# TIPS TO REMEMBER

❑ **CONGRUENT :** Let us start by choosing a **base** n, then each integer $N = q.n + r$ where q is **quitent** and r is **remainder** ($0 \leq r < n$). We write this as $N \equiv r \pmod{n}$ and say that **N is congruent to r modulo n. e.g.** $78 \equiv 66 \pmod{12}$.

**Ex. January 1, 2000 falls on a saturday. What day of the week will January 1, 2020 be ?**

**Sol.** Because there are 20 years in the range 2000 to 2019 of which 5 are leap-years, January 1, 2020 falls on day $20 + 5 \equiv 4 \pmod{7}$, i.e. **Wednesday.**

❑ **MATHEMATICAL INDUCTION :**

(1) **Weak Version :** Let P(n) be a statement such that **(i)** $P(n_0)$ is true and (ii) P(k) implies P(k+1) for any $k \geq n_0$, then P(n) is true for every $n \geq n_0$.

(2) **Strong Version :** Let P(n) be a statement such that **(i)** $P(n_0)$ is true and **(ii)** if $P(n_0), P(n_0+1), \ldots, P(k)$ are true for any $k \geq n_0$, then P(k+1) is also true. Then P(n) is true for every $n \geq n_0$.

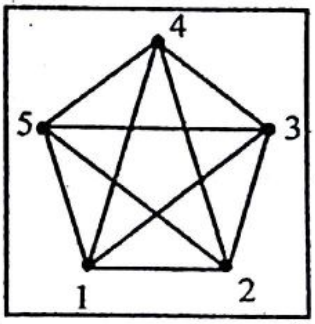
❑ **PIGEONHOLE PRINCIPLE :** If m pigeons are assigned to n pigeonholes , where $m > n$ , then at least two pigeons must occupy the same pigeonhole.

**Proof :** Let the given conclusion is false, that is, no two pigeons occupy the same pigeonhole. Then every pigeon must occupy a distinct pigeonhole, so $n \geq m$, which is a contradiction. Thus two or more pigeons must occupy some pigeonhole.

❑ **ROUND-ROBIN TOURNAMEN :** In **round-robin tournament** every team plays every other team exactly once. Suppose there are n teams , labeled 1 through n. Then the tournament can be represented by a polygon with n vertices with every pair of vertices connected, every vertex represents a team and every line segment with endpoints i and j represents a game between teams i and j.

Let $g_n$ = number of games by n teams in a tournament. It can be defined recursively :

$$g_1 = 0$$
$$g_n = g_{n-1} + (n-1), \text{ where } n \geq 2$$




Solve this recurrence relation, we get

$g_n = \dfrac{n(n-1)}{2} = {}^nC_2$. e.g. 5 teams will play 10 games shows in given figure.

❑ **FERMAT'S LITTLE THEOREM :** Let p be a prime and a is any integer such that $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

**Ex. Find the remainder when $24^{1947}$ is divided by 17.**

**Sol.** $24 \equiv 7 \pmod{17}$. $\therefore 24^{1947} \equiv 7^{1947} \pmod{17}$.

From Farmat's little theorem,

$7^{16} \equiv 1 \pmod{17}$. $\therefore 7^{1947} = (7^{16})^{121} \cdot 7^{11} = 1^{121} \cdot 7^{11} \equiv 7^{11} \pmod{17}$.

But $7^2 \equiv -2 \pmod{17}$. $\therefore 7^{11} \equiv (7^2)^5 .7 \equiv (-2)^5.7 \equiv -32.7$

$\equiv 2.7 \equiv 14 \pmod{17}$.

Hence the reqd. remainder is 14.

❑ **PRINCIPLE OF INCLUSION AND EXCLUSION (P.I.E.) OR SIEVE FORMULA :** It is very important principle is a generalization of the sum-rule to sets which need not be disjoint. Venn-diagrams show that $|A \cup B| = |A| + |B| - |A \cap B|$ and $|A \cup B \cup C| =$

$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

We generalize to n sets as follows :

$$|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$- \ldots, (-1)^{n+1} |A_1 \cap \ldots \cap A_n|.$$

e.g. We consider all n! permutations of 1,2,...n. If an element i is on place number i, then we say i is a fixed-point of the permutation . Let $p_n$= number of fixed point free permutations and $q_n$ = number of permutations with at least one fixed point . Then $p_n = n! - q_n$.

Let $A_i$ =number of permutations with i fixed point , then

$$q_n = |A_1 \cup .... \cup A_n| = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + ... + (-1)^{n+1}\binom{n}{n}0! =$$

$$= n!\left\{1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + ..... + \frac{(-1)^{n-1}}{n!}\right\}$$

$$\therefore p_n = n! - q_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + .... + \frac{(-1)^{n-1}}{n!}\right).$$

❑ **DIVISION WITH REMAINDER :** For polynomials $f(x)$ and $g(x)$ there exist unique polynomials q and r such that $f(x) = g(x)\, q(x)$ · $r(x)$, deg $r(x) <$ deg $g(x)$ or $r(x) = 0$ , where q(x) and r (x) are **quotient** and **remainder** respectively on division of $f(x)$ by $g(x)$. If $r(x) = 0$, then we say that $g(x)$ divides $f(x)$ and we write $g(x) \mid f(x)$.

❑ **ARITHMETIC-GEOMETRIC-HERMONIC-MEAN INEQUALITY:**

Take any n different positive numbers $x_1, x_2, ....., x_n$ where n is positive integer , then Arithmetic mean $A = (x_1 + x_2 + .... + x_n)/n$, Geomeric mean $G = (x_1 x_2 ... x_n)^{1/n}$ , Harmonic mean H =

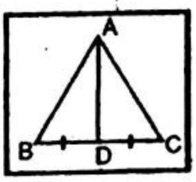
$$\left\{\left(\frac{1}{x_1} + \frac{1}{x_2} + .... + \frac{1}{x_n}\right)/n\right\}^{-1}$$ and Root mean square

$$[\mu_2']^{1/2} = \left\{\left(x_1^2 + x_2^2 + ........ + x_n^2\right)\frac{1}{n}\right\}^{1/2}.$$

Verify that whatever the values of n you take and whatever be the positive numbers you choose $[\mu_2']^{1/2} \geq A \geq G \geq H$, but if $x_1, x_2, ....., x_n$ are equal, then all there means are also equal .

## ❑ FOLLOWING RESULTS OF HEIGHTS AND DISTANCES:

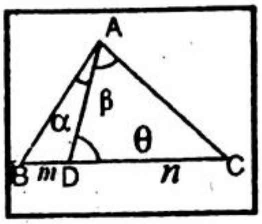
**(i) Appolonius theorem :** It in a $\Delta ABC$, AD is median, then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$.



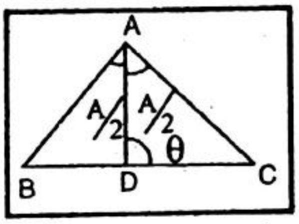

**(ii) m–n theorem :** If BD :DC = m:n, then

$(m+n)\ \text{Cot}\theta = m\cot\alpha - n\cot\beta$

$= n\cot B - m\cot C$.




**(iii) Angle bisector :** If AD is the angle bisector of $\angle BAC$ . ∴ $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{c}{b}$

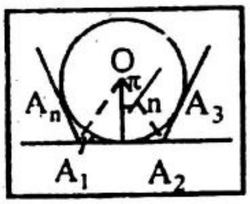



**(iv)** The exterior angle is equal to sum of interior oppoisite angles.
**(v)** If a line is perpendicular to a plane, then its perpendicular to every line in that plane.

## ❑ REGULAR POLYGON : Let $A_1, A_2, \ldots, A_n$ be a regular polygon of n sides each of length a .

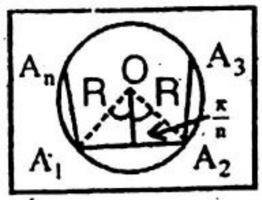
**(i) Inscribed circle of a regular polygon of n sides :**

Area $= \pi r^2 \tan\dfrac{\pi}{n}$ and Radius $r = \dfrac{a}{2}\cos\dfrac{\pi}{n}$.



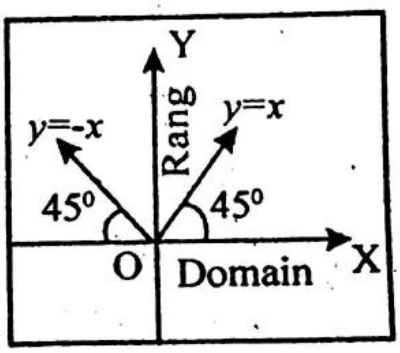

**(ii) Circumscribed circle of a regular polygon of n sides :**

Area $= \dfrac{1}{2} nR^2 \text{Sin}\dfrac{2\pi}{n}$ and Radius $R = \dfrac{a}{2}\text{cosec}\dfrac{\pi}{n}$.




## ❑ MODULUS OR ABSOLUTE VALUE FUNCTION :

It is defined as $y = f(x) = |x| = \sqrt{x^2} = \begin{cases} x: & x \ge 0 \\ -x: & x < 0 \end{cases}$.




$\because f(-x) = x = f(x)$.
$\therefore f(x)$ is symmetric about y-axis.
Here $|0| = 0$ ; $|-5| = 5$ ; $|5| = 5$. Domain : $x \in R$ .
Range : $y \in [\,0, \infty)$. Continuous everywhere .
Non differentiable at $x = 0$, elsewhere differentiable. Even as well as Many-one function. Monotonically

increasing and decreasing for all x>0 and x<0 respectively.

**Note :** $|x|$ is read modulus or mod x. $|x| = 2 \Leftrightarrow x = \pm 2$ ; $|x|<2 \Leftrightarrow -2<x<2$ and $|x|>2 \Leftrightarrow x<-2$ or $x>2$.

**Properties :**

**(i)** $|x+y| = |x| + |y| \Leftrightarrow x, y \geq 0$ or $x, y \leq 0$.

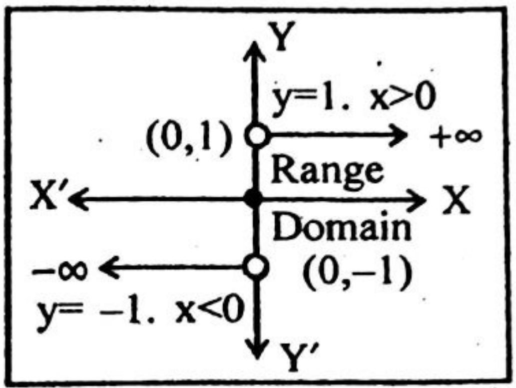
**(ii)** $|x-y| = |x| - |y| \Leftrightarrow x \geq 0, |x| \geq |y|$ or $x \leq 0, y \leq 0$ and $|x| \leq |y|$.

**(iii)** $|x \pm y| \leq |x| + |y|$.

**(vi)** $|x \pm y| \geq |x| - |y|$.

## ❑ SIGNUM FUNCTION : It is defined as y=f(x) =

$$\text{Sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} = \begin{cases} -1; & x < 0 \\ 0; & x = 0 \\ 1; & x > 0 \end{cases}$$




Domain : $x \in R$ . Range : $y \in \{-1, 0, 1\}$ .

Continuous everywhere except at x=0 and discontinuous at x = 0 . Odd as well as many-one function. Neither monotonically increasing nor decreasing.

**Note :** O indicates that the points ( 0,1) and (0,−1) are not included and denote the point (0,0) is included in the graph.

## ❑ GREATEST INTEGER OR FLOOR OR STEP FUNCTION:

It is defined as $y = f(x) = [x] = \lfloor x \rfloor =$ gint $(x) = n$ where $n \leq x < n+1$.

**e.g.** $[\pi] = 3$, $[\log_{10}3] = 0$, $[-3.5] = -4$, etc and $[0.7] = 0$ etc.

$\therefore$ Real number = Integral Part (I) + fractional part (f), where I=[x] and $0 \leq f < 1$. $\therefore$ $y = f(x) = [x]$

For $-3 \leq x < -2$; $y = -3$

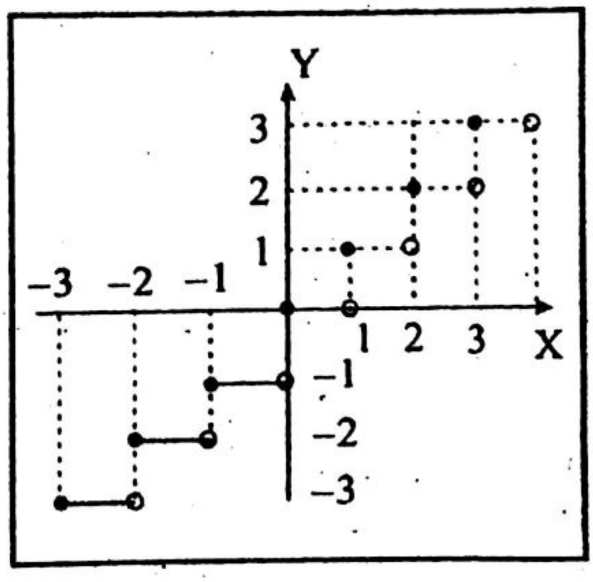
,, $-2 \leq x < -1$; $y = -2$

,, $-1 \leq x < 0$; $y = -1$

,, $0 \leq x < 1$; $y = 0$

,, $1 \leq x < 2$; $y = 1$ and so on.

- Domain : $x \in R$ • Range : $y \in I$
- Continuous and differentiable everywhere expect at x=n, $n \in I$
- Manyone and into function for co-domain as all real numbers.



**Properties :**

**(i)** $[x+n]=n+[x],\ n\in I$.

**(ii)** $[-x]=-[x]\ \forall\ n \in 1$ and $[-x]= -[x] -1,\ x \notin I$.

**(iii)** $x=[x]+\{x\}$, $\{\}$ denote fractional part of x .

**(iv)** $[x_1+x_2]\geq[x_1]+[x_2]$.

**(v)** $\left[\dfrac{[x]}{n}\right]=\left[\dfrac{x}{n}\right]\forall\ n\in N$.

**(vi)** If $[f(x)] \geq I$, then $f(x) \geq I$ and for $[f(x)] \leq I$, then $f(x) < I+1$.

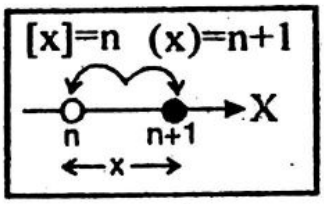
**(vii)** $[p+[q+[r+[s+[t]]]]]=[p]+[q]+[r]+[s]+[t]$.

**(viii)** $[x]+\left[x+\dfrac{1}{n}\right]+\left[x+\dfrac{2}{n}\right]+.....+\left[x+\dfrac{n-1}{n}\right]=[nx]$.

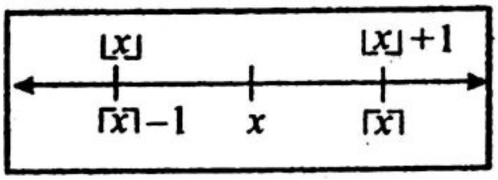
## ❑ LEAST INTEGER OR CEILING FUNCTION : It is defined

as $y = f(x) = (x)=\lceil x\rceil=\text{lint}\ (x) = n+1$,

where $n < x \leq n+1$.




e.g. $(\pi) =4$, $(\text{Log}_{10}3) =1$, $(-3.5) = -3$ etc.




$\therefore\ y=(x)= 0$ if $-1< x \leq 0$

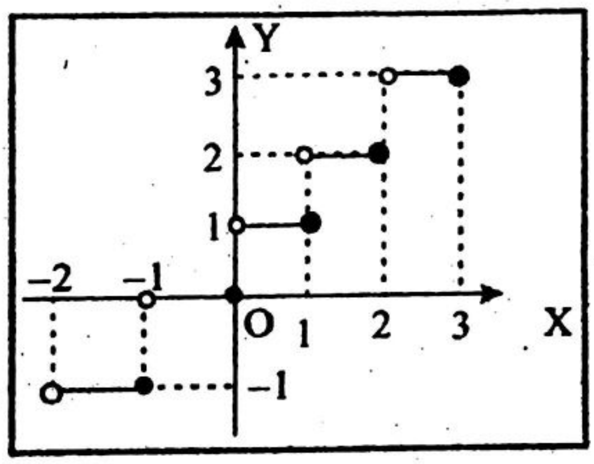
$= 1$ if $0 < x \leq 1$

$= 2$ if $1< x \leq 2$

$= 3$ if $2< x \leq 3$ and so on.

• Domain : $x \in R$ • Range : $y \in I$

• Many one into function for co-domain of all real numbers. Discountinuous at all integral values of x.




**Properties :**

**(i)** $(x+n) = (x)+n,\ n\in I$

**(ii)** $(-x) = -(x),\ x \in I$ and $(-x) = -(x)+1,\ x \notin I$.

**(iii)** $x = (x)+\{x\}-1, \{x\}$ denoes the fractional part of x.

**(iv)** $(x_1+x_2) \leq (x_1)+(x_2)$.

**(v)** $\left(\frac{(x)}{n}\right)=\left(\frac{x}{n}\right), n \in N$.

**(vi)** $(x)+\left(x+\frac{1}{n}\right)+\left(x+\frac{2}{n}\right)+.......+\left(x+\frac{n-1}{n}\right)=(nx)+n-1, n \in N$.

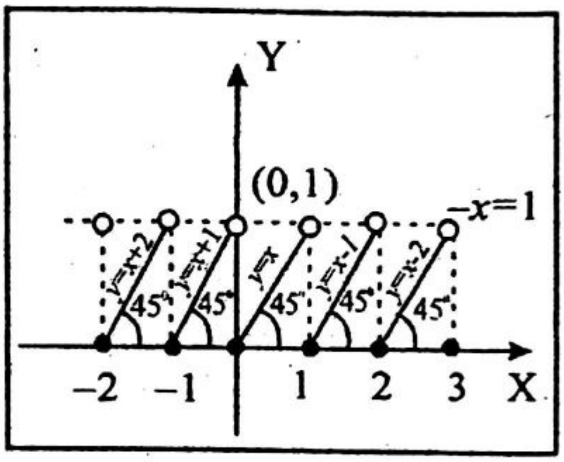
❑ **FRACTIONAL PART FUNCTION :** It is defined as $y = f(x) =\{x\}=x-[x] = f$ (where $0 \leq f < 1$) $=x-n, n \leq x < n+1$.

e.g. $\{1.6\} = 0.6$ , $\{1\}= 0$, $\{-3.6\} = 0.4$

$\therefore y = \{x\} = x$ if $0 \leq x < 1$

$= x-1$ if $1 \leq x < 2$

$= x-2$ if $2 \leq x < 3$ and so on.




• Domain : $x \in R$ , • Range : $y \in [0,1)$,

• Discontinuous and Non-differentiable at all integers, • Monotonically increasing throughout • Many-one and into function for co-domain of all real numbers.

**Properties :**

**(i)** If $0 \leq x < I$ , then $\{x\} = x$.

**(ii)** If $x \in I$, then $\{x\} = 0$.

**(iii)** If $x \notin I$ and $x > 0$, then $\{-x\} = I- \{x\}$.

**(iv)** Domain and Range of $\frac{1}{\{x\}}$ are R–I and (0,1) respectively.

**Note :** $f(x+T)=f(x)$ $(x \in R)$ , where T be a (+)ve real number and $f(x)=\{x\}$. $\Rightarrow x+T-[x+T]=x-[x] \ \forall \ x \in R$ , $\Rightarrow [x+T]-[x]=T \ \forall x \in R$, $\Rightarrow T=1,2,3,....$Thus there exist $T > 0$ such that $f(x+T) = f(x) \ \forall \ x \in R$, so f(x) is periodic and the smallest value of T satisfies $f(x+T) = f(x) \ \forall x \in R$ is 1 .

❑ **DIRICHLET FUNCTION :** Suppose c and d be real numbers (usually taken as c=1 and d = 0), then the **Dirichlet function** is defined by $y=f(x)=\begin{cases} c \text{ for } x \text{ rational} \\ d \text{ for } x \text{ irrational} \end{cases}$ and is discontinuous everywhere.

**Analytically :** $f(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\cos^{2n}(m!\pi x)\right\}$

• Domain : $x \in R$ • Range : $y \in \{c,d\}=\{0,1\}$ • Neither increasing nor decreasing • Not a periodic function. • Discontinuous everywhere • Manyone and into function for co-domain of all real numbers.

## WORKED OUT EXAMPLES

**Ex.1. Let max (x,y) denote the maximum of x, y and min (x,y) their minimum , where x and y are any integer, prove that max (x,y) – min (x,y)=|x–y|.**

**Sol. Case 1.** Let $x \geq y$, then $|x-y|=x-y$, max $(x,y) = x$ and min $(x,y) = y$. Thus max $(x,y)$–min $(x,y)=x-y=|x-y|$.

**Case.2.** Let $x<y$, then $|x-y|=y-x$, max $(x,y)= y$ and min $(x,y)=x$. Thus max $(x,y)$ – min $(x,y) = y-x=|x-y|$.

**Ex.2.** Solve : **(i)** $|x+1|^{\log(x+1)(3+2x-x^2)} = (x-3)|x|$

**(ii)** $|x-1| + \quad = 4$ , $x - |y-1| = 2$ .

**Sol. (i)** Given : $|x+1|^{\log(x+1)^{(3+2x-x^2)}} = (x-3)|x|$........(1)

$\because 3 + 2x - x^2 > 0$ (for log x, x>0)

$\therefore -1 < x < 3$...(2).

Let $x+1>1$, then $x>0$.....(3).

$\therefore$ From (1): $(x+1)^{\log_{(x+1)}(3+2x-x^2)} = (x-3)x$

$\Rightarrow 3+2x-x^2 = x^2-3x, \Rightarrow 2x^2-5x-3=0. \therefore x=-\frac{1}{2}, 3.$

Now , from (2) and (3) : $x \neq -\frac{1}{2}, 3.$

Hence it has no solution.

**(ii)** Given : $|x-1| +3y=4, \Rightarrow \begin{cases} x+3y=5, x\geq 1 ........(1) \\ -x+3y=3, x<1 ........(2) \end{cases}$ and also given : $x- |y-1|=2$

$\Rightarrow \begin{cases} x-y=1, y\geq 1 ......(3) \\ x+y=3, x<1 ......(4). \end{cases}$ Solving (1), (3): x=2, y=1 and solving(1),(4): x=2, y=1.

$\therefore$ No solution as $x \geq 1, y < 1$.

Solving (2), (3) : x=3, y=2, i.x. no solution as $x < 1, y \geq 1$ and solving

(2), (4) : $x = \frac{5}{2}, y = \frac{3}{2}$, i.e. no solution as $x < 1, y < 1$.

Hence the reqd. solution is unique, i.e. x=2, y=1.

**Ex. 3 If f(x)=3[x]+5 and f(x)=5[x-2] +7, find the value of [x+f(x)].**

**Sol.** given : $3[x] +5 = 5[x-2] +7, \Rightarrow 3[x]+5=5[x]-10+7, \Rightarrow 2[x]=8,$

$\Rightarrow [x]=4. \therefore 4 \leq x < 5$, i.e. x=4+fractional part .

$\therefore$ f(x)=3[x]+5=3.4+5=17. $\therefore$ [x+f(x)]=[17+fractional part +4]=21.

**Ex. 4. Solve : {x+1}+2x =4[x+1]-6 where {.}=f.p; [.]=G.I.**

**Sol.** $x+1-[x+1]+2x=4[x+1]-6$ ($\because \{x\}=x-[x]$), $\Rightarrow 3x+1=5[x+1] - 6$

$=5([x]+1) - 6, \Rightarrow 3x=5[x]-2...(i), \Rightarrow 3(\{x\}+[x])=5[x] - 2.$

$\therefore 3\{x\}=2[x]-2...(ii).$

Now , $0\leq \{x\}<1, \Rightarrow 0\leq 3\{x\}<3, \Rightarrow 0\leq 2[x]-2<3$ [using (ii)],

$\Rightarrow 2 \leq 2[x]<5, \Rightarrow 1\leq [x]< \frac{5}{2}, \Rightarrow [x] = 1,2.$

$\therefore$ From (i): $[x]=1$, i.e. $x = 1$ and $[x]=2$, i.e. $x = \frac{8}{3}$.

Hence reqd. solutions are 1,8/3.

**Ex. 5. Solve : $(x)^2 + (x+2)^2 = 20$, where (.) = least integer.**

**Sol.** Let x=I+f, then from the given equation, we have $(I+f)^2 + (I+f+2)^2$

$=20, \Rightarrow \{I+1\}^2 + \{I+3\}^2 = 20, \Rightarrow I^2 + 2I + 1 + I^2 + 6I + 9 = 20,$

$\Rightarrow I^2 + 4I - 5 = 0, \Rightarrow I = 1, -5.$

$\therefore x = 1+f, -5+f, \Rightarrow 0 < f < 1, \Rightarrow 1 < I+f < 2$ and $-5 < -5+f < -4$.

Further x = I.

$\therefore (x)^2 + (x+2)^2 = 20$, i.e. $x \in (-5,-4] \cup (1,2]$.

**Ex. 6. Solve :** $y=\frac{1}{3}[\text{Sin } x + [\text{Sin } x + [\text{Sin } x]]]$ **and** $[y+[y]]=2\cos x$**.**

**Sol.** we have

$y=\frac{1}{3}[\text{Sinx}+[\text{Sinx}+[\text{Sinx}]]]=\frac{1}{3}[\text{Sinx}+[+[\text{Sinx}]+[\text{Sinx}]]$ ($\because [I+x]=I+[x]$ for any integer I) $=\frac{1}{3}([\text{Sinx}]+[\text{Sinx}]+[\text{Sinx}])$.....(1)

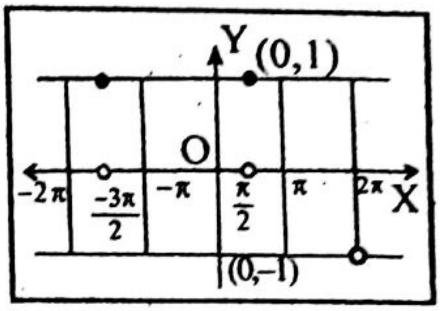
and $[y+[y]]=2\cos x$, $\Rightarrow 2[y]=2\cos x$ , $\Rightarrow [y]=\cos x$.....(2).

$\therefore$ From (1) and (2) : $[\text{Sinx}] = \cos x$ ,

$\Rightarrow [\text{Sinx}] = \cos x$.

Now, ploting the curves [Sin x]and cos x on the same frame, we see that the two curves have no intersection points.

Hence the given equation has no solution.




**Ex,7. How many integers between 1 and 300 (inclusive) are (i)divisible by at least one of 3,5and 7? (ii) divisible by 3 and by 5, but not by 7 ? (iii) divisible by 5 but by neither 3 nor 7 ?**

**Sol.** Let A, B and C be the set of those integers between 1 and 300 which are divisible by 3, 5, 7 respectively.

$A=\{n|1\le n\le 300,\ 3|n\}$; $B=\{n|1\le n\le 300, 5|n\}$;

$C=\{n|1\le n\le 300, 7|n\}$.

To be divisible by 3 or 5 or 7 is to be at least in one of A, B, or C.

Now, $|A|=\left[\frac{300}{3}\right]$, [.] is the greatest integer = 100.

Similarly, $|B|=\left[\frac{300}{5}\right]=60$ and $|C|=\left[\frac{300}{7}\right]=42$, where |A|,|B|,|C| being the cardinalities of sets A, B, C. Again , $A\cap B$ is the set of integers between 1 and 300, which are divisible by both 3 and 5, since 3 and 5 are relatively prime, any number divisible by them must be divisible by product of them,

$\therefore A\cap B$ is the set of integers divisible by 15.

Similarly $B\cap C$, $A\cap C$, $A\cap B\cap C$ are the sets of integers between 1 and 300 which are divisible by 35, 21 and 105.

$|A \cap B| = \left[\frac{300}{15}\right] = 20, |A \cap C| = \left[\frac{300}{21}\right] = 14, |B \cap C| = \left[\frac{300}{35}\right] = 8,$

$|A \cap B \cap C| = \left[\frac{300}{105}\right] = 2.$

**(i)** $|A \cap B \cap C| = |A|+|B|+|C| - |AB| - |BC| - |CA|+|ABC| = 100+60+42-20-14-8+2=162.$

**(ii)** The numbers divisible by 5 and 3, but not by 7, are preciesly those numbers in (ABC)/C, whose cordinality is $|A \cap B| - |A \cap B \cap C|$ $=20-2=18.$

**(iii)** The numbers divisible by 5 but by neither 3 nor 7 are those in $B\backslash(A \cup C)$, having cardinality $|B|-|B\cap(A\cup C)|$. Since $B\cap(A\cup C)| = (B\cap A)\cup(B\cap C)$, the principle of inclusion and exclusion gives $(B\cap(A\cup C)) = |B\cap A|+|B\cap C|-|(B\cap A)\cap(B\cap C)| =$ $|B\cap A|+(B\cap C)-|B\cap A\cap C = 20+8-2=26$ and the number we are looking for is $|B|-|B\cap(A\cup C)|=60-26=34.$

**Ex. 8. Let n be a positive integer. Suppose a function L is defined recursively as follows:**

$$L(n) = \begin{cases} 0 \text{ if } n=1 \\ L\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \text{if } n>1 \end{cases} \quad ([.] \text{ is the floor function }).$$

Find L (25).

**Sol.** L (25) is found recursively as follows:

L (25) = (12) +1......(i); L (12)=L (6) + 1......(ii);

L (6) = L(3) +1....(iii); L (3) = L (1) +1.......(iv);

Back calculating L(6)=L(3)+1=1+1=2; L(12)=L(6)+1=2+1=3.

$\therefore$ L(25) = L(12) +1=3+1=4.

**Ex. 9. If $f(x) = x^3 - 9x^2 + 24x + c$ has three real and distinct roots $\alpha$, $\beta$, $\gamma$ then find the possible values of c. Hence, otherwise show that $[\alpha]+[\beta]+[\gamma]$ can take only two values and determine these values, [.] denotes the greatest integer function.**

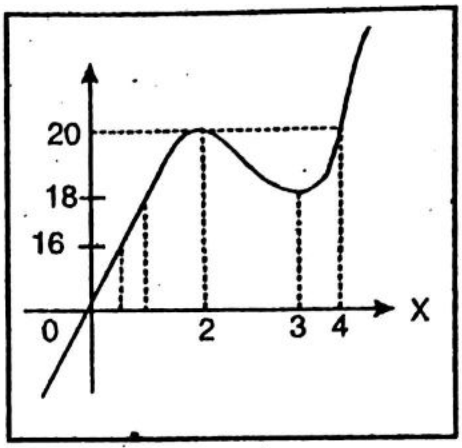
**Sol.** Suppose $y = x^3 - 9x^2 + 24x$.

$\therefore y_1 = 3(x-2)(x-4)$. $\therefore$ For turning point $y_1 = 0$.

$\therefore$ x = 2, 4 are turning points.

Thus $y(2)=8-36+48=20$ and $y(4)=64-144+96=16$.

From the graph we can see that x-axis will cut the graph 3 times , if it shifted downward by 16 to 20 units, i.e. $c\in(-20, -16)$. For $[\alpha]+[\beta]+[\gamma]$, if $c\in(-20,-16)$, then $\alpha \in (1,2)$, $\beta \in (3,4)$, $\gamma\in (4,5)$.




$\therefore [\alpha]+[\beta]+[\gamma]=1+3+4=8$. Again, if $c\in (-20,-18)$, then $\alpha\in(1,2,)$ $\beta \in (2,3)$, $\gamma\in(4,5)$.

$\therefore [\alpha]+[\beta]+[\gamma]=1+2+4=7$.

Finally, We can say that $[\alpha]+[\beta]+[\gamma]=\begin{cases} 8 & \text{if } -18 < c < -16 \\ 7 & \text{if } -20 < c < -18. \end{cases}$

**Ex. 10. If a, b, c are the cube root of p, (p<0) then for any permisible value of x, y, z which is given by $\left|\dfrac{xa+yb+zc}{xb+yc+za}\right| +(a^2_1-2b^2_1)\,\omega+\omega^2\,([x]+[y]+[z])=0$, where $\omega$ is cube root of unity and $a_1$, $b_1$, are real positive numbers, $b_1$ is a prime, find the value of $[x+a_1]+[y+b_1]+[z]$ ([.]=G.I.F.).**

**Sol.** Given : $1+\omega+\omega^2=0$....(i) and $\left|\dfrac{xa+yb+zc}{xb+yc+za}\right| + (a^2_1-2b^2_1)\,\omega + ([x]+[y]+[z])\omega^2=0$....(ii).

From (i) and (ii) : $\left|\dfrac{xa+yb+zc}{xb+yc+za}\right| = (a^2_1-2b^2_1)=[x]+[y]+[z]$ ........(iii).

Since a, b, c are the roots of $(p)^{1/3}$ and let $a = t$, $b = t\omega$ and $c = t\omega^2$.

$$\therefore \left|\frac{xa+yb+zc}{xb+yc+za}\right| = \left|\frac{xt+y+\omega+t\omega^2.z}{x.t\omega+y.t\omega^2+x.t}\right| = \left|\frac{x+y\omega+z\omega^2}{x\omega+y\omega^2+z}\right| = \left|\frac{1}{\omega}\right| = 1.$$

$\therefore$ From (iii) : $a^2_1-2b^2_1 = 1$, i.e. $a^2_1=2b^2_1+1$ (odd)......(iv)

Thus $a_1$ can be written in the from $(2n+1)$, $\Rightarrow (2n+1)^2=1+2b^2_1 \Rightarrow 4n^2+4n=2b^2_1 =2n\,(n+1), \Rightarrow\ b^2_1 =2n(2n+1)$, an even number and given prime, so $b^2_1$ is also prime. $\therefore\ b_1=2$($\because$ 2 is only even prime)

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$\therefore a_1^2=9, \Rightarrow a_1=3$. From (iii) : $[x]+[y]+[z]=a_1^2-2b_1^2=1$.

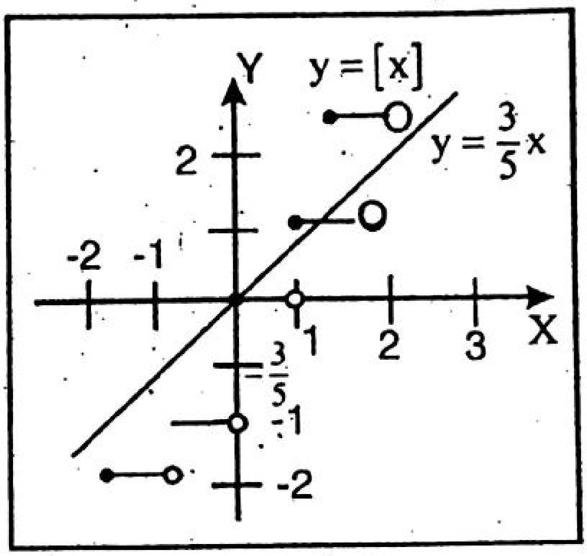
$\therefore [x+a_1]+[y+b_1]+[z]=[x+3]+[y+2]+[z]=[x]+[y]+[z]+5=1+5=6$.

**Ex. 11. Find the number of solutions of $4\{x\}=x+[x]$, where $\{.\}$, $[.]$ denotes the fractional part, greatest integer function respectively.**

**Sol.** Given : $4\{x\}=x+[x]$, $\Rightarrow 4(x-[x])=x+[x]$ ($\because x=[x]+\{x\}$), $\Rightarrow 3x=5[x]$.

$\therefore [x]=\frac{3}{5}x$.....(i)

$\therefore$ To plot the graph of both $y=[x]$ and $y=\frac{3}{5}x$.




Thus the two curves intersects when $[x]=0$ and $[x]=1$......(ii).

$\therefore$ From (i) and (ii) : $x=\frac{5}{3}[x]$, i.e. $x=\frac{5}{3}.0$ and $x=\frac{5}{3}.1$.

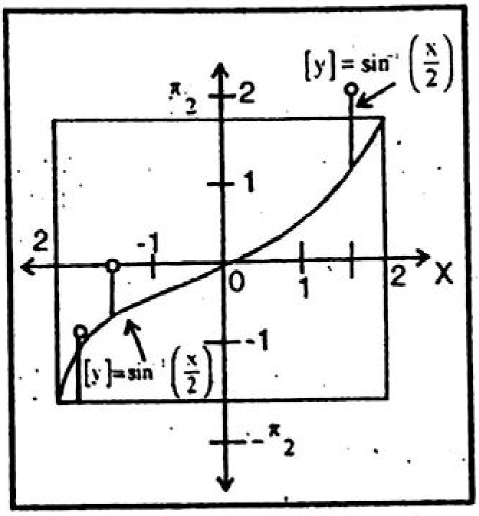
Hence $x=0, \frac{5}{3}$, i.e. the only two solutions.

**Ex. 12. Draw the graph of $[y]=\text{Sin}^{-1}\left(\frac{x}{2}\right)$, where $[.]=$ G.I.F.**

**Sol.** First draw the graph of

$y=\text{Sin}^{-1}\left(\frac{x}{2}\right)$, $\Rightarrow -1\le\frac{x}{2}\le 1$, i.e. $-2\le x\le 2$

and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.




$[y]=-2,-1,0,1$. $\therefore \text{Sin}^{-1}\left(\frac{x}{2}\right)=-2,-1,0,1$. or

$\frac{x}{2}=-\text{Sin}2, -\text{Sin}1, 0, \text{Sin}1$,

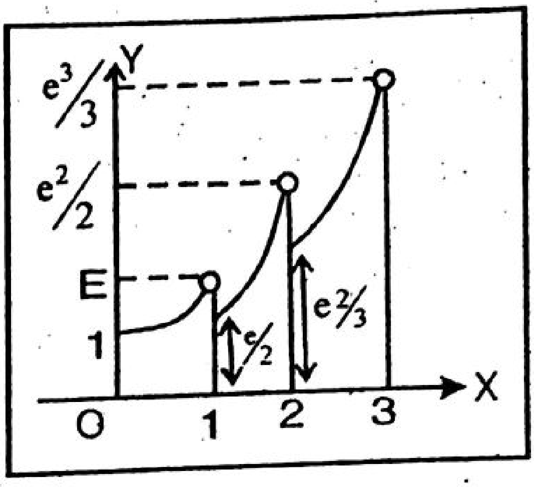
$\Rightarrow x=-2\,\text{Sin}2, -2\,\sin 1, 0, 2\sin 1$.

Now when $x=-2\sin 2$, $-2\le y<-1$.

$\therefore x=-2\,\text{Sin}1, -1\le y<0$; $x=0, 0\le y<1$; $x=2\,\text{Sin}1, 1\le y<2$.

**Ex. 13. Find the domain and range of $f(x) = \frac{e^x}{1+x} \forall x \in R$.**

**Sol.** Let $f(x) = \frac{g(x)}{F(x)}$, where $g(x) = e^x$ and $F(x) = 1+[x]$, now domain of $g(x) = e^x$ is set of all values of x as it is defind $\forall x \in R$.




$\therefore D_f$ of $e^x$ = Set of all real numbers.

Also, domain of $f(x) = 1+[x]$ is set of all values of $x \in R$. But

$1+[x] = 0 \ \forall \ -1 \le x < 0$.

$\therefore$ Domain of f(x) will be $R - \{-1 \le x < 0\} = (-\infty, -1) \cup [0, \infty)$.

For range $x \ge 0$, $1+[x] \ge x, \forall x \in [0, \infty)$ and $e^x \ge 1 \ \forall x \in [0, \infty)$.

For $x \in [0,1)$; $f(x) = e^x$ and for $x \in [1,2)$, $f(x) = \frac{e^x}{2}$ and so on.

Thus all values of y or $e \ge 1$. $\therefore$ Range $f(x) \in [1, \infty)$.

**Ex. 14. Evaluate :** $\underset{n \to \infty}{Lt} \frac{\{x\}+\{2x\}+\{3x\}+....+\{nx\}}{n^2}$ **where $\{x\}$ is fractional part of x.**

**Sol.** $\because 0 \le \{rx\} < 1$ where $r = 1, 2, 3, \ldots, n$.

$$\therefore 0 \le \sum_{r=1}^{n} \{rx\} < \sum_{r=1}^{n} (1), \Rightarrow 0 \le \sum_{r=1}^{n} \{rx\} < n.$$

Dividing throughout by $n^2$, we get

$$\frac{0}{n^2} \le \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \frac{1}{n}, \Rightarrow \underset{n \to \infty}{Lt} 0 \le \underset{n \to \infty}{Lt} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \underset{n \to \infty}{Lt} \frac{1}{n},$$

$$\Rightarrow 0 \le \underset{n \to \infty}{Lt} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < 0.$$

$$\therefore\ 0 \le \lim_{n\to\infty} \frac{\{x\}+\{2x\}+....+\{nx\}}{n^2} < 0.$$

According to Sandwich theorem or Squeeze principle

$$\lim_{n\to\infty} \frac{\{x\}+\{2x\}+......+\{nx\}}{n^2} = 0.$$

**Ex. 15. If [x] denotes the integral part of x and**

$$f(x) = [x]\left\{\frac{\text{Sin}\frac{\pi}{x+1} + \text{Sin}\pi\ x+1}{1+\ x}\right\}$$, then show that f(x) is

discountinuous at all integral points.

**Sol.** Sin $\pi[x+1] = 0$ and $[x+1] = [x] + 1$.

$$\therefore\ f(x) = \frac{[x]}{1+[x]}\text{Sin}\frac{\pi}{[x]+1}.$$

At $x = n$, $n \in I$, then $f(x) = \frac{n}{1+n}\ \text{Sin}\frac{\pi}{n+1}$. For $n < x < n+1$, $n \in I$,

$f(x) = \frac{n}{1+n}\ \text{Sin}\frac{\pi}{n+1}$. $\therefore\ \lim_{x\to n+0} f(x) = \frac{n}{1+n}\ \text{Sin}\frac{\pi}{n+1}$.

For $n-1 < x < n$, $[x] = n-1$. $\therefore f(x) = \frac{n-1}{n}\text{Sin}\frac{\pi}{n}$.

$\therefore \lim_{x\to n-0} f(x) = \frac{n-1}{n}\text{Sin}\frac{\pi}{n}$ and $f(x) = \frac{n}{1+n}\ \text{Sin}\frac{\pi}{n+1}$.

Hence f(x) is discontinuous at all $n \in I$.

**Ex. 16. Let [x] stands for the g.i.f. Find the derivative of f(x) = $(x+[x^3+1])^{x^2+\text{Sin}x}$, wherever it exists in (1,3/2). Indicate the points where it does not exist.**

**Sol.** We see that $[x^3+1]$ takes a jump from 2 to 3 at $2^{1/3}$ and again from 3 to 4 at $3^{1/3}$ in (1, 3/2).

$\therefore$ It is discontinuous at $x=2^{1/3}$ and $x=3^{1/3}$, i.e. it is not differentiable at

$x=2^{1/3}$ and $x=3^{1/3}$.

Now, $f(x)=(x+[x^3+1])^{x^2+\sin x} = \begin{cases} (x+2)^{x^2+\sin x} & \text{if } 1<x<2^{1/3} \\ (X+3)^{X^2+\sin x} & \text{if } 2^{1/3}<x<3^{1/3} \\ (X+4)^{X^2+\sin x} & \text{if } 3^{1/3}<x<\frac{3}{2} \end{cases}$

$$f'(x)=\begin{cases} (x+2)^{x^2+\sin x}\left\{(2x+\cos x)\log\left((x+2)+\dfrac{x^2+\sin x}{x+2}\right)\right\} & \text{if } 1<x<2^{1/3} \\ (x+3)^{x^2+\sin x}\left\{(2x+\cos x)\log\left((x+3)+\dfrac{x^2+\sin x}{x+3}\right)\right\} & \text{if } 2^{1/3}<x<3^{1/3} \\ (x+4)^{x^2+\sin x}\left\{(2x+\cos x)\log\left((x+4)+\dfrac{x^2+\sin x}{x+4}\right)\right\} & \text{if } 3^{1/3}<x<\frac{3}{2} \end{cases}$$

**Ex. 17. Evaluate :**

**(i)** $\dfrac{\int_0^n [x]\,dx}{\int_0^n \{x\}}$; **(ii)** $\int_0^{[x]}\left(\int_0^{[x]}[x]-[x-\frac{1}{2}]dx\right)dx$;

**(iii)** $\int_0^{n\pi}\sin\left[\frac{2x}{n}\right]dx$; **(iv)** $\int_{-10}^{0}\dfrac{\left|\dfrac{2[x]}{3x-[x]}\right|}{\dfrac{2[x]}{3x-[x]}}dx$; **(v)** $\int_{-1}^{1}\dfrac{\sin^2 x}{\left[\frac{x}{\sqrt{2}}\right]+\frac{1}{2}}dx$

**(vi)** $\int_0^{\infty}[ne^{-x}]dx$, where $[x]$ = G.I.F.; $\{x\}$ = F.P. and n is natural numbers.

**17. (i)** Let $I=\dfrac{\int_0^n x\,dx}{\int_0^n \{x\}dx}=\dfrac{\int_0^1 x\,dx+\int_1^2 x\,dx+\int_2^3 x\,dx+\ldots\ldots+\int_{n-1}^n x\,dx}{n\int_0^1\{x\}dx}$

$$= \frac{0+1.\int_1^2 dx + 2\int_2^3 dx + \dots + (x-1)\int_{n-1}^{n} dx}{n\int_0^1 x\,dx} = \frac{(n-1)n/2}{n/2} = (n-1).$$

**(ii)** $\int_0^{[x]}\left(\int_0^{[x]}\left(x - \left[x + f\left(-\frac{1}{2}\right)\right]\right)dx\right)dx$ (let $x = [x] + f, 0 \le f < 1$)

$= \int_0^{[x]}\left(\int_0^{[x]} -[f\left(-\frac{1}{2}\right)]dx\right)dx$. $\because 0 \le f < 1 \therefore -\frac{1}{2} \le f\left(-\frac{1}{2}\right) < \frac{1}{2}$, i.e. $[f\left(-\frac{1}{2}\right)] = -1, 0.$

**Case 1.** If $[f\left(-\frac{1}{2}\right)] = 0$, then $\int_0^{[x]}\left(\int_0^{[x]} 0.dx\right)dx = 0.$

**Case 2.** If $[f\left(-\frac{1}{2}\right)] = -1$, then $\int_0^{[x]}\left(\int_0^{[x]} 1.dx\right)dx = 0$

$\int_0^{[x]}[x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \dots + \int_{[x]-1}^{[x]} [x]dx = 0+1+2+$

$\dots\dots + ([x]-1) = -\frac{([x]-1)[x]}{2}.$

**(iii)** $\int_0^{x\pi} \text{Sin}\left[\frac{2x}{\pi}\right]dx = \int_0^{\frac{\pi}{2}} \text{Sin}0\,dx + \int_{\frac{\pi}{2}}^{\pi} \text{Sin}\,1\,dx + \int_{\pi}^{\frac{3\pi}{2}} \text{Sin}\,2\,dx + \dots\dots$

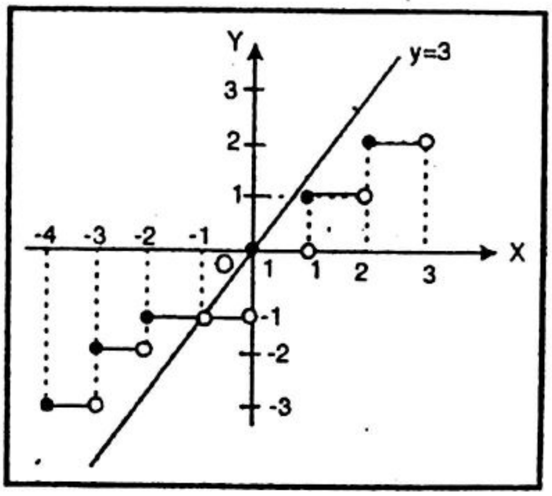
$+ \dots\dots + \int_{n\pi-\frac{\pi}{4}}^{n\pi-\frac{\pi}{2}} \text{Sin}\,(2n-2)dx + \int_{n\pi-\frac{\pi}{2}}^{n\pi} \text{Sin}(2n-1)dx =$

$\frac{\pi}{2}$ {Sin 1+Sin 2+Sin 3+......+Sin (2n–2) +Sin (2n–1)} =

$$\frac{\pi}{2}\left[\frac{\text{Sin}\left(a+\frac{(2n-2)}{2}\right)h}{\text{Sin}\frac{h}{2}}\times\text{Sin}\left\{\frac{(2n-1)h}{2}\right\}\right] \text{ where a =1, h=1,}$$

$$= \frac{\pi}{2}\left[\frac{\text{Sin}\left\{1+\frac{2(n-1)}{2}\right\}.\text{Sin}\frac{(2n-1)}{2}}{\text{Sin}\frac{1}{2}}\right]=\frac{\pi}{2}.\frac{\text{Sinn}}{\text{Sin}\frac{1}{2}}\text{Sin}\frac{(2n-1)}{2}$$

**(iv)** Let $f(x) = \dfrac{\left|\dfrac{2[x]}{3x-[x]}\right|}{\dfrac{2[x]}{3x-[x]}}$.




Clearly f is not defined if x=0
and when 3x=[x].

∴ In (-10,0), f is not defined at $x = -\frac{1}{3}$.

**Case. 1.** $x \in \left(-10,-\frac{1}{3}\right)$, $[x]<0$ and $3x-[x]<0$

$\therefore \frac{[x]}{3x-[x]}>0$, i.e. $f(x)=1$.

**Case. 2.** $x \in \left(-\frac{1}{3},0\right)$, $[x]<0$ and $3x-[x]>0$, i.e. $f[x]=-1$.

$$\text{Now}, \int_{-10}^{0} f(x)dx = \int_{-10}^{-1/3} dx + \int_{-1/3}^{0} (-1)dx = (x)_{-10}^{-1/3} - (x)_{-1/3}^{0} = \left(-\frac{1}{3}+10\right)$$

$$-\left(0+\frac{1}{3}\right) = \frac{28}{3}$$

(v) Let $I = \int_{-1}^{1} f(x)dx$, where $f(x) = \dfrac{\text{Sin}^2 x}{[x/\sqrt{2}]+1/2}$

$$\therefore f(-x) = \frac{\text{Sin}^2 x}{[-x/\sqrt{2}]+1/2} = \frac{\text{Sin}^2 x}{-1-[x/\sqrt{2}]+1/2} \quad (\because [x]-[-x] = -1 \forall x \notin I)$$

$\because$ $x/\sqrt{2}$ is not an integer in $(-1,0)$ and $(0,1)$.

$$\therefore f(-x) = \frac{\text{Sin}^2 x}{[x/\sqrt{2}]-1/2} = -f(x), \text{ i.e. } f(x) \text{ is an odd function in } x$$

$$\therefore I = \int_{-1}^{1} f(x)dx = 0.$$

(vi) Let $f(x) = n.e^{-x} \ \forall \ n \in N, x \in R^+$.

$\therefore f'(x) = -ne^{-x} < 0 \ \forall \ x \in R.$

This is decreasing 0 to $\infty$ and in such type of problems we always break the internal for $x > \log_e n$, we have $[ne^{-x}] = 0$.

$$\therefore \int_0^{\infty} [xe^{-x}]dx = \int_0^{\log\frac{n}{n-1}} (x-1)dx + \int_{\log\frac{n}{n-1}}^{\log\frac{n}{n-2}} (x-2)dx + \int_{\log\frac{n}{n-2}}^{\log\frac{n}{n-3}} (x-3)dx + \ldots\ldots\ldots + $$

$$\int_{\log\frac{n}{2}}^{\log} 1\,dx + \int_{\log n}^{\infty} 0\,dx = (n-1)\left(\log\frac{n}{n-1} - 0\right) + (n-2)\left(\log\frac{n}{n-2} - \log\frac{n}{n-1}\right)$$

$$+\ldots\ldots+1.\left(\log\frac{n}{1} - \log\frac{n}{2}\right) + \ldots\ldots + 0(\infty - \log n).$$

Now writing the terms in reversed order, we have

$$\int_0^\infty ne^{-x}\,dx = 1\left(\log\frac{n}{1}-\log\frac{n}{2}\right)+2\left(\log\frac{n}{2}-\log\frac{n}{3}\right)+3\left(\log\frac{n}{3}-\log\frac{n}{4}\right)+$$

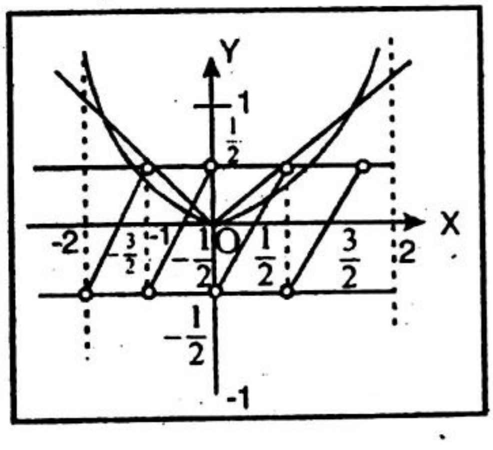
$$.....+(n-2)\left(\log\frac{n}{n-2}-\log\frac{n}{n-1}\right)+(n-1)\log\frac{n}{n-1}$$

$$=\log n+(2-1)\log\frac{n}{2}+(3-2)\log\frac{n}{3}+.......+[(n-1)-(n-2)]\log\frac{n}{n-1}$$

$$=\log n+\log\frac{n}{2}+\log\frac{n}{3}+....+\log\frac{n}{n-1}=\log\left(\frac{n}{1}\cdot\frac{n}{2}\cdots\frac{n}{n-1}\right)$$

$$=\log\left\{\frac{n^{n-1}}{(n-1)!}\right\}=\log\left\{\frac{n^n.n^{-1}}{(n-1)!}\right\}=\log\left(\frac{n^n}{n!}\right)$$

**Ex. 18. Let** $f(x)=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x\notin I \\ 0 & \text{if } x\in I\end{cases}$ **where [.] denotes the G.I.F. If $g(x)=\max\{x^2, f(x), |x|\}$, $\forall\ x\in[-10, 10]$, then find the value of $\int_{-2}^{2} g(x)dx$.**




**Sol.** It is clear from the graph

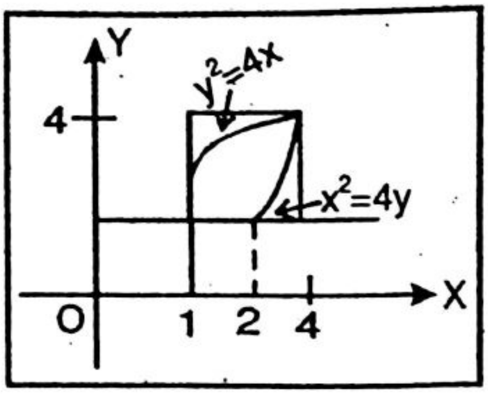
$$g(x)=\begin{cases} X^2 & \text{if } -2\le x\le -1 \\ -X & \text{if } -1\le x\le -\frac{1}{4} \\ X+\frac{1}{2} & \text{if } -\frac{1}{4}\le x\le 0 \\ X & \text{if } 0\le x\le 1 \\ X^2 & \text{if } 1\le x\le 2\end{cases}$$

$$\therefore \int_{-2}^{2} f(x)dx=\int_{-2}^{-1} x^2dx+\int_{-1}^{-1/4}(-x)dx+\int_{-1/4}^{0}\left(x+\frac{1}{2}\right)dx+\int_0^1 x\,dx=\frac{275}{98}$$

**Ex. 19. Let the curves $C_1: y^2=4[\sqrt{y}]x$ and $C_2: x^2=4[\sqrt{x}]y$ where [.] denotes the G.I.F., find the area of the region enclosed by these two curves with in the square formed by the lines, x=1,y=1, x=4, y=4.**

**Sol.** Since 1<x<4, so $1<\sqrt{x}<2$. $\therefore [\sqrt{x}]=1$ and similarly $[\sqrt{y}]=1$.

Now, $c_1 : y^2=4x$ and $c_2 : x^2=4y$.




$\therefore$ Reqd. area $= \int_1^4 2\sqrt{x}\,dx-(2-1).1-\int_2^4 \frac{x^2}{4}dx$

$=\frac{4}{3}\, x^{3/2}\,\Big|_1^4 -1-\frac{1}{4}\, x^3\,\Big|_2^4=\frac{11}{3}$ sq.units.

**Ex. 20. If x is a real number in [0,1], then show that the value of $\underset{m\to\infty}{Lt}\ \underset{n\to\infty}{Lt}\left\{1+\cot^{2m}(n!\pi x)\right\}$ is given by 2 or 1 according as x is rational or irrational.**

**Sol. Case.1.** If $x \in Q$, then n! $\pi$x will be an integral multiple of $\pi$ for large values of $\pi$.

$\therefore$cos (n!$\pi$x) will be either 1 or –1, i.e. $\cos^{2m}(n!\pi x)=1$

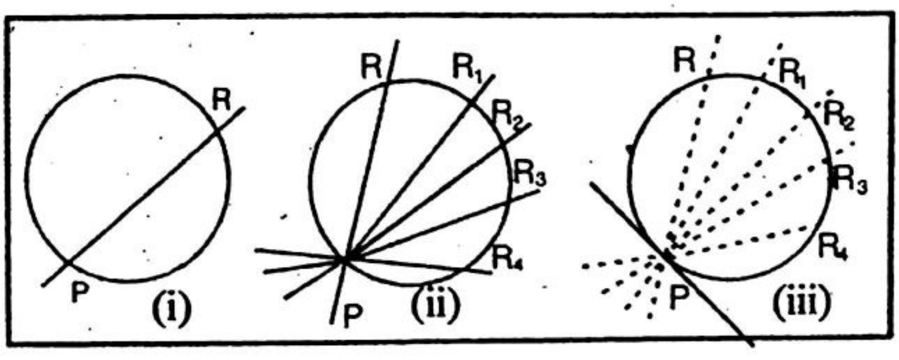
$\therefore \underset{m\to\infty}{Lt}\ \underset{n\to\infty}{Lt}\ \{1+\cos^{2m}(n!\pi x)\} = 1+1=2.$

**Case. 2.** If $x\in Q$, then x!$\pi$x will not be an integral multiple of a $\pi$.

$\therefore$cos (x!$\pi$x) will be between –1 or I, i.e. $\cos^{2m}(n!\pi x)=0$.

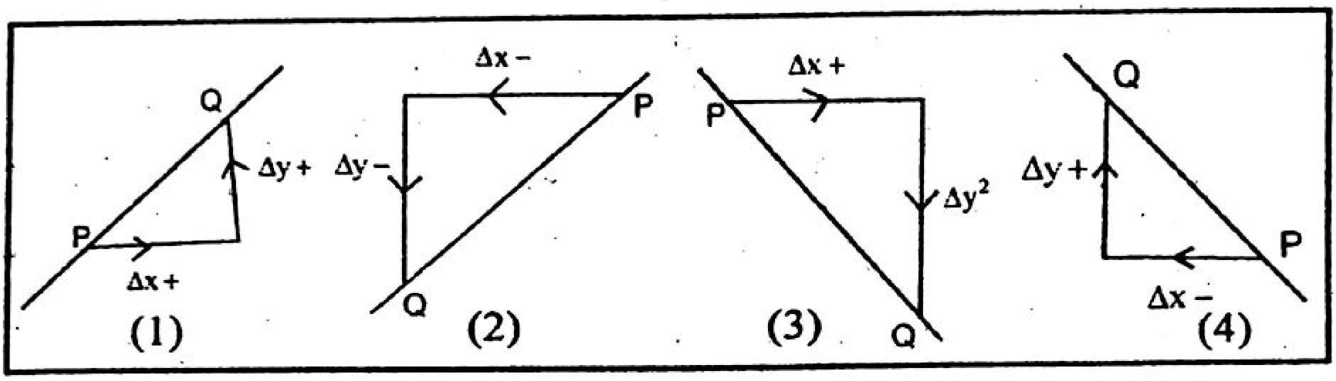
Hence, $\underset{m\to\infty}{Lt}\ \underset{n\to\infty}{Lt}\{1+\cos^{2m}(n!\pi x)\} = 1+0=1.$

❑ **LIMING POSITION OF A SECANT :** A **secant** (Latin **seco**, I cut) is a straight line that cuts a curve at two points. If the secant in fig (i) moves in such a way that P remains fixed in position, but R approching P more and more closely, successively occupies the position $R_1, R_2, R_3$ and $R_4$, shown in fig.(ii) . It will be seen that the length of chord PR and the length arc PR both becomes successively less and less as R approaches P, while all the time the secant of which PR is part approaches closer and closer to its **limiting position,** which is that of the tangent to the curve at point P as shown in fig(iii).

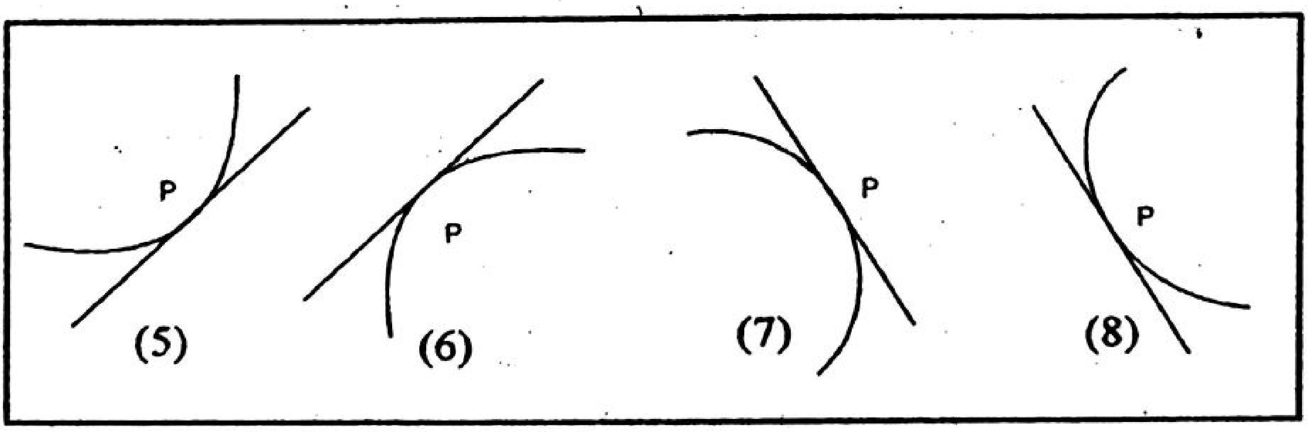


**❑ SIGN OF $\frac{dy}{dx}$ :** Let y=f(x). If $\frac{\Delta y}{\Delta x}$ is(+)ve $\Delta x$ and $\Delta y$ have the same sign, i.e. an increase in x produces an increase in y and a decrease in x productes a decrease in y.

If y=f(x) be drawn, P being the point (x,y) and Q $(x+\Delta x, y+\Delta y)$. Then if $\Delta x$ and $\Delta y$ both (+)ve P and Q will be placed as in (1), if both negative as in (2). In each case the chord PQ has a (+)ve gradient. But if $\Delta x$ be (+)ve and $\Delta y$ (–)ve, P and Q will be placed as in (3), if $\Delta x$ (–)ve, and $\Delta y$ (+)ve, as in (4).




So, if $\frac{dy}{dx}$ is (+)ve for a given value of x, x and y are both increasing or both decreasing, but if $\frac{dy}{dx}$ is (–)ve, x is increasing and y decreasing or vice-versa. In the graph if $\frac{dy}{dx}$ is (+)ve at the point (x,y), the gradient of



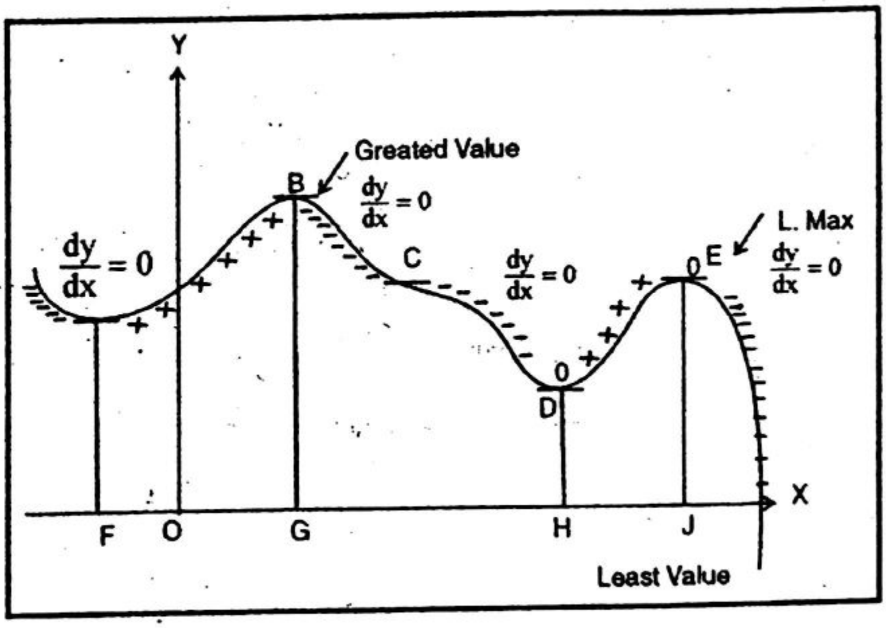

the tangent is (+)ve and the curve in the neighbourhood of P is shaped like (5) or (6), but if $\frac{dy}{dx}$ is (-)ve, the shape is like (7) or (8).

## ❑ MAXIMA AND MINIMA :

At A and D, the gradient is changing from (–)ve to (+)ve and these are called **minimum points**, FA and HD are **minimum** (or **minima**) **values** of y . At B and E, the gradient is changing from (+)ve to (–)ve and these are called **maximum** (or **maxima**)**points**, GB and JE are **maximum values** of y.

The words max. and min. are used in the sense of greatest and least only in the immediate vicinity of the point, this local meaning is bought out clearly in this curve, since a maximum value, JE as in fact less than a minimum value FA.

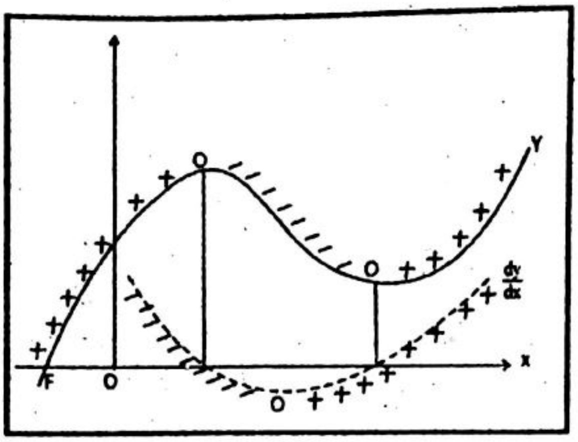
At C the gradient is zero, but is not changing sign, this point is a **point of inflexion,** which may be linked to the point on an S-bend at which a road stops turning left and begins to turn right or vice-versa. The gradient of a curve at a point of inflexion need not be zero (the reader should be able to spot four more in given fig.), however at this stage we are concerned only with searching for maximum and minimum. We need to bear in mind points of inflexion only as a third possibility at points where the gradient of a curve is zero, At any point where the gradient of a curve is zero, y is said to be a **stationary value.** Any max or min point is called a **turning point** and y is said to be a **turning** value there.




**Note:** While a function has at most only one **greatest value** or **absolute max.** or **global max** and at most only one **least value** or **absolute min.** or **global min.**. It may have several **max. value** or **local max** and several **min valueor local min.**

## ❑ DISTINGUISHING BETWEEN MAX AND MIN POINTS :

At a max.point $\frac{dy}{dx}=0$. Just before the point $\frac{dy}{dx}$ is (+)ve and just after if $\frac{dy}{dx}$ is (–)ve. Thus in passing from one side of the point to the other $\frac{dy}{dx}$ decrease. If a graph of $\frac{dy}{dx}$ against x is plotted, it has a downword slope in the region of the point under consideration. Now the slope of the $\left(\frac{dy}{dx}-x\right)$ graph is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$. So, at the max. point $\frac{d^2y}{dx^2}$ is (-)ve (except as below). In fig. the graph of y is represented by a continuous line and the graph of $\frac{dy}{dx}$ by a dotted line. In a similar way it can be seen that at a min.point $\frac{d^2y}{dx^2}$ will be (+)ve (except as below) This provides a method of investigating the nature of a turning point.




If, at a point, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$,i.e. **max.point.**

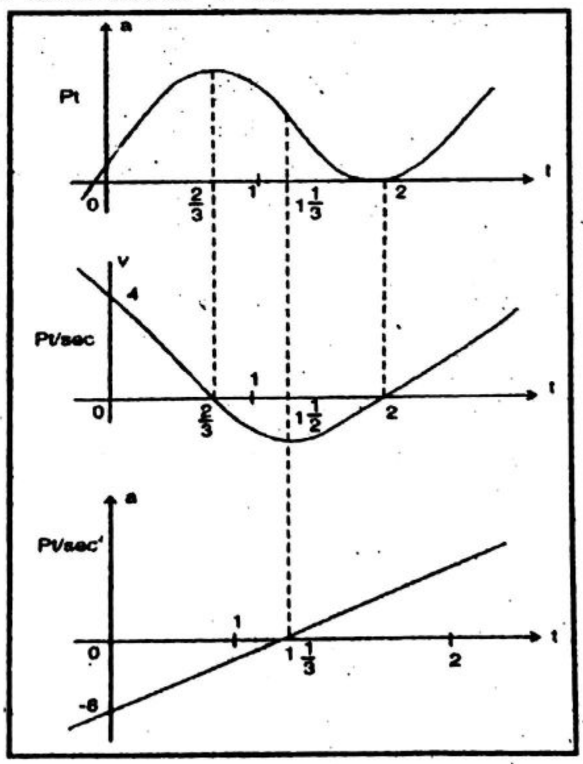
If, at a point, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$,i.e. **min.point.**

**Ex. O is a point on a straight line. A particle moves along the line so that it is s ft. from O,t see. after a certain instant, where $s=t(t-2)^2$. Describe the motion before and after t=0.**

**Sol.** Here $s=t^3-4t^2+4t$.

$\therefore \frac{ds}{dt}=3t^2-8t+4=(3t-2)(t-2)=v.$

This graph has a min.point $\left(1\frac{1}{3},-1\frac{1}{3}\right)$ and passes through $\left(\frac{2}{3},0\right)$, (2,0) and (0,4), it is the middle sketch and upper sketch the max pt. $\left(\frac{2}{3},\frac{32}{27}\right)$ and min. point (2,0).



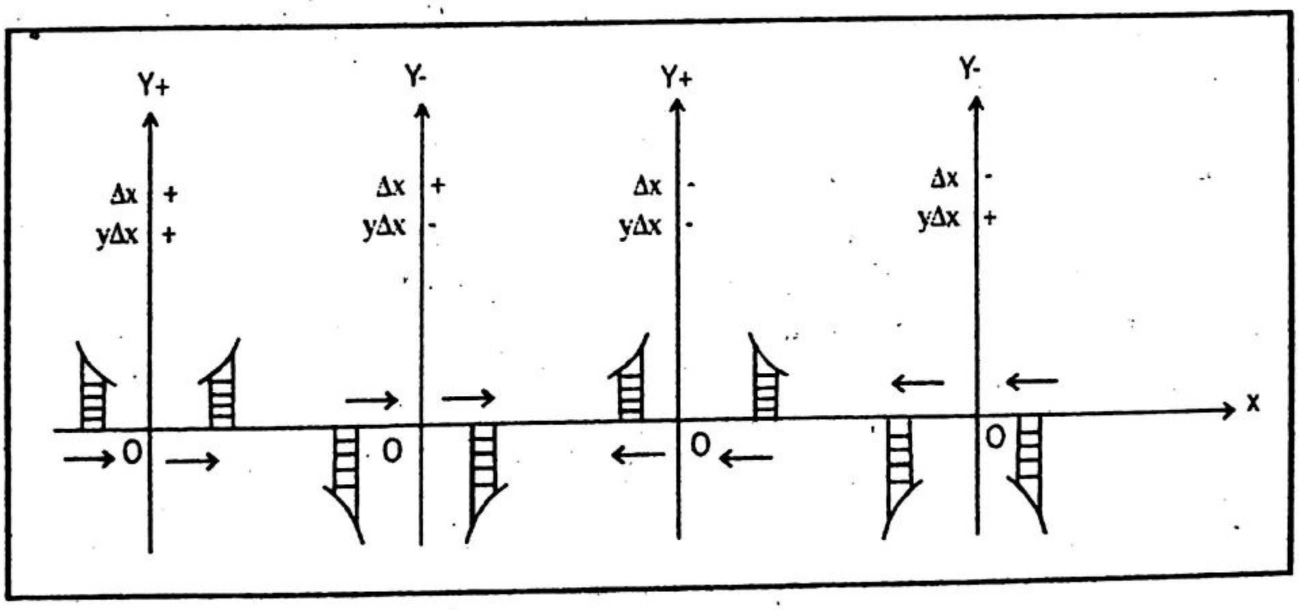
Again, $\frac{dv}{dt}$=a=6t–8 and is the bottom sketch.

Notice that the max. and min. values of s occur when v=$\frac{ds}{dt}$=0 and that the min. value of v occurs when a=$\frac{dv}{dt}=\frac{d^2s}{dt^2}$=0.

Before t=0, the particle is approaching 0 from the (-)ve side, at t=0 it is passing through 0 with velocity 4ft./Sec. and acceleration – 8ft/Sec$^2$.

Hence its speed is decreasing and it comes momentarily to rest $\frac{32}{27}$ ft. from O (on the (+)ve side), when t=$\frac{2}{3}$, it returns to 0, where it is momentarily at rest when t=2.and thereafter it moves away from 0 in the (+)ve direction.

❑ **SIGN OF AREA :** If we advance from left to right , i.e. if Δx is (+)ve, yΔx representing the area of our typical strip is (+)ve if y is




(+)ve, i.e. if the strip lies above the x-axis and (–)ve if the strip lies below the x-axis is given fig.

If we advance from right to left, i.e.if Δx is (–)ve, yΔx is (+)ve if the strip lies below and (–)ve if the strip lies above the x-axis.

❑ **DIVISION BY ZERO :** Values like $\frac{0}{0},\frac{3}{0},\frac{a}{0}$ are inadmissible. There have no meaning. Some students are under the impression that $\frac{5}{0}$ is

**infinity** which is wrong. **Infinity** ($\infty$) is not a number it is a symbol.

Nor is $\frac{0}{0}=1$.If we let $\frac{0}{0}=1$, then we shall be led to absurdities.Consider, for example, what would be the result of dividing by zero both side of an equation like $5\times0=2\times0$.

Thus the statement $x=\infty$ is meaningless unless it stands for the statement $x\to\infty$.

Corresponding to the notion of infinity, the symbol (zero) denotes the continuous decrease without limit. The student should understand the following facts also $\frac{1}{x}\to0$ if $x\to+\infty$ and $\frac{1}{x}\to0$ if $x\to-\infty$.

**Ex. Show that the number 0(zero) has no reciprocals.**

**Sol.** Suppose that 0 has the reciprocals $\frac{1}{0}$, then $0.\left(\frac{1}{0}\right)=1$

($\because$ product of two reciprocals = 1).

Also, $0.\left(\frac{1}{0}\right)=0$ ($\because$ product of any number by zero is 0). It follows that $1=0$, which brings contradiction.

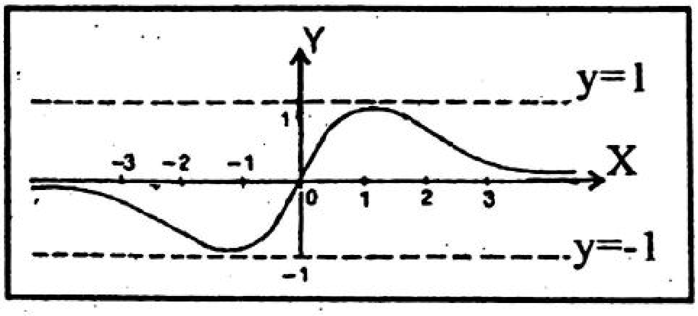
Hence 0 has no reciprocal.

# NOTES ON GRAPHS

❑ **Introduction :** The graphs of a function provides many information regarding the function f(x). All the information we need to draw the graph of a function accurately can be put together to a name and that is **DR. T. SAM.** This means **Domain (D), Range (R), Transformation (T), Symmetry (S), Asymptotes (A), Maximum and Minimum (M),** Now we shall discuss some important graphs.

**Draw the graphs of each of the following functions:**

**(a)** $f(x)=\frac{2x}{1+x^2}$.




(i) f(x) is a odd function,

(ii) graph is symmetrical about origin,

(iii) f(x) attains minimum value f(x)=0,

(iv) Maximum value of the function f(x) equals 1.

(v) f(0)=0 and $|f(x)|\le 1$ indeed $(1-|x|)^2 \ge 0$ or $1+x^2 \ge |x|$,

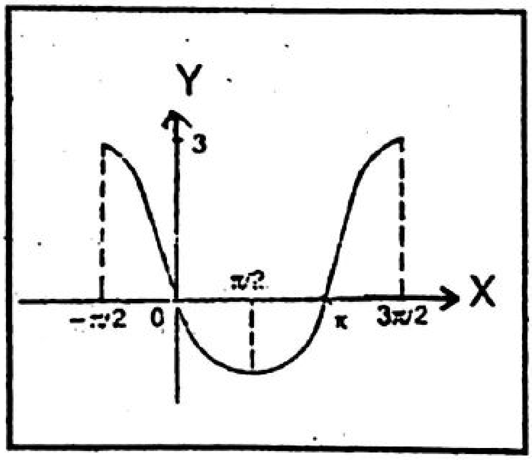
or $1\ge \frac{2|x|}{1+x^2}=|f(x)|$

Since $f(x)\ge 0$ at $x\ge 0$ and $f(1)=1$ in the interval $[0,\infty)$ the maximum value of the function f(x) equals 1, the minimum value being zero.

Domain :$-\infty < x < \infty$.

Range: $-1 \le f(x) \le +1$

**(b)** $f(x)=\sin^2 x-2\sin x=(\sin x-1)^2-1$.



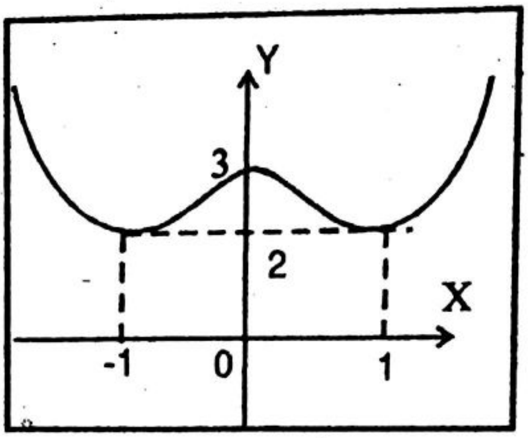

When f(x) increases then sin x decreases. Similarly, f(x) decreases when sin x increases.

When sin x increases

$(-\pi/2\le x\le \pi/2)$ sin x decreases

$(\pi/2\le x\le 3\pi/2)$.

**(c)** $f(x) = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2$




(i) f(x) is even function,

(ii) f(x) is symmertric about y-axis,

(iii) f(x) is attains min.value at $x = \pm 1$,

(iv) f(x) decreases x=0 to 1 and similarly x=0 to–1,

(v) f(x) increases x=1 to $\infty$,

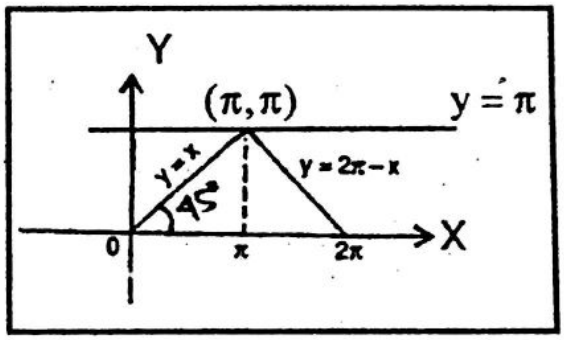
(vi) $f(0)=3$, $D_f : -\infty < x < \infty$ and $R_f : 2 \le y < \infty$.

**(d)** $y = f(x) = \cos^{-1}(\cos x)$

f(x) is periodic with period $2\pi$.

$f(x) = x, 0 \le x \le \pi$ [from definition of $\cos^{-1} x$]

$= 2\pi - x, \quad \pi \le x \le 2\pi$




Let $2\pi - x = x^1$

or $2\pi - x^1 = x$

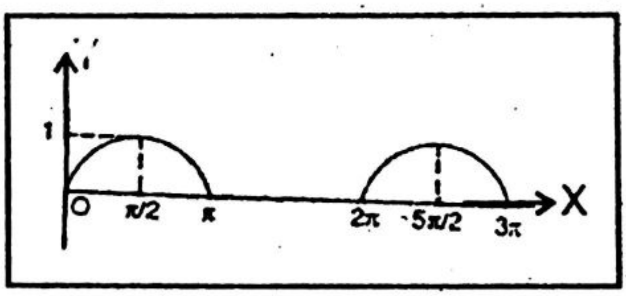
$$\pi \le 2\pi - x^1 \le 2\pi$$
$$-\pi \ge x^1 - 2\pi \ge -2\pi$$
$$\pi \ge x^1 \ge 0$$
$$0 \le x^1 \le \pi$$

$\cos^{-1} \cos(2\pi - x^1) = \cos^{-1} \cos x'^1 = x^1 = 2\pi - x$

Domain: $-n\pi \le x \le n\pi$ [1,2,....=n]

Range : $0 \le f(x) \le \pi$

**(e)** $y = f(x) = \sqrt{\sin x}$




Here $\sin x \ge 0, 0 \le x \le \pi$

$$2\pi \le x \le 3\pi$$
$$2n\pi \le x \le (2n+1)\pi$$

[n=0,1,2....]

Domain : $2n\pi \le x \le (2n+1)\pi$ [n = 0,1,2...]

Range : $0 \le f(x) \le 1$

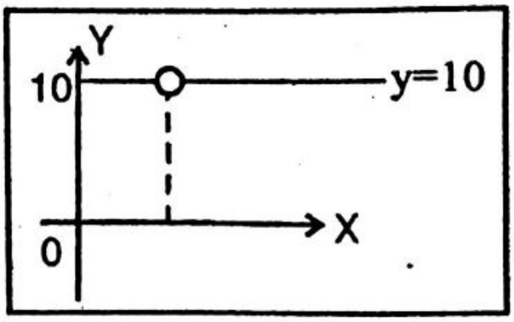
**(f)** $y=f(x)=x^{1/\log x}$

Domain : $0<x<1$ and $1<x<\infty$

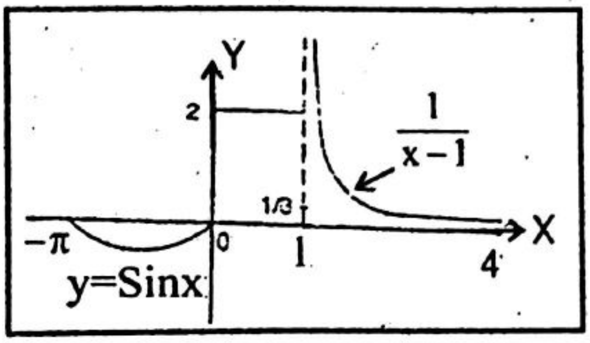
$f(x)=x^{1/\log x}$

$=x^{(\log_{10}10/\log_{10}x)}=x^{\log_x 10}$.

$\therefore$ $f(x)=10$.




**(g)** $y=\begin{cases}\sin x \text{ at } -\pi\le x\le 0\\ 2 \text{ at } 0<x\le 1\\ 1/(x-1) \text{ at } 1<x\le 4\end{cases}$




**(h)** $y=[x]^2$, we have

$y=0$ when $0\le x<1$

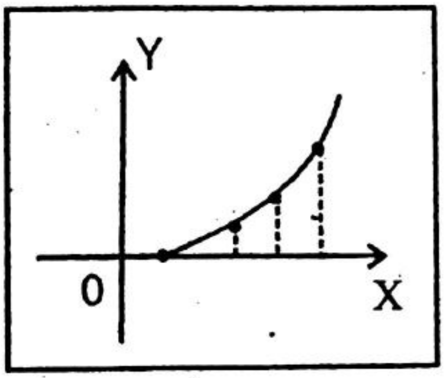
$=1$ when $1\le x<2$

$=4$ when $2\le x<3$

$=9$ when $2\le x<4$

and so on.

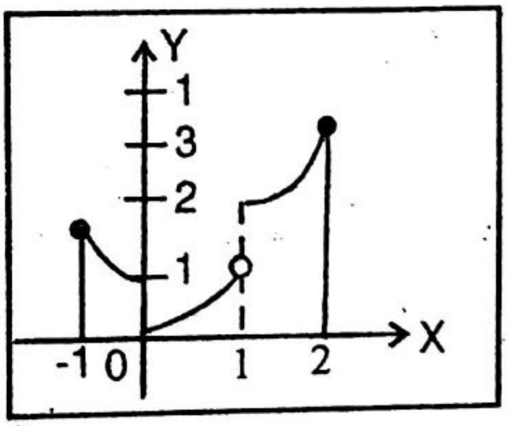
Similarly in the case of negative values.



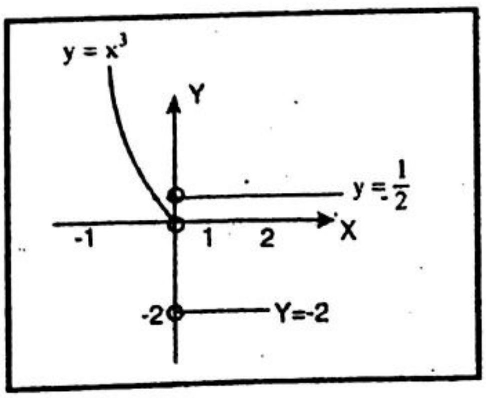

**(i)** $y=x^2+[x]^2$, we have

$y=\begin{cases}x^2 \text{ if } x\in[0,1[\\ x^2+1 \text{ if } x\in[1,2[\\ x^2+4 \text{ if } x\in[2,3[\\ x^2+1 \text{ if } x\in[-1,0[\end{cases}$

and so on.




**(j)** $y=\begin{cases}-2 \text{ at } x>0\\ 1/2 \text{ at } x=0\\ -x^3 \text{ at } x<0\end{cases}$



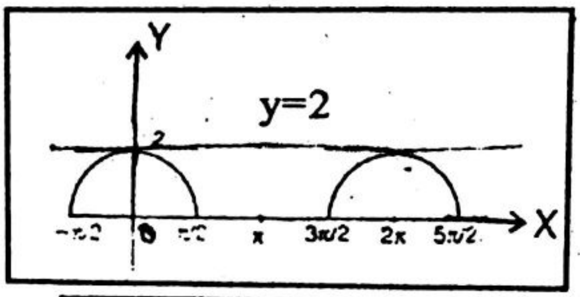
**(k)** $y = \text{Cos}x + |\text{Cos}x|$

$y = \cos x + \cos x \quad \cos x \geq 0$

$y = \cos x - \cos x \quad \cos x < 0$

$y = \cos 2x \quad -\pi/2 \leq x \leq \pi/2$

$y = 0 \quad \pi/2 \leq x \leq 3\pi/2$




**(i)** $y = |x+2|x$

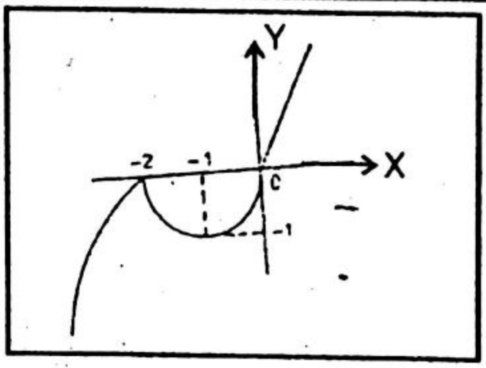
$y = (x+2)x \quad x \geq -2$

$= -(x+2)x \quad x \leq -2$

$y = x^2 + 2x$

$= (x+1)^2 - 1 \quad x \geq -2$

$y = 1 - (x+1)^2 \quad x \leq -2$




**(m)** $y = 2|x-2| - |x+1| + x$ at $x \geq 2$

$y = 2(x-2) - (x+1) + x$

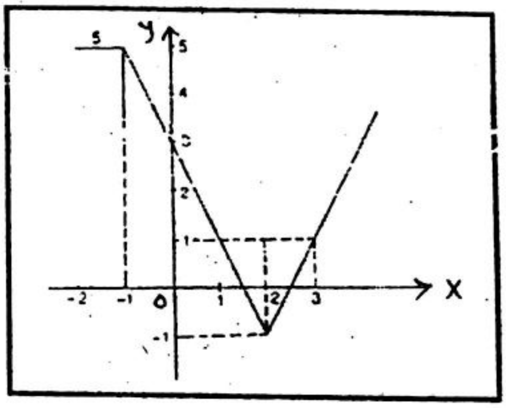
$= 2x - 5 \quad -1 \leq x \leq 2$

$y = 2(2-x) - (x+1) + x$

$= -2x + 3 \quad x \leq -1$

$y = 2(2-x) + (x+1) + x$

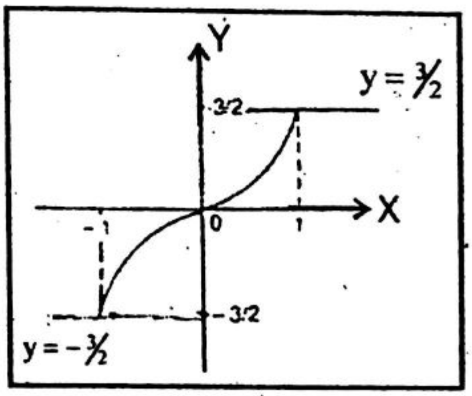
$= 5$




**(n)** $y = 2^x - 2^{-x}$

Let us define domain as $(-1,1)$

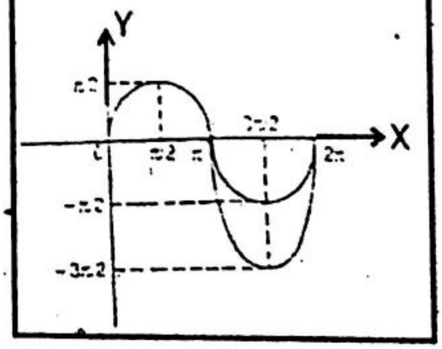
$$y = 2^x - 2^{-x}(-1,1)$$
$$y = 2^{-1} - 2^{1}$$
$$= -3/2$$
$$y = 2^{1} - 2^{-1}$$
$$= 3/2$$
$$y = 2^0 - 2^{-0}$$
$$= 0$$




**(o)** $y = x\text{Sin}x \;:\; 0 \leq x \leq \pi;$

$y = x\text{Sin}x \;:\; 0 \leq y \leq \dfrac{\pi}{2} \;:\; 0 \leq x \leq \dfrac{\pi}{2};$

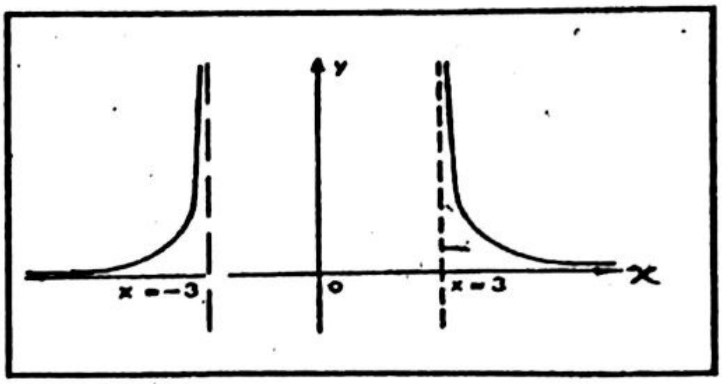
$: \; \dfrac{\pi}{2} \leq y \leq 0 \;:\; \dfrac{\pi}{2} \leq x \leq \pi;$



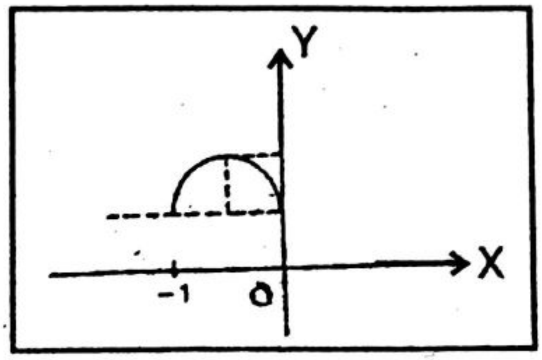
(p) $y = \frac{1}{(x^2-9)}$

$= \frac{1}{(x+3)(x-3)}$

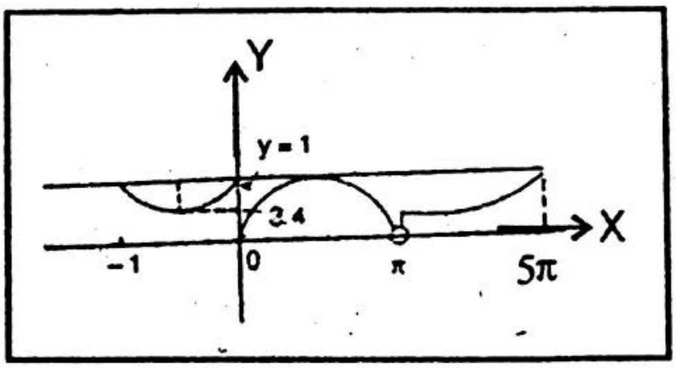
At x = 3, -3 the curve is undefined, i.e. the two asymptotes are x=3 and x=-3.




(q) $y = \begin{cases} (x^2+x+1) : -1 \le x \le 0 \\ \sin^2 x : \le x \le \pi \\ (x-1)/(x+1) : \pi < x \le 5) \end{cases}$




$\therefore y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$




$y = \sin^2 x \quad : 0 \le x \le \pi$

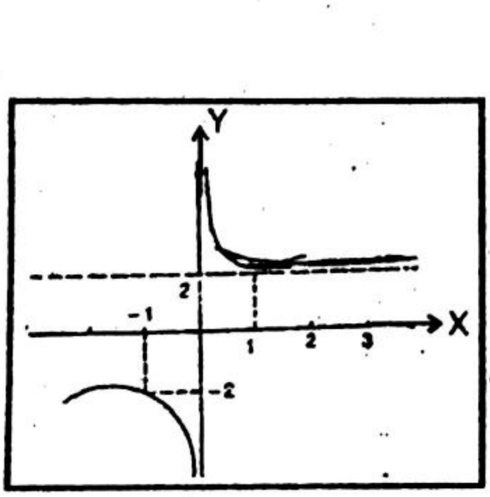
$y = \frac{x-1}{x+1} \quad : \pi < x \le 5\pi$

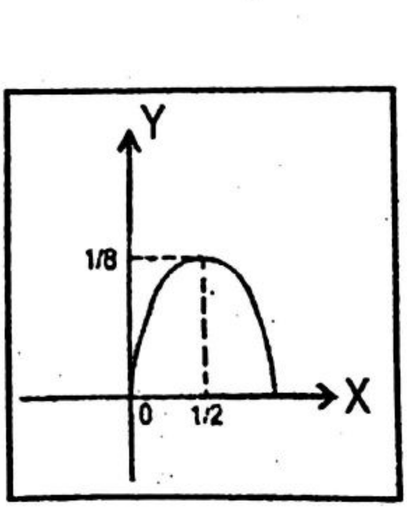
$= \frac{x+1-2}{x+1} \;;\; = 1 - \frac{2}{x+1}$

(r) $y = x + \frac{1}{x}$

(s) $y = x^2 - x^3$

$y = x^2(1-x)$

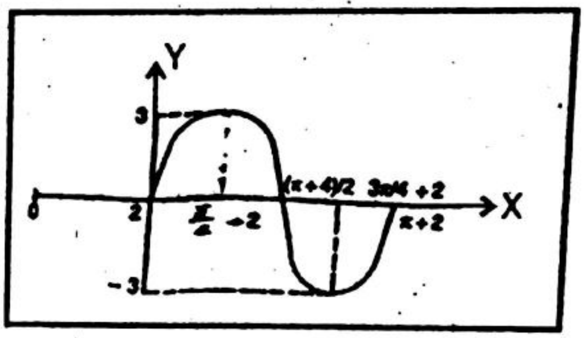


**(t)** y=3 sin (2x-4) $0 \le 2x-4 \le 2\pi$

$4 \le 2x \le 2\pi+4$

$2 \le x \le \pi+2$




**(u)** $y = 2\sqrt{[(-3(x+1.5))-1.2]}$

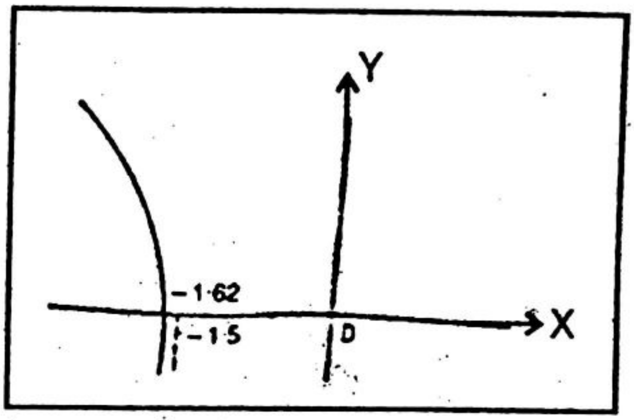
$y = 0$

$1.2 = 2\sqrt{(-3(x+1.5))}$

$\frac{1.44}{4} = -3(x+1.5)$

$-12 - 1.5 = x$

$-1.62 = x$




**(v)** $y = |x^2 - 2x - 1|$ $\quad (x^2-2x-1) > 0$ or $x \ge 1+\sqrt{2}$

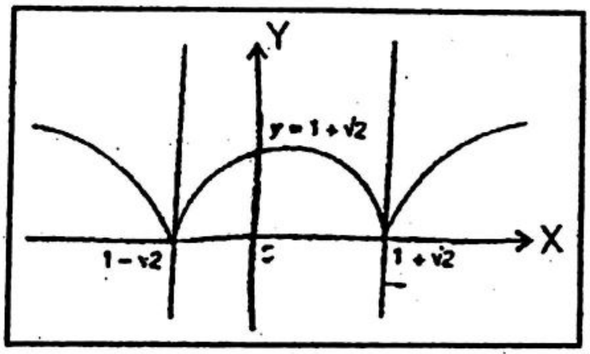
$= (x^2-2x-1)$ $\quad (x^2-2x-1) < 0$ or $x \le 1-\sqrt{2}$

$= -(x^2-2x-1$

$1-\sqrt{2} \le x \le 1+\sqrt{2}$

$y = (x-1)^2-2$

$y = -(x-1)^2+2$




**(w)** $y = ||x|-1|$ $\quad |x|-1 \ge 0$ $\quad x \ge 1$ or $x \le -1$

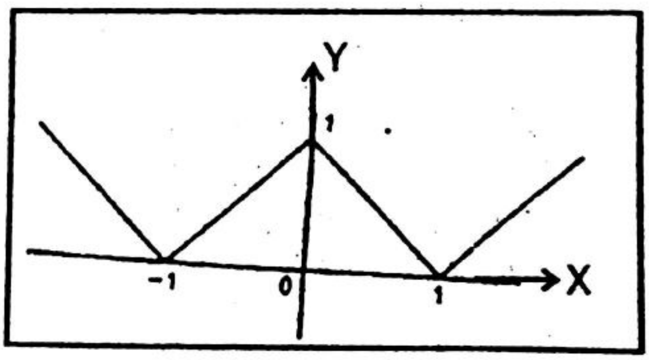
$= |x|-1$ $\quad |x| \ge 1$

$= 1-|x|$ $\quad -1 \le x \le 1$

$y = |x|-1$ $\quad x \ge 1$ or $x \le -1$

$y = x-1$ $\quad x \ge 1$

$y = -x-1$ $\quad x \le -1$



Similarly $y = 1 - x \qquad 0 \le x \le 1$

$y = 1 + x \qquad -1 \le x \le 0$

Arranging all parts $y = -x - 1 \qquad x \le -1$

$y = 1 + x \qquad -1 \le x \le 0$

$y = 1 - x \qquad 0 \le x \le 1$

$y = x - 1 \qquad x \ge 1$

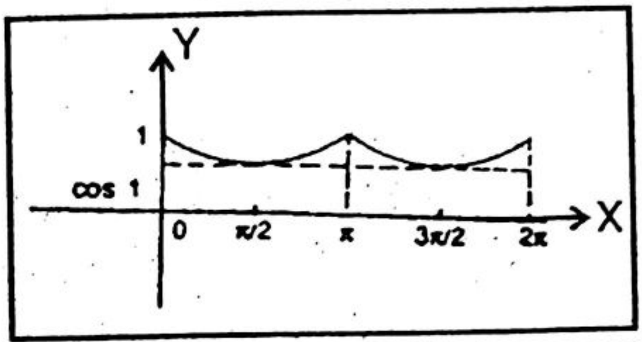
**(x)** $y = |\sin x| + \sin x$ **on the interval** $[0, 3\pi]$

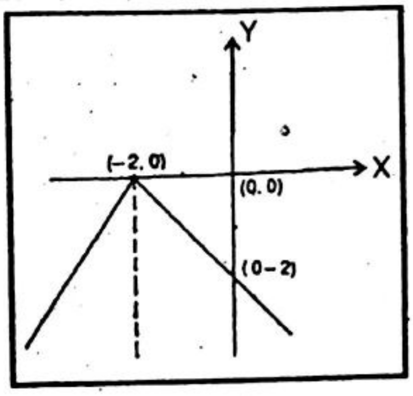
$= \sin x + \sin x \quad \sin x \ge 0 \qquad 0 \le x \le \pi$

$= 2\sin x \qquad 2\pi \le x \le 3\pi$

$= -\sin x + \sin x \quad \sin x \le 0 \qquad \pi \le x \le 2\pi$

$= 0$

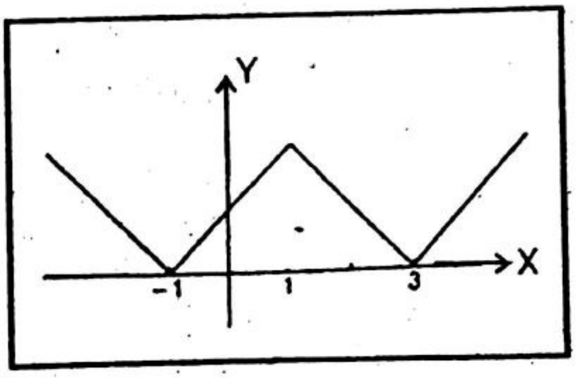



**(y)** $y = -|x + 2|$

$y = -(x + 2) x + 2 \ge 0;\ x \ge -2$

$= (x + 2) x + 2 < 0; x \le -2$




**(z)** $y = ||x - 1| - 2|$

$y = |x - 1| - 2 \qquad |x - 1| \ge 2$

$= -\{|x - 1| - 2\} \qquad |x - 1| < 2$

$y = (x - 1) - 2 \qquad x \ge 3$

$y = x - 3$

$y = -x - 1 \qquad x \le -1$

$y = -x + 3 \qquad +1 \le x \le 3$

$y = x + 1 \qquad -1 \le x \le 1$

**(A)** $y = |x+2| + |x-3|$

$y = (x+2) + (x-3)$ $\quad x+2 \geq 0; x \geq -2$

$y = 2x-1$ $\quad x \geq 3$

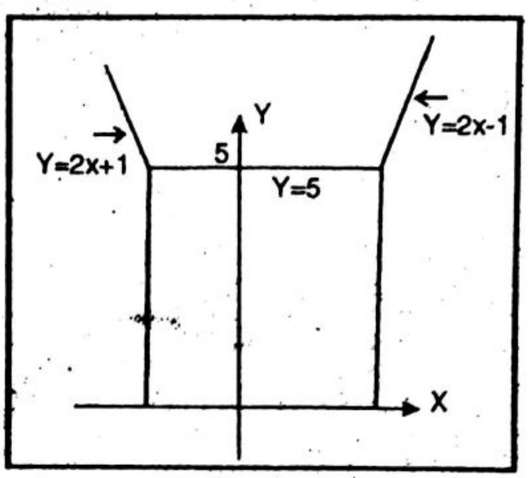
$y = (x+2) + (3-x)$ $\quad x+2 \geq 0 \quad x \geq -2$

$y = 5 \quad x \leq 3$ $\quad x-3 \leq 0 \quad x \leq 3$

$y = -(x+20+(x-3)$ $\quad x \leq -2$

$x \geq 3$

Not possible



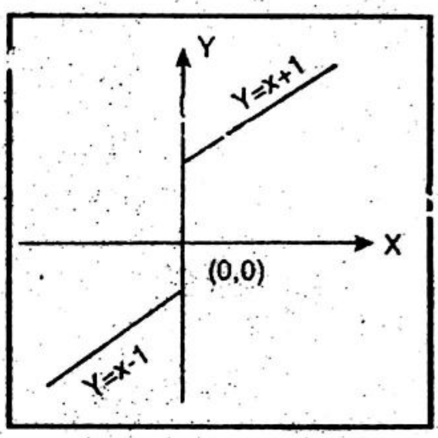

$y = -(x+2) + (3-x) \quad x \leq -2$ and $x \leq 3$

$y = -2x+1 \quad x \leq -2.$

**(B)** $y = x + \frac{x}{|x|}.$ $\quad y = x + \frac{x}{x} \quad x \geq 0$

$= x+1 \quad x \geq 0$

$= x-1 \quad x \leq 0$




$y = x + |x-1| + \frac{|x-2|}{x-2}$

**(C)** $y = x + |x-1| + 1 \quad x \geq 2$

$= x + x - 1 + 1 \quad x \geq 2$

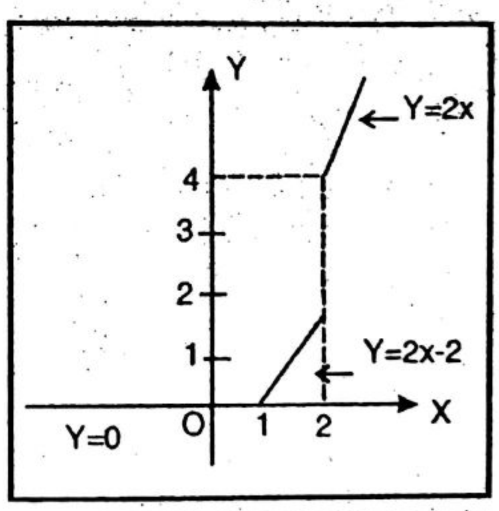
$y = 2x \quad x \geq 2$

$y = x + x - 1 - 1 \quad 1 \leq x \leq 2$

$= 2x - 2$

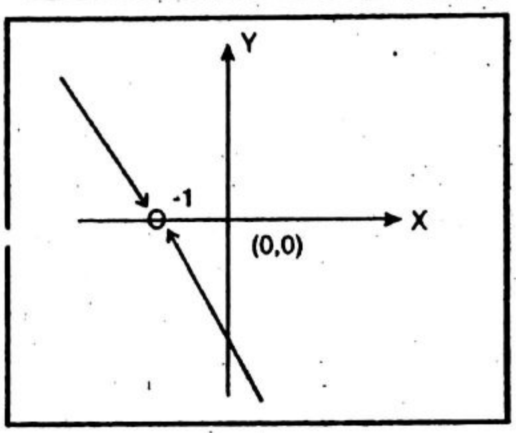
$y = x + 1 - x - 1 \quad x \leq 1$

$= 0$




**(D)** $\frac{y}{x+1} = 1$

At $x = -1$, the curve does not exist, i.e. undefined.



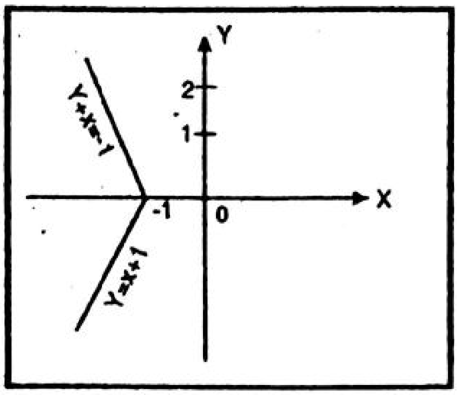
(E) $|y| + x = -1$

$y + x = -1$

$y = -x - 1 \quad y \geq 0$

$-y + x = -1 \quad y \leq 0$

$= x + 1$



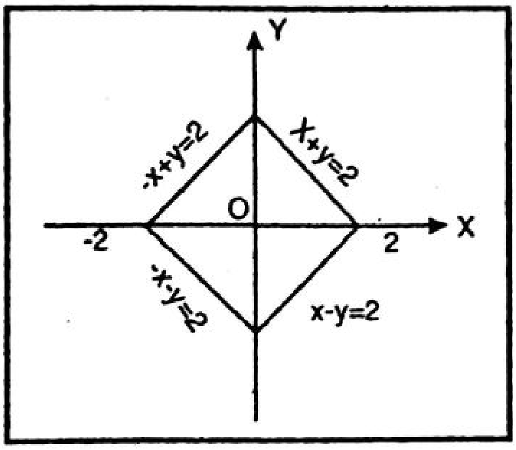

(F) $|x| + |y| = 2$

$x + y = 2 \quad x \geq 0; y \geq 0$

$x - y = 2 \quad x \geq 0; y \leq 0$

$-x + y = 2 \quad x \leq 0; y \geq 0$

$-x - y = 2 \quad x \leq 0; y \leq 0$




(G) $|y - 3| = |x - 1|$

$y - 3 = x - 1 \quad y \geq 3; x \geq 1$

$y = x + 2$

$y - 3 = 1 - x \quad y \geq 3; x \leq 1$

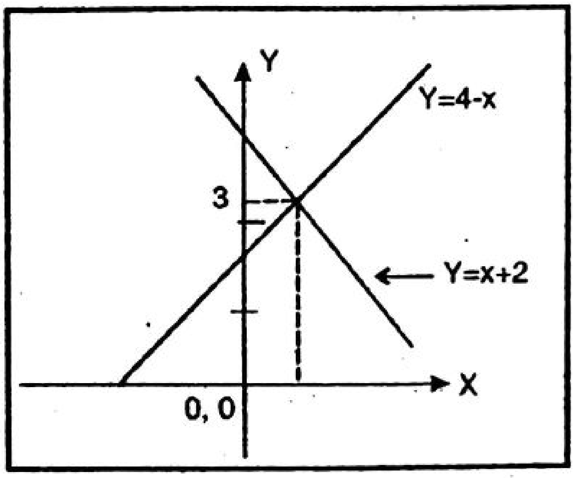
$y = 4 - x$

$3 - y = x - 1 \quad y \leq 3; x \geq 1$

$y = 4 - x$

$3 - y = 1 - x \quad y \leq 3; x \leq 1$

$y = 2 + x$




(H) $|x + y| + |x - y| = 4 \quad (x+y) > 0$

$(x+y)+(x-y)=4 \quad x-y>0$

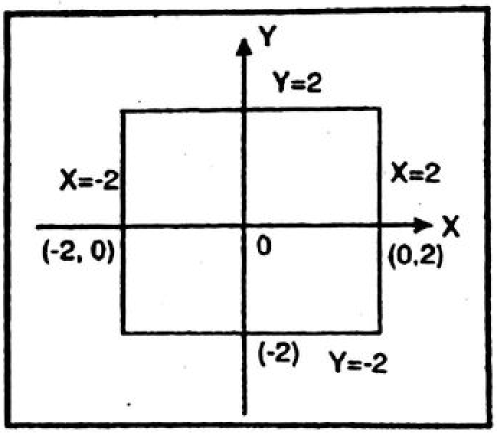
$2x=4 \quad x<-y$

$x=2 \quad x>y$

$(x+y)-x+y=4 \quad x+y>0$

$2y=4 \quad x-y<0$

$y = 0$



$(x+y)-x+y=4 \qquad x+y>0$

$x-y<0$

$2y=4 \qquad x-y<0$

$y=2$

$-(x+y)-(x-y)=4 \qquad -(x+y)+(x-y)=4$

$-x-y-x+y=4(x+y)<0 \qquad -2y=4$

$-2x=4 \qquad y=-2$

$x=-2$

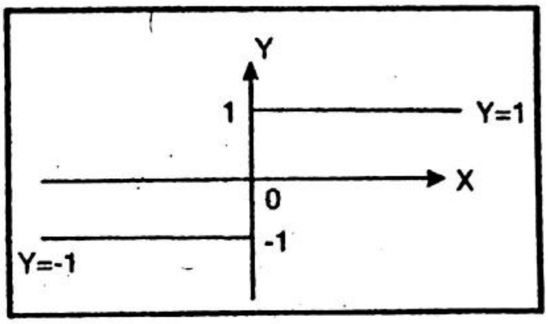
**(I)** $|y|x = x$

$yx = x \quad y>0 \quad x(y-1)=0$

$-yx = x \quad y<0 \quad x=0;\ y=1$

$x+yx=0$

$x=0 \qquad y=-1$




**(J)** $|x-y|+y=0$

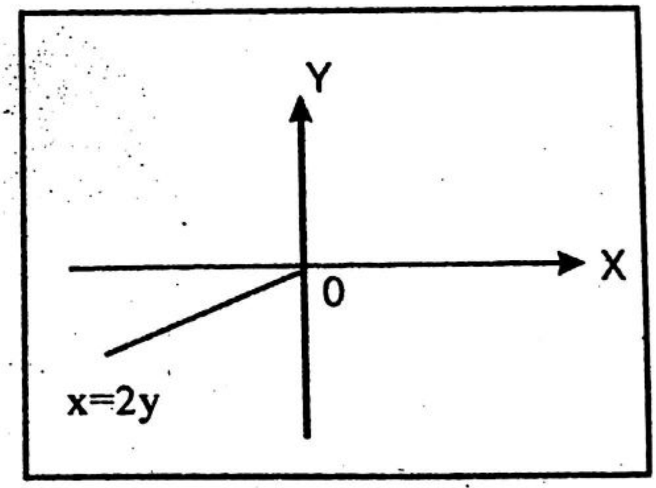
$x-y+y=0 \qquad x-y>0$

$x=0 \qquad x>y$

$-x+y+y=0 \qquad x<y$

$x=2y$

$$y=\frac{x}{2}$$




**(K)** $y = 2-|x-x^2|,\ x>0$

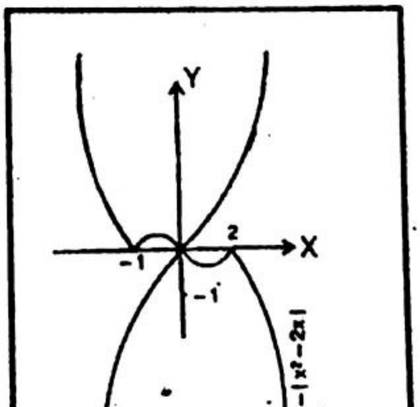
$=2-x-x^2$

$y=2+x-x^2,\ x<0$

$=-(x^2+x-2),\ x>0$

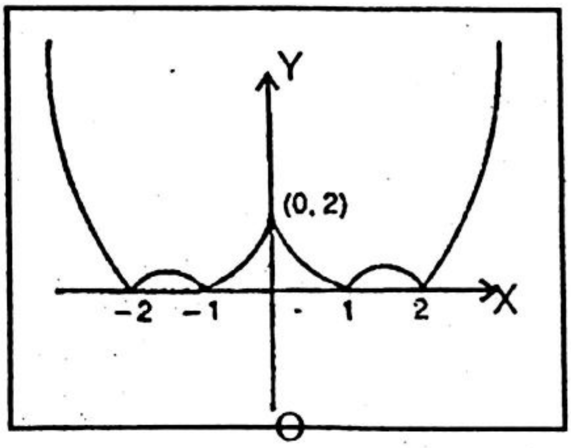
$$=-\left\{\left(x+\frac{1}{2}\right)^2-\frac{9}{4}\right\}, x>0$$

$y=-(x^2-x-2)$



$$y = -\left[\left(x - \frac{1}{2}\right)^2 - \frac{9}{4}\right] x < 0$$

**(L)** $y = \left| x^2 + x \right|$




$$= \left|(x^2 + \frac{1}{2})^2 - \frac{1}{4}\right|$$

$$y = \left(x + \frac{1}{x}\right)^2 - \frac{1}{4}, \left(x + \frac{1}{x}\right)^2 - \frac{1}{4} \geq 0$$

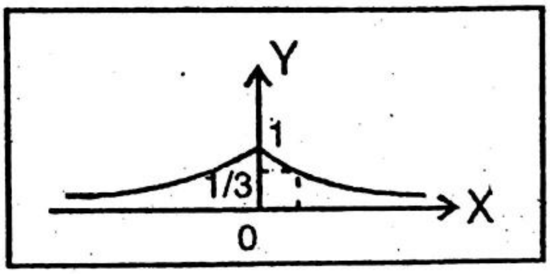
$$= -\left\{\left(x + \frac{1}{x}\right)^2 - \frac{1}{4}\right\}, \left(x + \frac{1}{x}\right) - \frac{1}{4} < 0$$

$$\frac{1}{4} - \left(x + \frac{1}{2}\right)^2 \geq 0$$

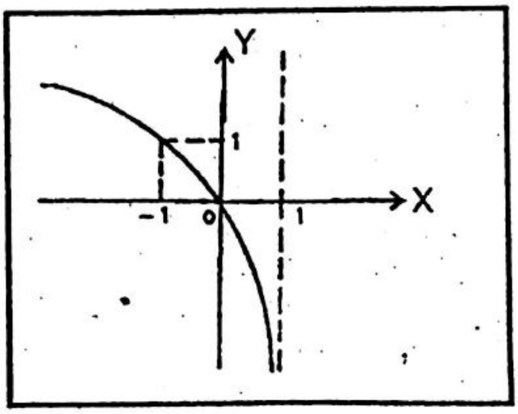
$$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \leq 0$$

$$\left(x + \frac{1}{2} + \frac{1}{2}\right)\left(x + \frac{1}{2} - \frac{1}{2}\right) \leq 0$$

**(M)** $y = 3^{-|x|}$
$= 3 - x : x \geq 0$
$= 3 + x : x < 0$




**(N)** $y = \log_2(1 - x) : (1 - x) = 2$
$x = -1$
$1 - x = 0$
$x = 1$



(O) $y = |\log_2(1-x)|$

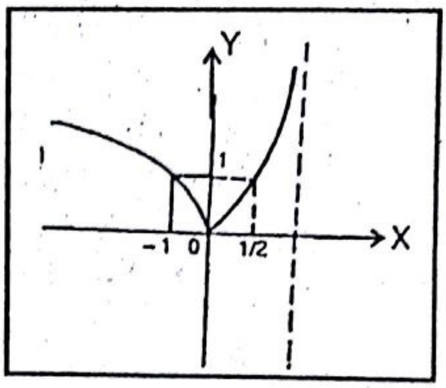
$= \log_2(1-x) \quad \log_2(1-x) \geq 0$

$\geq \log\ 1$

$..\infty \leq x \leq 0$

$= -\log_2(1-x) \log_2(1-x) \leq 0$

$0 \leq x < 1$




(P) $y = x \sin x$

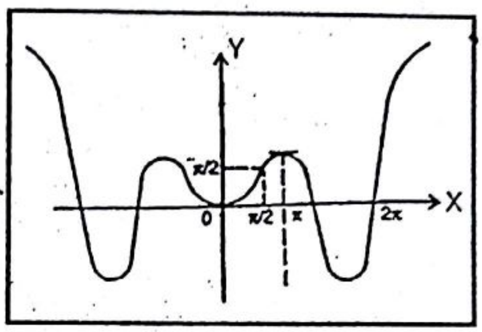
$-x \leq x \sin x$

$\leq x$

$\Rightarrow -x \leq y \leq x$

$y = x \sin \pi x$, same graph but instead of

$x = 0, \frac{\pi}{2}, \pi; \quad \pi = 0, 1, \frac{1}{2}$




(Q) $y = \frac{|\ln x|}{\ln x}$ $\quad \ln x \geq 0$

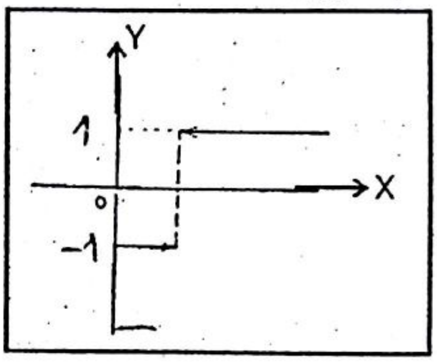
$= \frac{\ln x}{\ln x}$ $\quad x \geq 1$

$= 1$

$y = -\frac{\ln x}{\ln x}$ $\quad \ln x \leq 0$

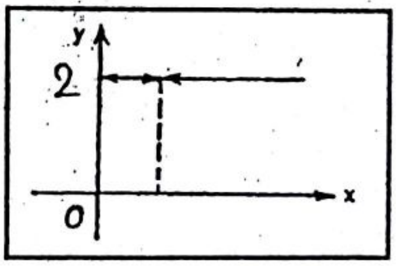
$0 \leq x \leq 1$

$= -1$




(R) $y = x^{\log_x 2}$

$y = 2$



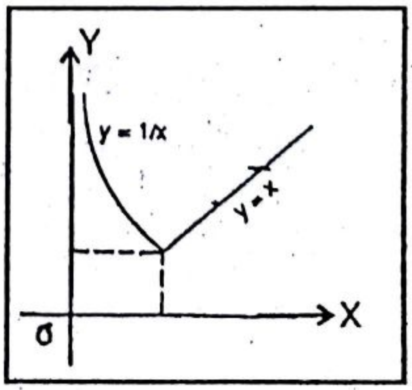

(S) $y = e^{|\ln x|}$

$= e^{\ln x}: x \geq 1$

$= x$

$= e^{-\ln x} = e^{-\ln x}$

$= \frac{1}{x}$

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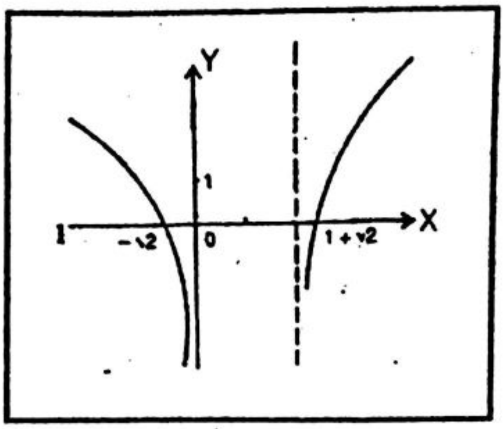
(T) $y = \log_2(x^2 - 2x)$

$$y = \log_2\{(x-1)^2 - 1\}$$
$$(x-1)^2 - 1 = 1$$
$$(x-1) = \pm\sqrt{2}$$
$$x = 1 \pm \sqrt{2}$$
$$(x-1)^2 - 1 = 0$$
$$x = 2,\ 0$$



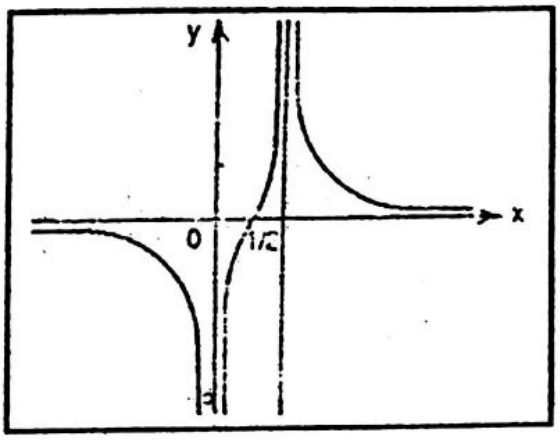

(U) $y = \log_2\left|\dfrac{x}{x-1}\right|$

$$= \log_2\left(\frac{x}{x-1}\right) : \frac{x}{x-1} \geq 0$$
$$x > 1$$
$$x < 0$$

$$= \log_2\left(\frac{x}{x-1}\right) : \frac{x}{x-1} \geq 0$$
$$0 < x < 1$$

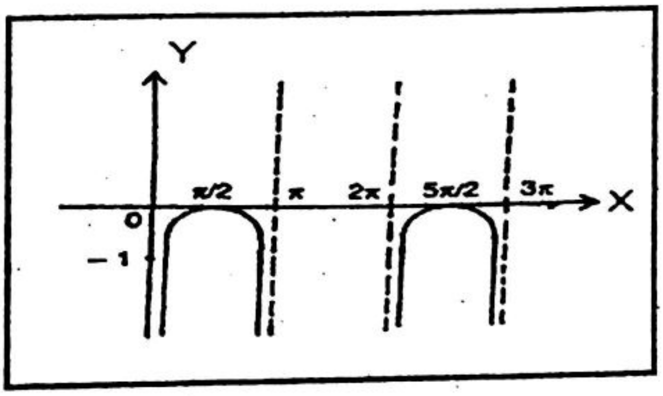
at $x = 1: y \to$

at $x = 0: y \to -\infty$



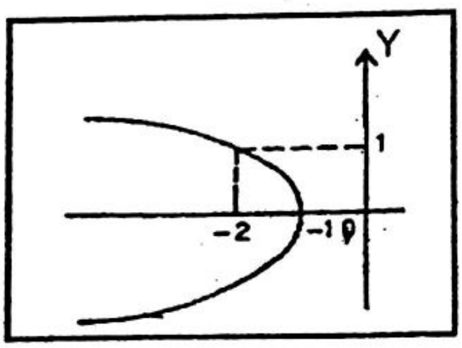

(V) $y = \log_2 \sin x : \sin x > 0$

$$0 < x < \pi$$
$$2\pi < x < 3\pi$$




(W) $|y| = \log_2(-x)$

$$y = \log_2(-x) : y > 0\ x < 0$$
$$y = -\log_2(-x) : y < 0\ x < 0$$



(X) $y = |\sin x|$

$= \sin x \quad \sin x > 0$

$0 < x < \pi$

$2\pi < x < 3\pi$

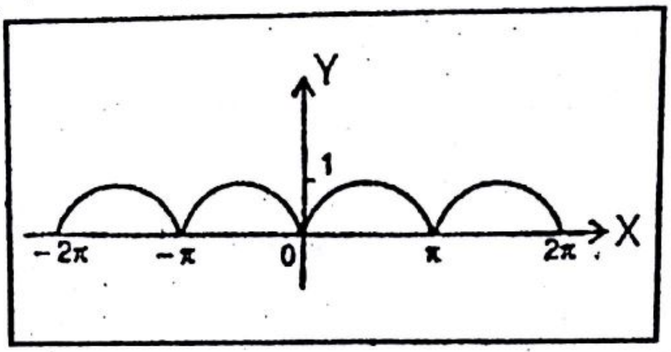
...............

$= -\sin x \quad \sin x < 0$

$\pi < x < 2\pi$

$3\pi < x < 4\pi$

...............



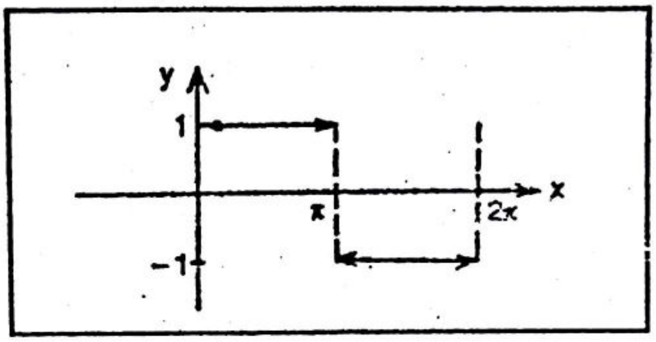

(Y) $y = \dfrac{|\sin x|}{\sin x}$

$= 1 \quad : \sin x > 0$

$0 < x < \pi$

$= -1 \quad : \sin x < 0$

$\pi < x < 2\pi$



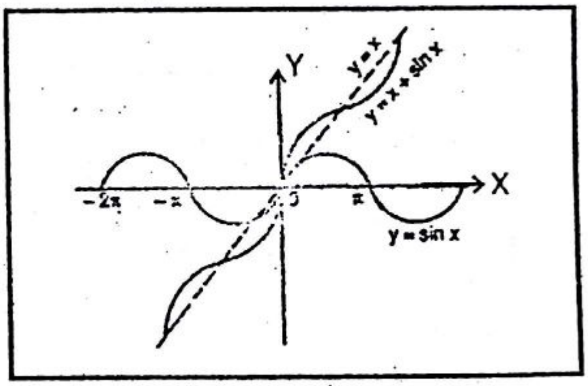

(Z) $y = x + \sin x$

$\because \; -1 \leq \sin x \leq 1$

$\therefore \; x - 1 \leq x + \sin x$

$\leq x + 1$

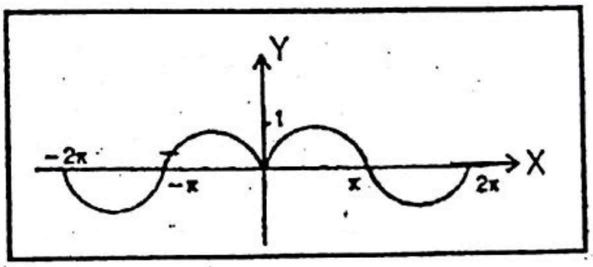
$\Rightarrow x - 1 \leq y \leq x + 1$




($Z_1$) $y = \sin|x|$

$= \sin x \quad x > 0$

$= -\sin x \quad x < 0$




($Z_2$) $y = \sin \quad \sin^{-1}\left(\log_{1/2} x\right)$

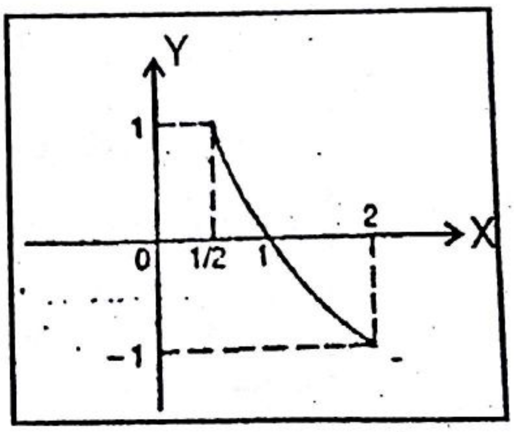
$x \quad = 1/2$

$y = \log_{1/2} x \quad \log_{1/2}(1/2) = 1$

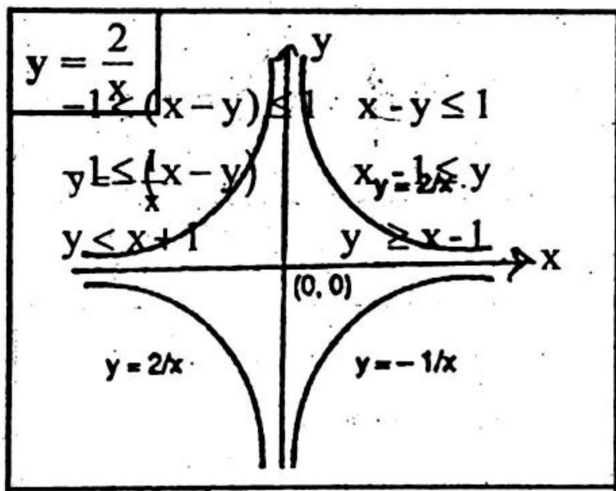
$\log_{1/2} 1 = 0$

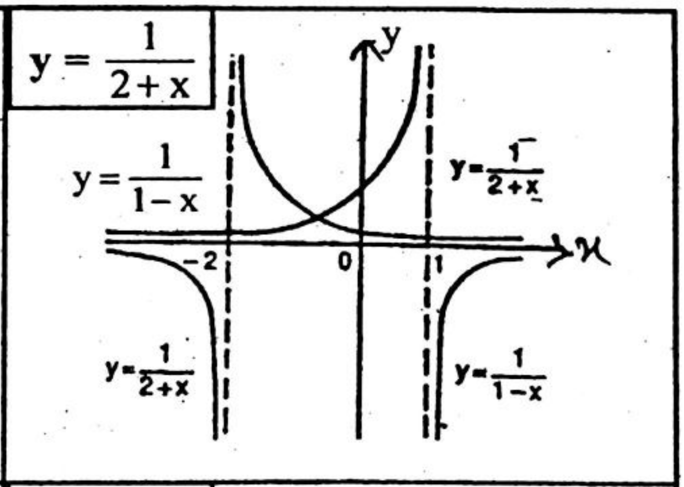
$\log_{1/2} 2 = \dfrac{\log_2 2}{\log_2(1/2)}$

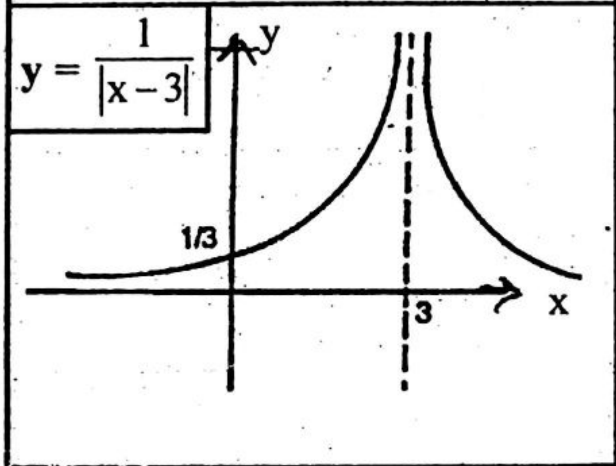
$= \dfrac{1}{0-1} = -1$

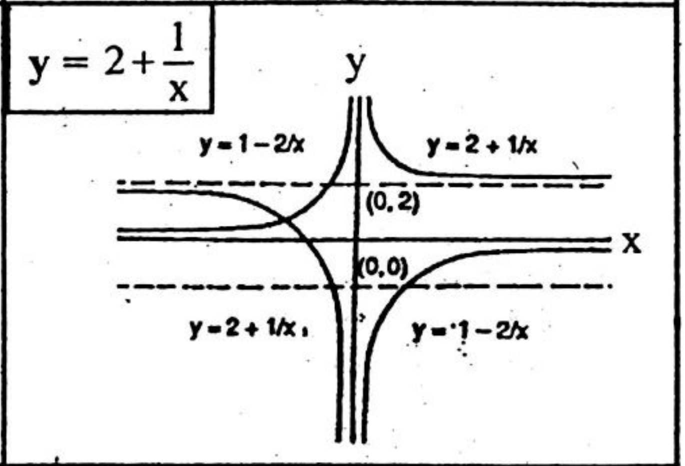


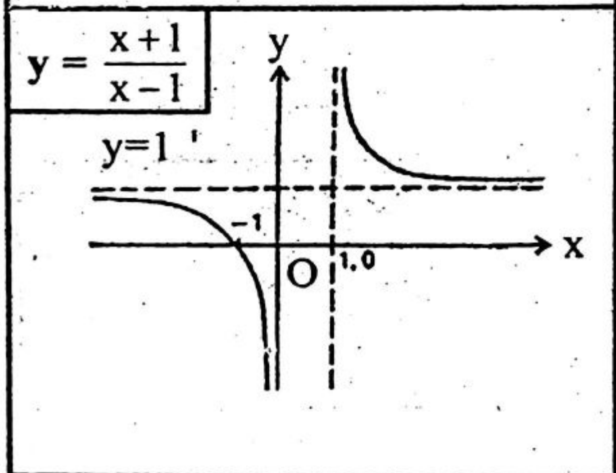
## ❑ SOME MORE GRAPHS OF THE FOLLOWING FUNCTIONS :

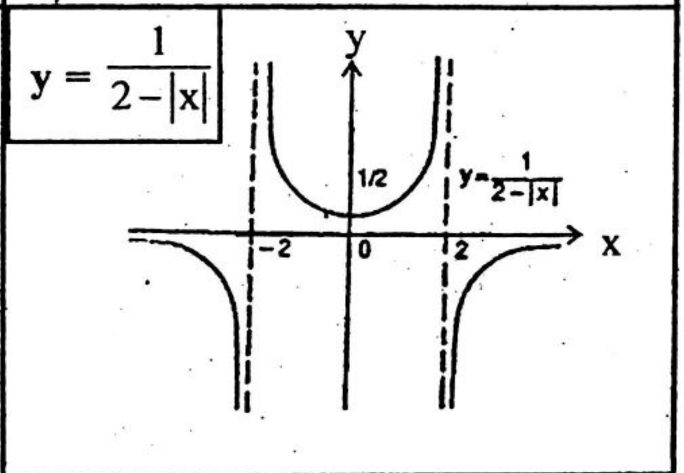


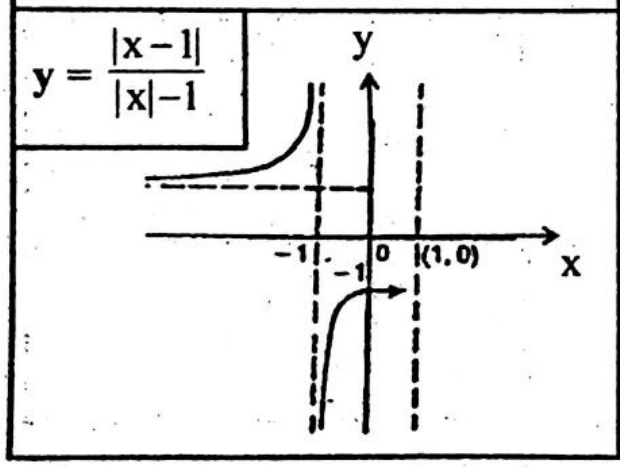



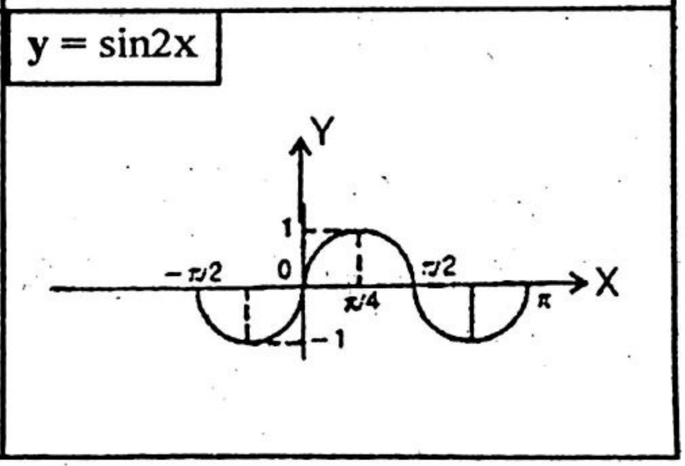



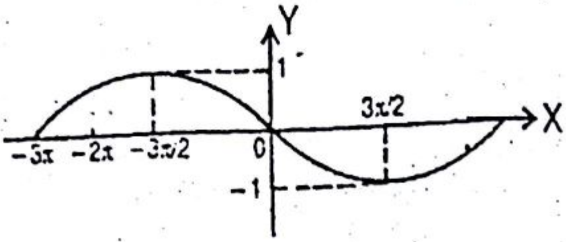



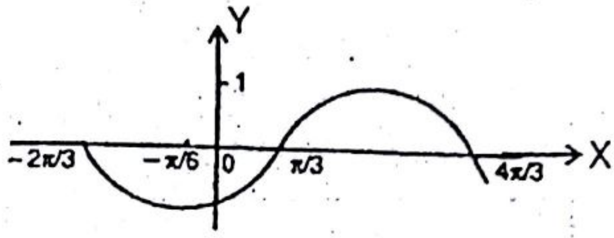



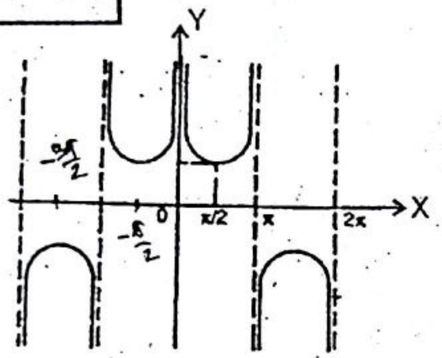
$y = -\sin x/3$



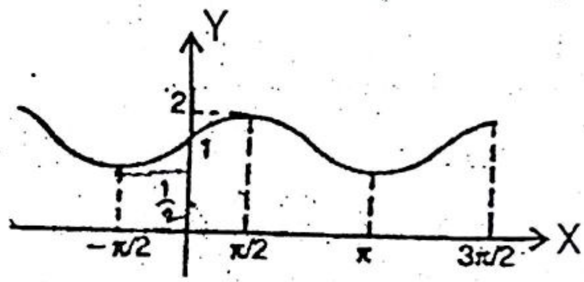

$y = \sin(x - \pi/3)$



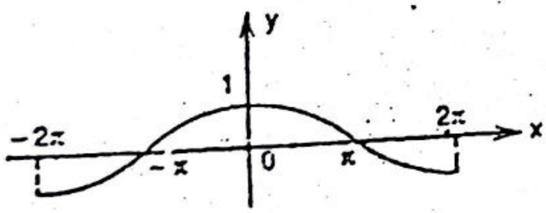

$y = \text{cosec}\ |x|$



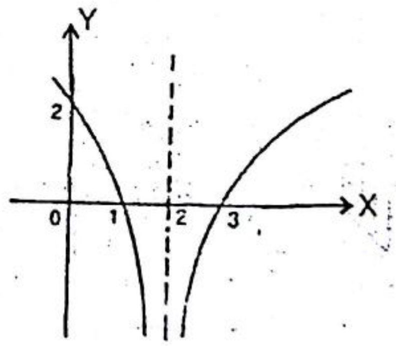

$y = 2^{\text{Sinx}}$



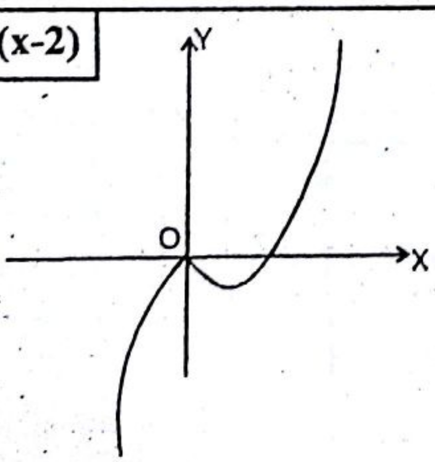

$y = \cos\left(-\frac{x}{2}\right)$



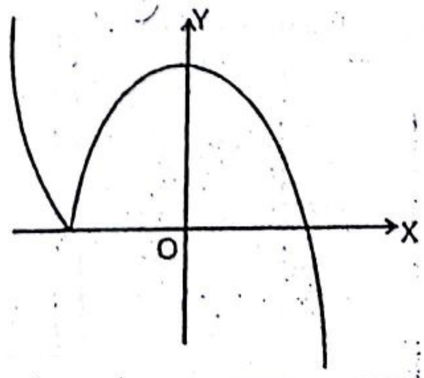

$y = 2\ \log_2\ (2-x)$




$y = |x|(x-2)$




$y = (3-x)|x+1|$



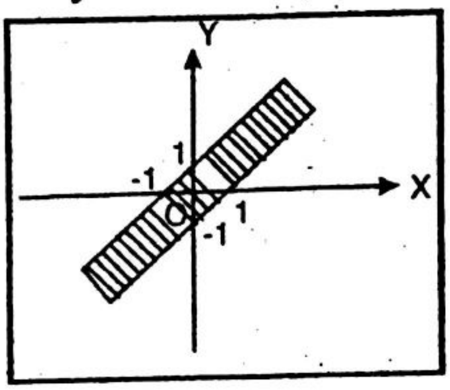
## ❑ SOME MORE GRAPHS OF THE FOLLOWING INEQUALITIES :

(a) $|x-y| \leq 1$

$-1 < (x-y) \leq 1$ $\quad$ $x-y \leq 1$

$-1 \leq (x-y)$ $\quad$ $x-1 \leq y$

$y < x+1$ $\quad$ $y \geq x-1$

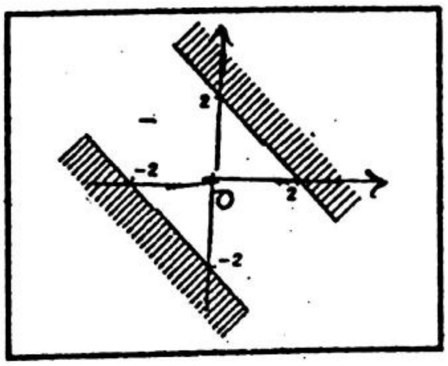



(b) $|x-y| \geq 2$

$(x+y) \geq 2$

$(x+y) \leq -2$

$y \leq 2-x,\ y \leq -2-x$



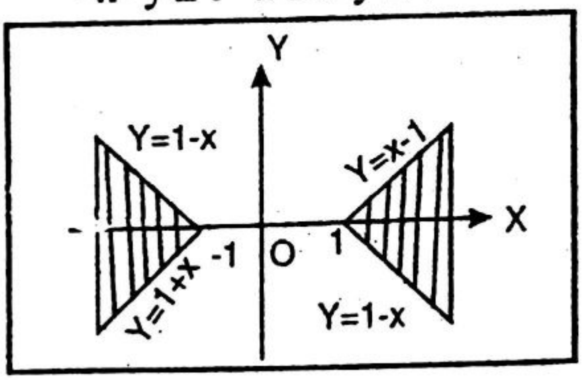

(c) $|x|-|y| \geq 1$

$x+y \geq 1 \quad x \geq 0\ y \geq 0$

$x-y \geq 1 \quad x \geq 0\ y \geq 0$

$-x+y \geq 1 \quad x \leq 0\ y \leq 0$

$-x-y \geq 1 \quad x \leq 0\ y \geq 0$



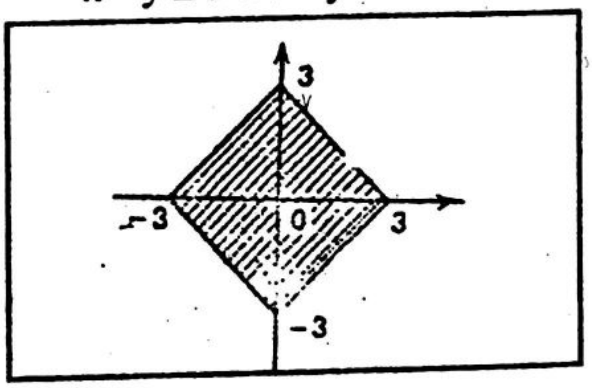

(d) $|x|+|y| \leq 3$

$x+y \geq 3 \quad x \geq 0\ y \geq 0$

$x-y \geq 3 \quad x \geq 0\ y \leq 0$

$-x-y \geq 3 \quad x \leq 0\ y \leq 0$

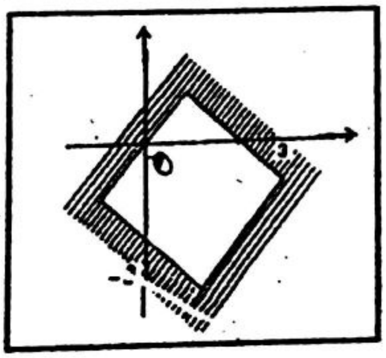
$-x+y \geq 3 \quad x \leq 0 y \geq 0$




(e) $|x-1| + |y+1| \geq 2$

$(x-1) + (y+1) \geq 2$

$(x-1) \geq 0 \quad (y+1) \geq 0$




(f) $|x+y| + |x-y| \leq 2$

$(x-1) + (y+1) \geq 2$

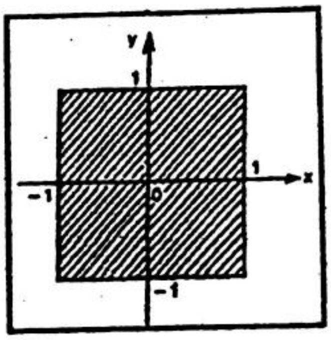
$(x-1) \geq 0 \quad (y+1) \geq 0$

$-(x-1) - (y+1) \geq 2$

$(x-1) \leq 0\ (y+1) \leq 0$

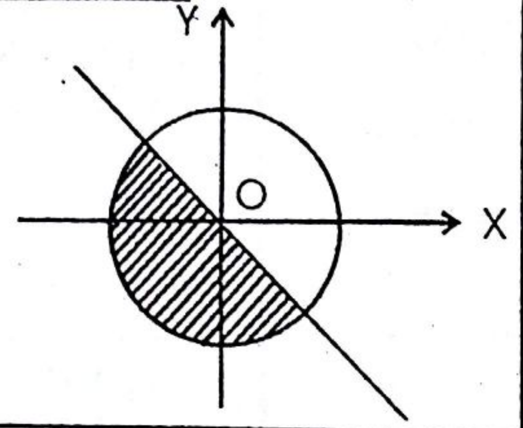
$-(x-1) - (y+1) \geq 2$

$(x-1) \leq 0\ (y+1) \geq 0$

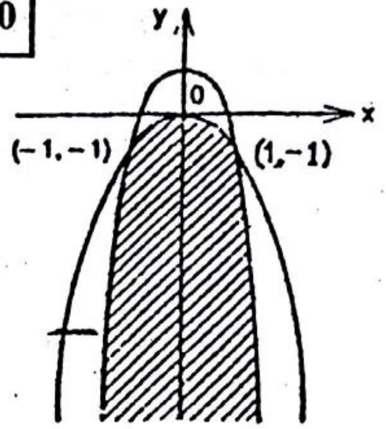


## ❑ SOME MORE GRAPHS OF THE FOLLOWING INEQUALITIES :

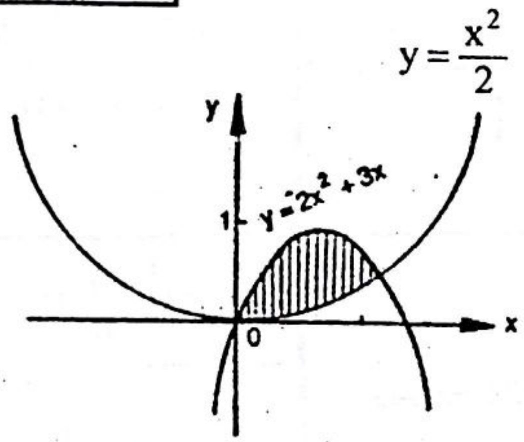
$y - 2x \leq 0$
$x^2+y^2-5 \leq 0$



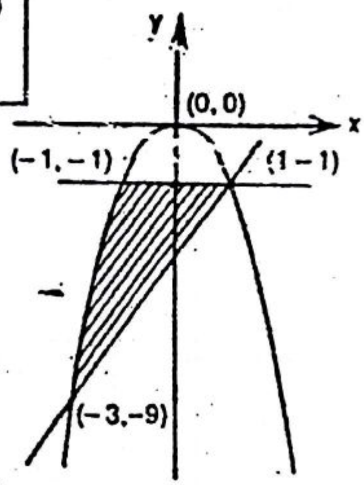

$x^2+y \leq 0$
$2x^2+y-1 \leq 0$



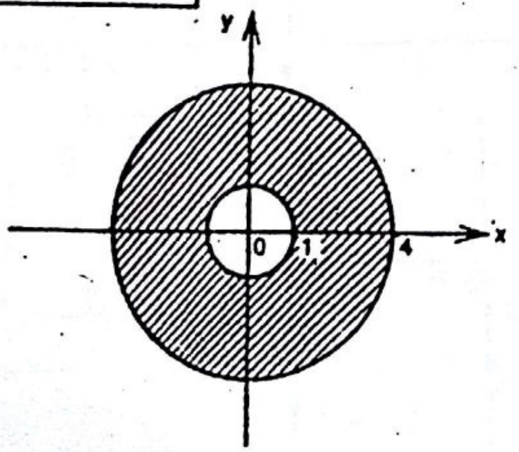

$x^2+y^2 \geq 1$
$x^2+y^2 \geq 16$




$y+x^2 \leq 0$
$y-2x+3 \geq 0$
$y+1 \leq 0$




$2y \geq x^2$
$y \leq -2x^2+3x$




$3y - x < 5$
$y + 2x < 11$
$4y + x > 9$