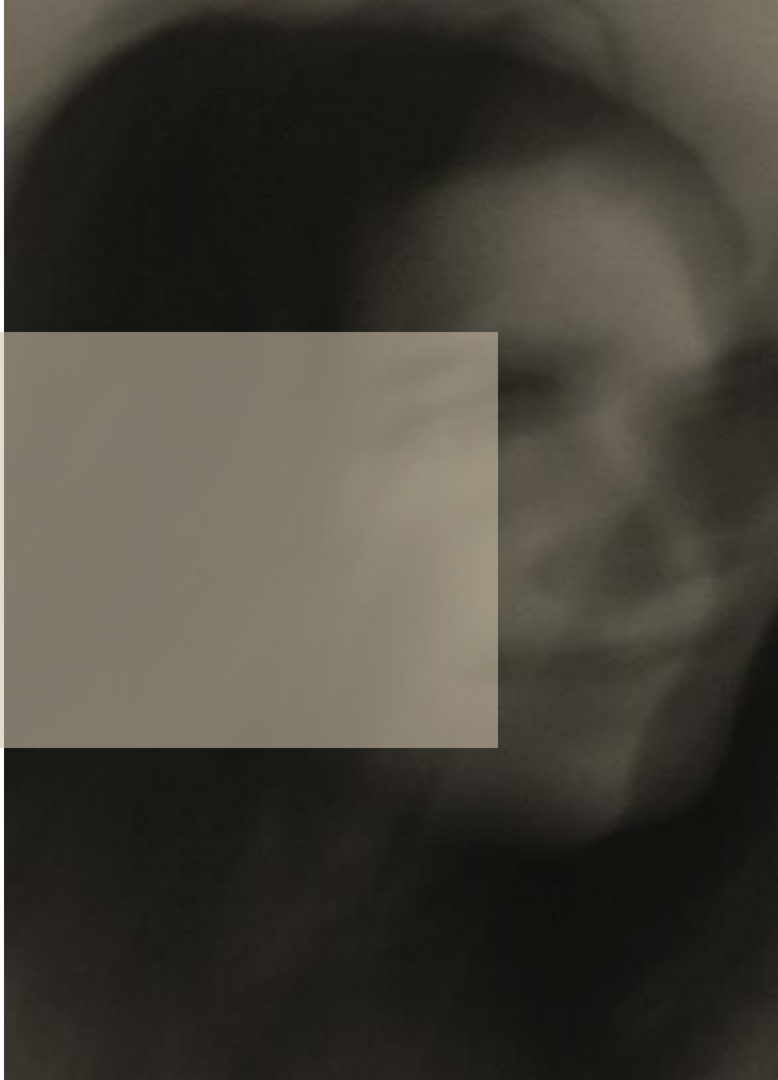
$$V(M_1 \geq M_2) \rightarrow V(M \geq M_1, M_2, \ldots, M_k)$$

- We want to determine how likely a fuzzy number M is to be greater than all other fuzzy numbers.

- The equation

$$V(M \geq M_1, M_2, \ldots, M_k) = \min V(M \geq M_i), \quad i = 1, 2, \ldots, k$$

means that the degree of possibility for M to be greater than all other fuzzy numbers is given by the minimum possibility when comparing it to each individual M_i.



Fuzzy AHP

Why take the minimum?

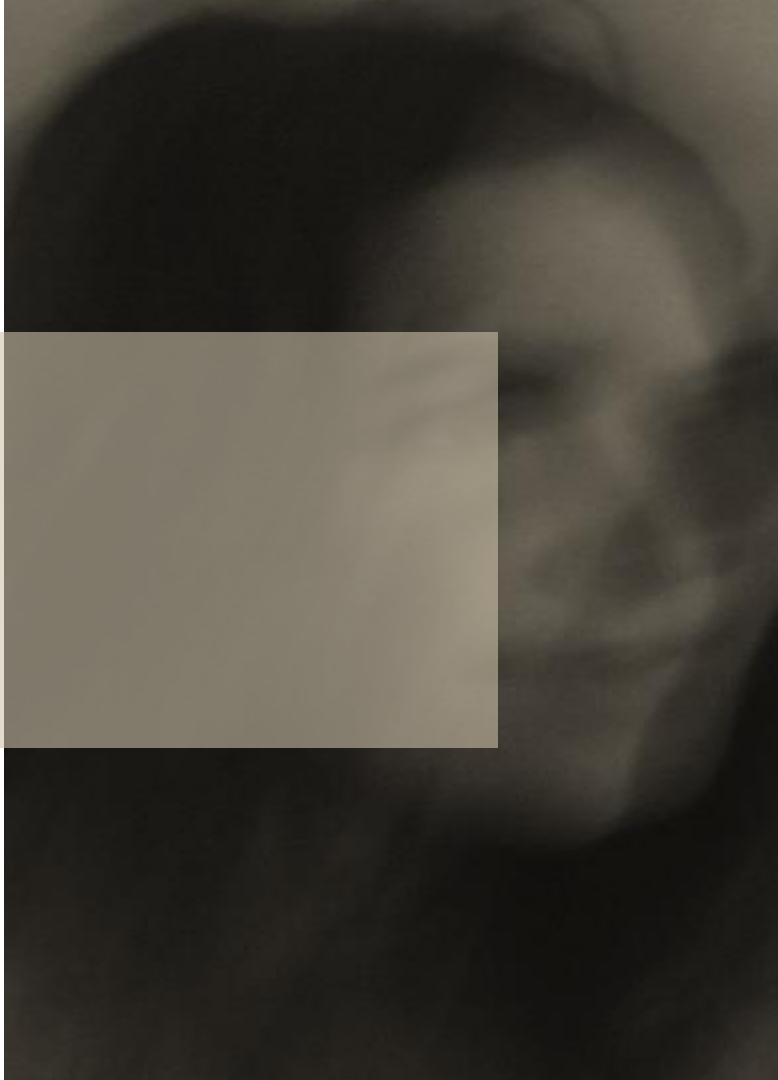
• If M is definitely greater than all others, its lowest comparison score still must be high.

• If even one comparison is low, then M is not strongly dominant.

If $\mathrm{V}(M_1 \geq M_2) = 0.5$ then M_1 is 0.5 more likely to be be better than M_2.

And if $\mathrm{V}(M_1 \geq M_3) = 0$ then it is definitely not better than M_3.

So, it is highly unlikely that M_1 is the best option, since its lowest (min value) is 0.



Fuzzy AHP

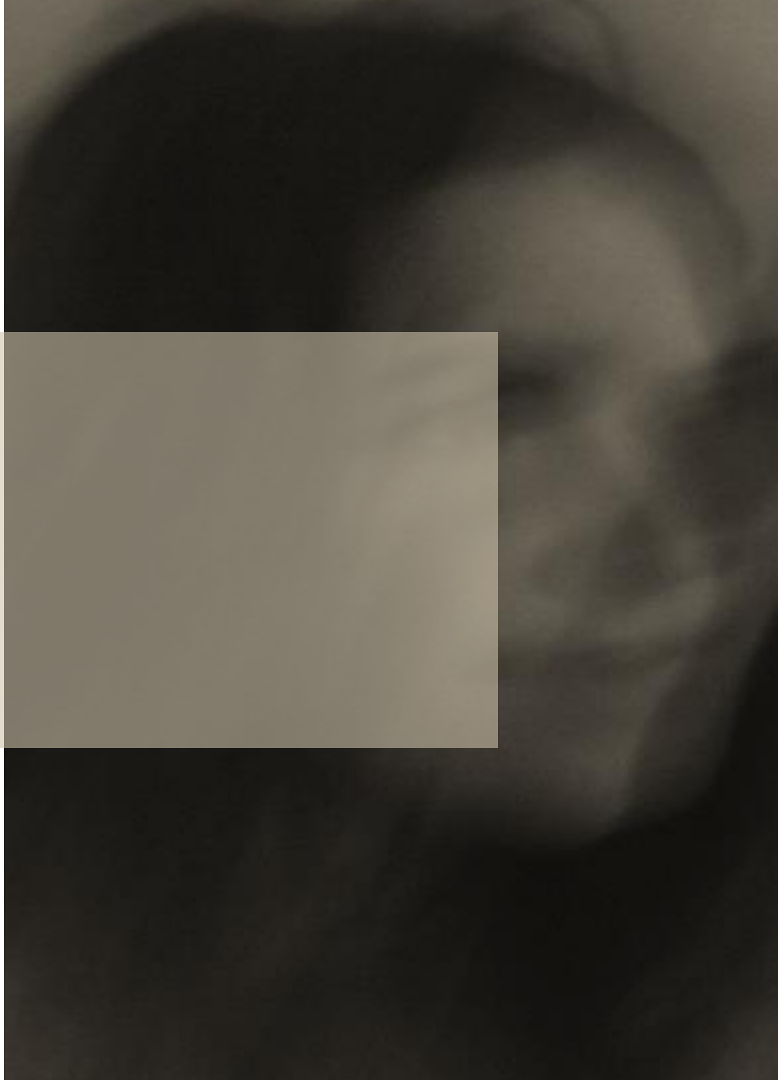
This gets clearer in the next formula:

$$d'(A_i) = \min V(S_i \geq S_k), \quad \text{for } k \neq i$$

1. The weight for each alternative takes into consideration the **worst-case comparison**.

2. If an alternative is **weak in even one comparison**, it lowers its overall ranking.

Intuition:
• If A_i is always better than the others, its lowest comparison score will be high.
• If A_i is worse in even one case, it will be ranked lower.

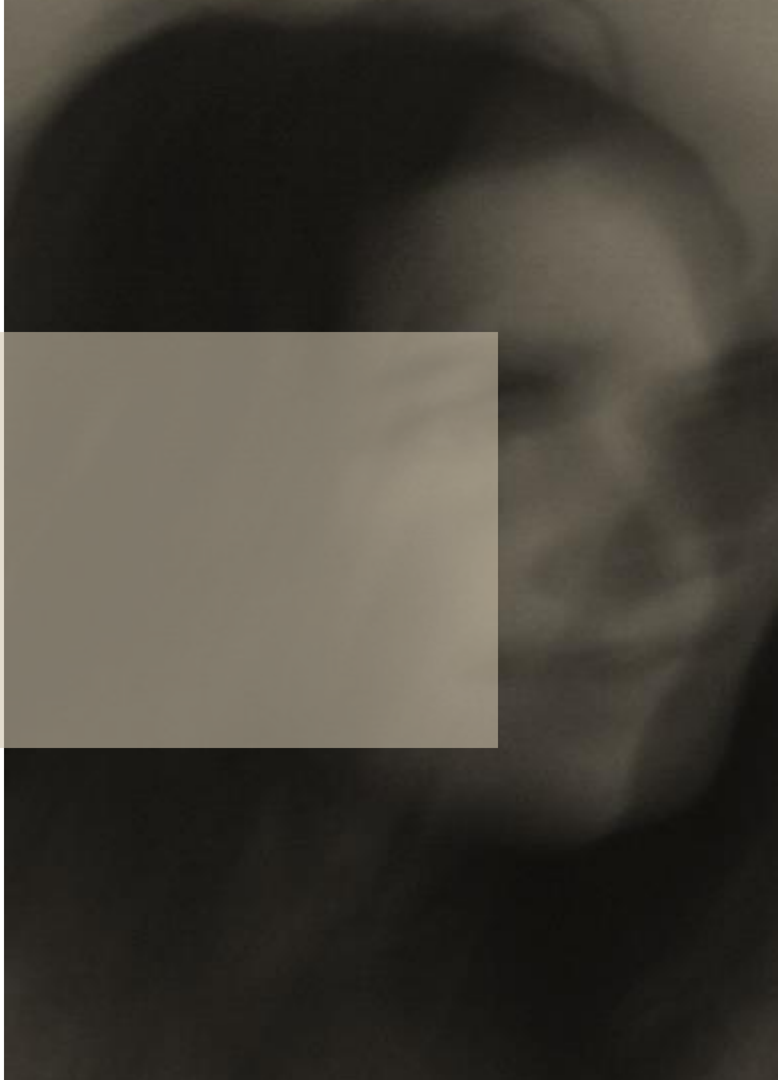


Fuzzy AHP

Once we have $d'(A_i)$ for all alternatives, we construct the weight vector:

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_n))^T$$

- **This stores the relative dominance of each alternative.**
- **Higher values mean better ranking.**
- **However, this is not yet normalized, meaning it doesn’t sum to 1.**



Fuzzy AHP

To ensure that the weights sum to **1**, we normalize them:

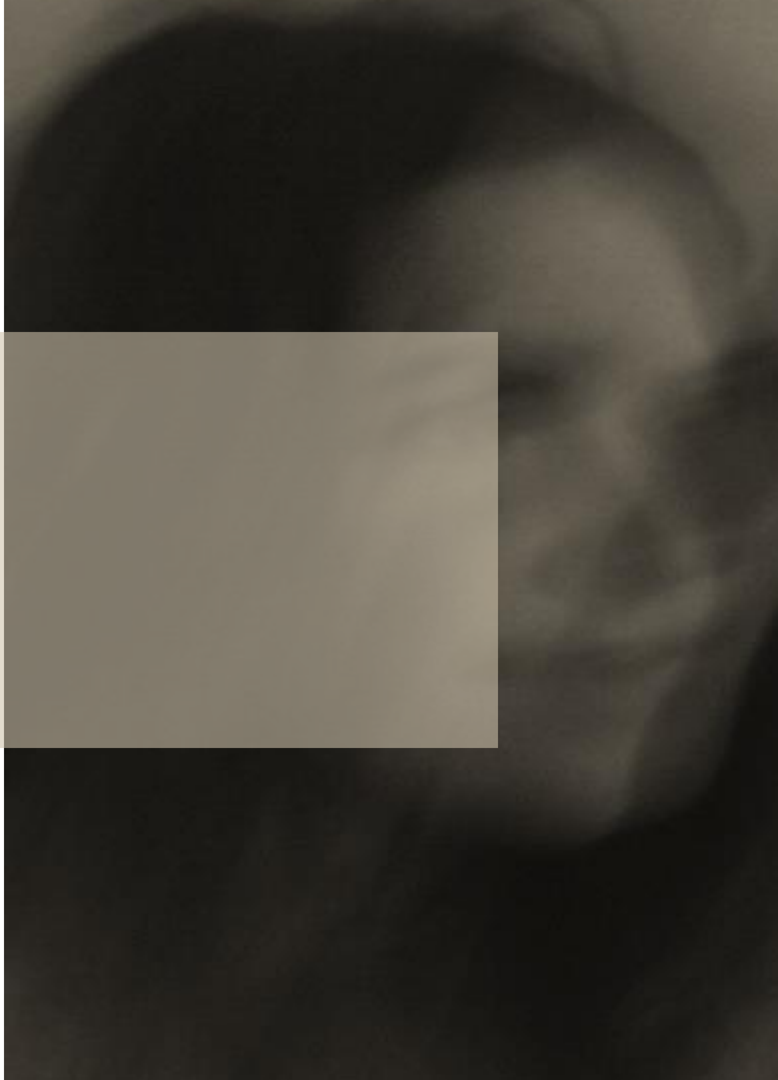
$$W = (d(A_1), d(A_2), \ldots, d(A_n))^T$$

where:

$$d(A_i) = \frac{d'(A_i)}{\sum d'(A_j)}$$

This ensures that:

$$\sum W = 1$$



Hands-on approach

- Let's try this on Google Sheets (same spreadsheet as before):

1 Fil the fuzzified pariwise comparison matrix according to the sc

1.1 Remember that there is a special rule for the reciprocals - see forn

	C1			C2			C3	
	l	*m*	*u*	*l*	*m*	*u*	*l*	*m*
C1	1.0	1.0	1.0	2.0	3.0	4.0	0.3	[illegible]
C2	0.3	0.3	0.5	1.0	1.0	1.0	2.0	[illegible]
C3	1.0	2.0	3.0	0.3	0.3	0.5	1.0	[illegible]

2 Obtain Si

2.1 Sum the rows (by *u,m,l*)

2.2 Sum the columns (by *u,m,l*)

2.3 Calculate the reciprocals

2.4 Multiply by the original sum of the rows

	C1			C2			C3	
	l	*m*	*u*	*l*	*m*	*u*	*l*	*m*
C1	1.0	1.0	1.0	2.0	3.0	4.0	0.3	[illegible]
C2	0.3	0.3	0.5	1.0	1.0	1.0	2.0	[illegible]
C3	1.0	2.0	3.0	0.3	0.3	0.5	1.0	[illegible]

3 Degree of possibility

3.1 Test every relationship both ways (A --> B; B. --> A)

3.2 V(m1>=m2) = 1; otherwise use formula

letter

Si	*m*	*0.370*	*0.356*

Hands-on approach

- This is a paid but with a free demo version of an implementation of FAHP:
- **https://onlineoutput.com/fuzzy-ahp-demo/**

Hands-on approach

- This is a simple app I developed on the weekend.
- It will be taken offline by the end of this course.
- <u>**I did not test exhaustively so use it with caution!**</u>
- **Also, I appreciate feedback (especially if you find errors).**
- **https://fuzzy-ahp-tool-fellipesilva3.replit.app/**

- I have hosted on my GitHub so you can use it later on your own:
- **https://github.com/fellipemartins/Enhanced-Fuzzy-AHP-Calculator**

ARGUING FOR MCDA PAPERS

Methods are slaves to theory

ARGUING FOR MCDA PAPERS

Why?

Multicriteria Decision Analysis (MCDA) is often associated with engineering, operations research, and technical decision-making.

However, in management studies, especially those focusing on strategy, organizational behavior, policy, and decision-making, MCDA is equally valuable.

The need to argue for MCDA in management research stems from several key challenges:

- **1. Overcoming the perception that MCDA is only for quantitative or technical problems**
- **2. Addressing the complexity and uncertainty in management decisions**
- **3. Strengthening theoretical contributions in management studies**
- **4. Providing decision support in non-technical fields**
- **5. Enhancing managerial decision-making beyond heuristics and intuition**

ARGUING FOR MCDA PAPERS

Why?

1. Overcoming the Perception That MCDA is Only for Quantitative or Technical Problems

Why is this important?

- Management studies often incorporate **qualitative judgments, subjective assessments, and social factors** that are difficult to quantify.
- Many scholars may view MCDA as **too rigid or numerical** for topics related to leadership, strategy, governance, or organizational behavior.

How do we argue for it?

- **MCDA is not purely quantitative**: Many MCDA methods (e.g., MACBETH, fuzzy AHP, PAPRIKA) allow for qualitative judgments to be structured and transformed into more robust decision models.
- **MCDA accommodates expert judgment**: It systematically integrates subjective evaluations, making it ideal for studying managerial and strategic decision-making.
- **Example Argument:** *"While strategic decision-making is often qualitative, MCDA enables us to structure and prioritize subjective criteria, making the decision process more transparent and replicable."*

Why?

2. Addressing the Complexity and Uncertainty in Management Decisions

Why is this important?

- Many management studies involve **uncertainty, multiple stakeholders, and conflicting criteria**—all of which are central to MCDA.
- Traditional decision-making approaches (e.g., case studies, surveys) may lack the **systematic comparison and trade-off analysis** that MCDA provides.

How do we argue for it?

- **MCDA helps manage complexity**: It allows for structured decision-making in uncertain and multi-criteria environments.
- **MCDA goes beyond intuition**: Instead of relying solely on heuristics or expert opinion, it provides a formalized decision framework.
- **Example Argument:** *"Organizational change involves balancing financial, cultural, and operational trade-offs. MCDA enables a structured comparison of these factors, reducing the reliance on intuition and improving decision transparency."*

ARGUING FOR MCDA PAPERS

Why?

3. Strengthening Theoretical Contributions in Management Studies

Why is this important?

- Many management papers focus on **conceptual frameworks** without empirical validation of construct importance.
- MCDA offers **a structured way to test and refine theoretical models**, enhancing their **empirical robustness**.

How do we argue for it?

- **MCDA refines theory**: By weighting criteria and analyzing sensitivity, researchers can refine theoretical constructs and ensure they are **empirically grounded**.
- **MCDA improves replicability**: Instead of relying solely on qualitative arguments, researchers can **quantify and justify construct importance**.
- **Example Argument:** *“Existing leadership theories suggest multiple drivers of effective decision-making, but their relative importance remains unclear. Using MCDA, we systematically prioritize these factors based on empirical data, enhancing the theoretical model’s robustness.”*

Why?

4. Providing Decision Support in Non-Technical Fields

Why is this important?

- Many managerial decisions (e.g., strategy, policy, governance) are **subjective and value-laden**, yet they still require **a structured decision-making process**.
- MCDA allows scholars to formalize decisions in areas **not traditionally considered 'technical'**.

How do we argue for it?

- **MCDA is applicable to social sciences**: It helps structure and evaluate trade-offs in decision-making fields like ethics, governance, and organizational behavior.
- **MCDA allows stakeholder inclusion**: By explicitly considering different viewpoints, it improves fairness and legitimacy in decision-making.
- **Example Argument:** *"Sustainability decisions in corporate governance involve balancing economic, social, and environmental priorities. MCDA provides a transparent framework for integrating these dimensions and aligning them with strategic objectives."*

Why?

5. Enhancing Managerial Decision-Making Beyond Heuristics and Intuition

Why is this important?

- Management decisions are often made under **time constraints, bias, and cognitive limitations**.
- Traditional methods (e.g., case studies, surveys, qualitative models) do not always **capture or mitigate biases** in decision-making.

How do we argue for it?

- **MCDA improves decision quality**: It forces decision-makers to explicitly define criteria, reducing cognitive biases.
- **MCDA enhances transparency**: It provides a formal methodology that justifies decisions rather than relying on intuition.
- **Example Argument:** *"Boards of directors often make strategic decisions based on heuristics and past experience. MCDA introduces a structured evaluation that mitigates bias and ensures a more comprehensive assessment."*

TIMELINE

Subject to change

03

Analytical Hierarchical Process (AHP)

04

Fuzzy AHP

05

TOPSIS

06

MACBETH

07

PAPRIKA

REFERENCES

REFERENCES

Today's content was mainly based on

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- **Greco, S., Figueira, J., & Ehrgott, M. (Eds.). (2016). Multiple criteria decision analysis: state of the art surveys. New York, Springer.**
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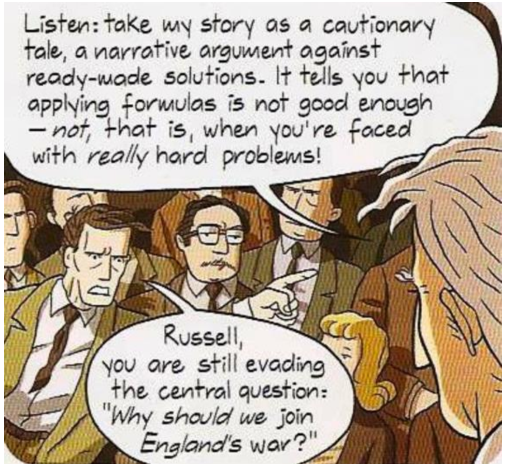

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FOOD FOR THOUGHT

This is an excerpt from (p. 297):

- **Doxiadis, A., Papadimitriou, C., Papadatos, A., & Di Donna, A. (2022). *Logicomix*. Vuibert.**

THANKS

Does anyone have any questions?
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