Pure Sector 4: Trigonometry 4

Aims:
- Understand and use double angle formulae
- Use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$
- Understand geometrical proofs of these formulae
- Understand and use expressions for $a \cos\theta + b \sin\theta$ in the equivalent forms of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$
- Construct proofs involving trigonometric functions and identities
- Apply trigonometric identities to find integrals

Addition Formulae

The addition formaulae are given in the formula booklet:

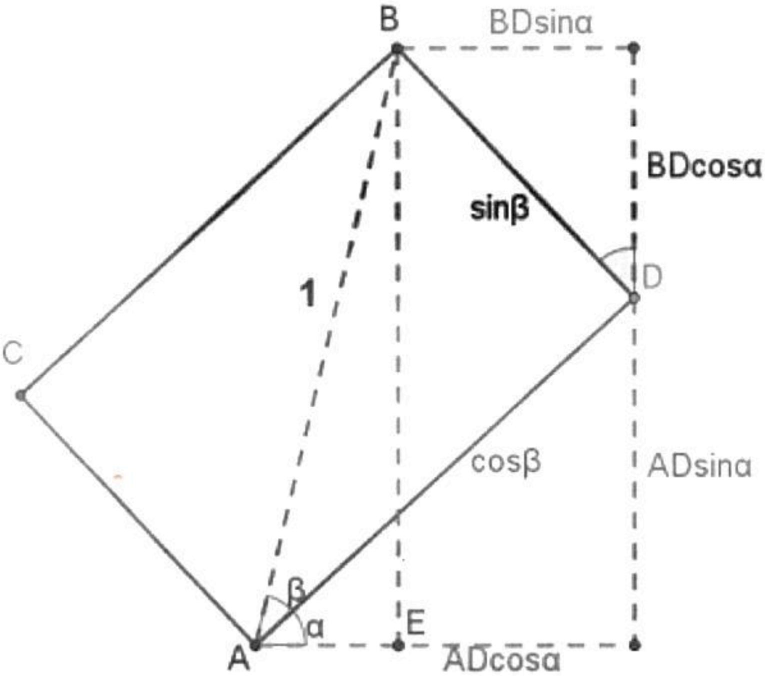
Trigonometric identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \qquad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

Geometric Proofs



$\cos(\alpha+\beta)=AE$

$=AD\cos\alpha-BD\sin\alpha$

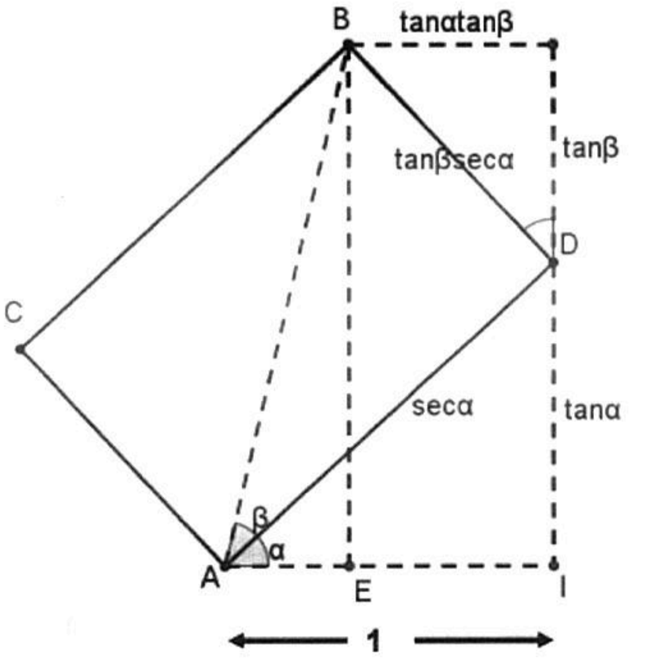
$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha+\beta)=BE$

$=AD\sin\alpha+BD\cos\alpha$

$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Use this diagram to prove that $\tan(\alpha+\beta) = \frac{BE}{AE} = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$



Example 1

Prove that $\frac{\sin(A-B)}{\cos A\cos B} = \tan A - \tan B$

$$\frac{\sin(A-B)}{\cos A\cos B} \equiv \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$$

$$\equiv \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$\equiv \tan A - \tan B$$

Example 2

Solve $\cos(\theta+60) = \sin\theta$ for $0 \le \theta \le 360$ (to 3sf)

$$\cos\theta\cos 60 - \sin\theta\sin 60 = \sin\theta$$

$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = \sin\theta$$

$$\cos\theta = (2+\sqrt{3})\sin\theta$$

$$\tan\theta = \frac{1}{2+\sqrt{3}} \qquad (\cos\theta \neq 0)$$

$\theta = 15°, 195°$

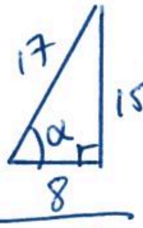
Example 3

Given that $\tan 60 = \sqrt{3}$ show that $\tan 15 = 2 - \sqrt{3}$

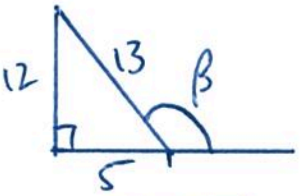
$$\tan 15 = \tan(60 - 45)$$
$$= \frac{\tan 60 - \tan 45}{1 + \tan 60 \tan 45}$$
$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{\sqrt{3} - 1 - 3 + \sqrt{3}}{1 - 3} = \frac{-4 + 2\sqrt{3}}{-2}$$
$$= 2 - \sqrt{3}$$

Example 4

Angle α is acute and cos $\alpha = \frac{8}{17}$. Angle β is obtuse and $\sin\beta = \frac{12}{13}$. Find the value of $\tan(\alpha + \beta)$.



$\sqrt{17^2 - 8^2} = 15$



$\sqrt{13^2 - 12^2} = 5$

$$\tan\alpha = \frac{15}{8}$$
$$\tan\beta = -\frac{12}{5}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{15}{8} - \frac{12}{5}}{1 - \left(\frac{15}{8}\right)\left(\frac{-12}{5}\right)} = \frac{-21}{220}$$

Double Angle Formulae

Use the addition formulae with B=A to find:

$\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$

$\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$

$\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$

The $cos2A$ formula can be written in terms of sin or cos by using $\sin^2 A + \cos^2 A \equiv 1$

So, $\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$

Or $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \begin{cases} \cos^2 A - \sin^2 A \\ 2\cos^2 A - 1 \\ 1 - 2\sin^2 A \end{cases}$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

These are **not** given in the formula booklet.

Example 5

Solve $3\cos 2\theta - 7\cos\theta - 2 = 0$ for $0 \le \theta \le 360$

$3(2\cos^2\theta - 1) - 7\cos\theta - 2 = 0$

$6\cos^2\theta - 7\cos\theta - 5 = 0$

$(3\cos\theta - 5)(2\cos\theta + 1) = 0$

$\cos\theta = \frac{5}{3}$ (NO ROOTS)

OR $\cos\theta = \frac{-1}{2}$

$\theta = 120, 240$

Example 6

Show that:

a) $2\,cosec 2\theta \equiv sec\theta cosec\theta$

$LHS \equiv 2\operatorname{cosec} 2\theta \equiv \frac{2}{\sin 2\theta} \equiv \frac{2}{2\sin\theta\cos\theta} \equiv \frac{1}{\sin\theta\cos\theta}$

$\equiv \sec\theta\operatorname{cosec}\theta \equiv RHS$

b) $\sin 3x \equiv 3\sin x - 4\sin^3 x$

$LHS \equiv \sin 3x$

$\equiv \sin(2x + x)$

$\equiv \sin 2x\cos x + \cos 2x\sin x$

$\equiv 2\sin x\cos^2 x + (1 - 2\sin^2 x)\sin x$

$\equiv 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x)\sin x$

$\equiv 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$\equiv 3\sin x - 4\sin^3 x \equiv RHS$

Exam Question – AQA C4 Jun 06

4 (a) (i) Express $\sin 2x$ in terms of $\sin x$ and $\cos x$. *(1 mark)*

(ii) Express $\cos 2x$ in terms of $\cos x$. *(1 mark)*

(b) Show that

$$\sin 2x - \tan x = \tan x \cos 2x$$

for all values of x. *(3 marks)*

(c) Solve the equation $\sin 2x - \tan x = 0$, giving all solutions in degrees in the interval $0° < x < 360°$. *(4 marks)*

a) (i) $\sin 2x \equiv 2\sin x \cos x$

(ii) $\cos 2x \equiv 2\cos^2 x - 1$

b) RHS $\equiv \tan x \cos 2x$

$\equiv \tan x (2\cos^2 x - 1)$

$\equiv \dfrac{2\sin x \cos^{\cancel{2}} x}{\cancel{\cos x}} - \tan x$

$\equiv 2\sin x \cos x - \tan x$

$\equiv \sin 2x - \tan x \equiv$ LHS

c) $\tan x \cos 2x = 0$

$\Rightarrow \tan x = 0$

$\Rightarrow \underline{x = (0), 180,}$

OR $\cos 2x = 0$

$2x = 90, 270, 450, 630$

$\underline{x = 45, 135, 225, 315}$

Application to Integration

Example 7

Find $\int \sin^2 x \, dx$

$\cos 2x \equiv 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

$\Rightarrow \int \sin^2 x \, dx = \frac{1}{2}\int 1 - \cos 2x \, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$= \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$

Example 8

Find $\int \sin 3x \cos 3x \, dx$

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\Rightarrow \frac{1}{2}\sin 6x \equiv \sin 3x \cos 3x$$

$$\Rightarrow \int \sin 3x \cos 3x \, dx = \frac{1}{2}\int \sin 6x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{6}\cos 6x\right] + C$$

$$= -\frac{1}{12}\cos 6x + C$$

Example 9

Find $\int 6\cos^2\frac{\theta}{2} \, d\theta$

$$\cos 2A \equiv 2\cos^2 A - 1$$

$$\Rightarrow 3\cos 2A \equiv 6\cos^2 A - 3$$

$$\Rightarrow 6\cos^2\frac{\theta}{2} \equiv 3\cos\theta + 3$$

$$\Rightarrow \int 6\cos^2\frac{\theta}{2} \, d\theta = 3\int \cos\theta + 1 \, d\theta$$

$$= 3\left(\sin\theta + \theta\right) + C$$

$$= 3\sin\theta + 3\theta + C$$

$R\sin(\theta+\alpha)$ and $R\cos(\theta+\alpha)$ form

This is useful when solving equations

Example 10

a) Express $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$

b) What is the maximum value of the expression $3\sin x + 4\cos x$?

c) What is the smallest positive value of x for which this value occurs?

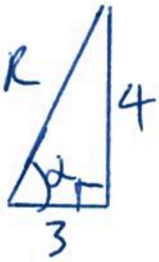
$$a)\ R\sin(x+\alpha) \equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$$
$$\equiv (3)\sin x + (4)\cos x$$

$\Rightarrow R\cos\alpha = 3$

$R\sin\alpha = 4$

$R = \sqrt{3^2+4^2} = 5$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.13\ldots$



$\therefore \underline{3\sin x + 4\cos x \equiv 5\sin(x+53.1)}$

b) $-1 \le \sin\theta \le 1$

$\therefore$ MAX VALUE IS 5.

c) $\sin(x+53.1) = 1$

$\Rightarrow (x+53.1) = 90$

$x = 90 - 53.1 = 36.869\ldots = \underline{36.9^\circ}$

Example 11

Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta+\alpha)$ and hence find its min and max values.

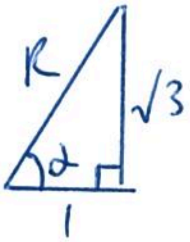
$$R\cos(\theta+\alpha) \equiv R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$$
$$\equiv \cos\theta - \sqrt{3}\sin\theta$$

$\Rightarrow R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}$

$R = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\alpha = \tan^{-1}\sqrt{3} = 60$



$\therefore \cos\theta - \sqrt{3}\sin\theta \equiv 2\cos(\theta+60)$

MAX VALUE: 2

MIN VALUE: −2

Example 12

Write $2\cos x - 3\sin x$ in the form $R\cos(x+\alpha)$ and hence solve $2\cos x - 3\sin x = 3$ for $0 \le x \le 2\pi$

$$R\cos(x+\alpha) \equiv R\cos\alpha\cos x - R\sin\alpha\sin x$$
$$\equiv 2\cos x - 3\sin x$$

$\Rightarrow R\cos\alpha = 2$

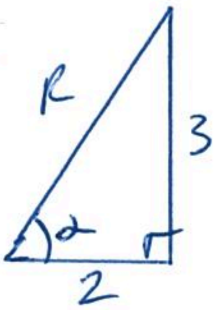
$R\sin\alpha = 3$

$R = \sqrt{2^2+3^2} = \sqrt{13}$

$\alpha = \tan^{-1}\left(\frac{3}{2}\right)$

$= \cancel{56.3099}\ldots 0.98279\ldots$

$= \cancel{56.3}\ 0.983$



$\Rightarrow \sqrt{13}\cos(x + \cancel{56.3}\ 0.983) = 3$

$x + \cancel{56.3}\ 0.983 = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$

$= \cancel{33.6}$

$(0.588),\ 5.695,\ 6.871$

$x = 4.712,\ 5.888$

$= 4.71,\ 5.89$

AQA C4 June 10

5 (a) (i) Show that the equation $3\cos 2x + 2\sin x + 1 = 0$ can be written in the form

$$3\sin^2 x - \sin x - 2 = 0$$

(3 marks)

(ii) Hence, given that $3\cos 2x + 2\sin x + 1 = 0$, find the possible values of $\sin x$.

(2 marks)

(b) (i) Express $3\cos 2x + 2\sin 2x$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving α to the nearest $0.1°$. *(3 marks)*

(ii) Hence solve the equation

$$3\cos 2x + 2\sin 2x + 1 = 0$$

for all solutions in the interval $0° < x < 180°$, giving x to the nearest $0.1°$.

(3 marks)

AQA Specimen Paper 2

5 (a) Determine a sequence of transformations which maps the graph of $y = \cos\theta$ onto the graph of $y = 3\cos\theta + 3\sin\theta$

Fully justify your answer.

[6 marks]

5 (b) Hence or otherwise find the least value and greatest value of

$$4 + (3\cos\theta + 3\sin\theta)^2$$

Fully justify your answer.

[3 marks]

AQA C4 JUNE 10

a) (i) $3\cos 2x + 2\sin x + 1 = 0$

$3(1 - 2\sin^2 x) + 2\sin x + 1 = 0$

$3 - 6\sin^2 x + 2\sin x + 1 = 0$

$6\sin^2 x - 2\sin x - 4 = 0$

$3\sin^2 x - \sin x - 2 = 0$

(ii) $(3\sin x + 2)(\sin x - 1) = 0$

$\sin x = \frac{-2}{3}$ **or** $\sin x = 1$

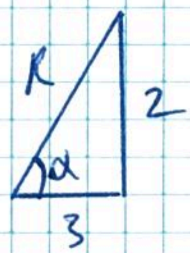
b) (i)

$R\cos(2x - \alpha) \equiv R\cos\alpha\cos 2x + R\sin\alpha\sin 2x$

$\equiv 3\cos 2x + 2\sin 2x$

$\Rightarrow R\cos\alpha = 3$

$R\sin\alpha = 2$



$R = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.690\ldots$

$= 33.7$

$3\cos 2x + 2\sin 2x \equiv \sqrt{13}\cos(2x - 33.7)$

(ii) $\sqrt{13}\cos(2x - 33.7) = -1$

$2x - 33.7 = \cos^{-1}\left(\frac{-1}{\sqrt{13}}\right)$

$= 106.102\ldots,\ 253.897\ldots$

$x = \frac{106.1\ldots + 33.7}{2},\ \frac{253.897\ldots + 33.7}{2}$

$= 69.896\ldots,\ 143.839\ldots$

$= 69.9,\ 143.8$

AQA SPECIMEN PAPER 2

a) $3\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$

$\equiv R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$

$\Rightarrow R\cos\alpha = 3$

$R\sin\alpha = 3$

$R = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{3}{3}\right) = 45^\circ$

$\therefore\ 3\cos\theta + 3\sin\theta \equiv 3\sqrt{2}(\cos(\theta - 45^\circ))$

$\Rightarrow$ STRETCH, y-DIRECTION, SCALE FACTOR $3\sqrt{2}$

TRANSLATION BY VECTOR $\begin{pmatrix} 45^\circ \\ 0 \end{pmatrix}$

b) $\forall\theta: -1 \leq \cos\theta \leq 1$

$-3\sqrt{2} \leq 3\sqrt{2}\cos(\theta - 45)) \leq 3\sqrt{2}$

$-3\sqrt{2} \leq 3\cos\theta + 3\sin\theta \leq 3\sqrt{2}$

$0 \leq (3\cos\theta + 3\sin\theta)^2 \leq 18$

$4 \leq 4 + (3\cos\theta + 3\sin\theta)^2 \leq 22$

$\Rightarrow$ LEAST VALUE = 4
GREATEST VALUE = 22