Pure Sector 2: Proof

Aims

Students should be able to:

- Set out a clear proof with the correct use of symbols, such as =, ≡, ⇒, ⇐, ⇔, ∴, ∵
- Understand that many examples can be useful in looking for structure, but they do not constitute a proof.

A proof is a logical argument for a mathematical statement. It shows that something must be either true or false.

Proof questions can look a bit terrifying. There are several techniques that can be used to aid the process. This section looks at a few but a starting point should be to get a feel of the problem and then establish a technique to be used.

Initially

Before starting there a few things to be aware of that will come in handy when trying to prove various problems.

In all of the following statements, n is any integer.

Any **even number** can be written as 2n – i.e. 2 x something.

> This can be extended to multiples of other numbers too – e.g. to prove something is a multiple of 5, show that it can be written as 5 x something.

Any **odd number** can be written as 2n + 1 (i.e. 2 x something + 1) or 2n – 1 (i.e. 2 x something - 1).

Consecutive numbers can be written as n, n + 1, n + 2, etc.

Consecutive even numbers can be written as 2n, 2n + 2, 2n + 4, etc.

Consecutive odd numbers can be written as 2n + 1, 2n + 3, 2n + 5, etc.

If two numbers are **not consecutive** then use two different variables such as m and n. E.g. two even numbers can be written as 2m and 2n.

The sum, difference and product of integers are always integers.

Direct Proof

Sometimes referred to as deductive proof.

In direct proof you rely on statements that are already established, or statements that can be assumed to be true, to show by deduction that another statement is true or false.

Statements that can be assumed to be true are sometimes known as axioms.

Examples of statements that can be assumed to be true include ‘you can draw a straight line segment joining any two points’ and ‘you can write all even numbers in the form 2n and all odd numbers in the form 2n+1 (where n is any integer)’.

To use direct proof:

- Assume that a statement, P, is true.
- Use P to show that another statement, Q, must be true.

Example

Use direct proof to prove that the square of any integer is one more than the product of the two integers either side of it.

Let the integer be n.

The two numbers on each side of n are n-1 and n+1

Assume that this statement is true.

The product of these two numbers is $(n-1)(n+1)$

Expand brackets to get n^2

$$(n-1)(n+1) = n^2 - 1$$

Rearrange to show the required result.

So $n^2 = (n-1)(n+1) + 1$

So the square of any integer is one more than the product of the two integers either side of it.

Example

Prove that the sum of any three odd numbers is odd.

Need to show that the sum of three odd numbers can be written as (2 x integer) + 1

Let the three odd numbers be 2a+1, 2b+1 and 2c+1 (they don't have to be consecutive).

The sum is therefore:

Note that 3 has been written as 2 + 1

$$\begin{aligned}(2a+1) + (2b+1) + (2c+1) &= 2a + 2b + 2c + 3\\ &= 2a + 2b + 2c + 2 + 1\\ &= 2(a + b + c + 1) + 1\\ &= 2n + 1 \text{ where } n \text{ is an integer } (a + b + c + 1)\end{aligned}$$

So the sum of any three odd numbers is odd.

Example

Prove that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$

≡ is the identity symbol and means that two things are identically equal to each other for all values.

Take one side of the equation and manipulate until you get the other side.

Taking the LHS:

$$\begin{aligned}(n + 3)^2 - (n - 2)^2 &\equiv (n^2 + 6n + 9) - (n^2 - 4n + 4)\\ &\equiv n^2 + 6n + 9 - n^2 + 4n - 4)\\ &\equiv 10n + 5\\ &\equiv 5(2n + 1) \qquad = \text{RHS}\end{aligned}$$

Use Direct Proof to answer the following questions:

1 Prove that the number 1 is not a prime number

A PRIME NUMBER, BY DEFINITION, HAS EXACTLY TWO FACTORS:
"1 AND THE NUMBER ITSELF"

THE NUMBER 1 HAS ONLY ONE FACTOR.

SO IT IS NOT A PRIME NUMBER.

2 Prove that the sum of two odd numbers is always even.

LET THE TWO ODD NUMBERS BE
$2m+1$ AND $2n+1$

SO:

$$(2m+1) + (2n+1) = 2m + 2n + 2$$
$$= 2(m+n+1)$$

NOW: $m + n + 1 = k$, AN INTEGER

SO THE SUM IS $2k$, WHICH IS THE DEFINITION OF AN EVEN NUMBER.

3 Prove that the product of two consecutive odd numbers is one less than a multiple of four.

TWO CONSECUTIVE ODD NUMBERS.

LET THE SMALLER ONE BE $2m+1$
THEN THE NEXT ONE IS $2m+3$

SO:

$$(2m+1)(2m+3) = 4m^2 + 8m + 3$$
$$= 4(m^2+2m+1) - 1$$

NOW: $m^2 + 2m + 1 = k$, AN INTEGER

SO THE PRODUCT IS $4k - 1$, WHICH IS ONE LESS THEN A MULTIPLE OF 4.

4 Prove that the mean of three consecutive integers is equal to the middle number.

LET THE CONSECUTIVE INTEGERS BE, m, $m+1$, $m+2$

$$\text{THE MEAN} = \frac{(m) + (m+1) + (m+2)}{3}$$

$$= \frac{3m+3}{3} = \frac{\cancel{3}(m+1)}{\cancel{3}} = m+1$$

WHICH IS THE MIDDLE NUMBER

5 a Prove that the sum of the squares of two consecutive integers is odd.

TWO CONSECUTIVE INTEGERS, m AND $m+1$

SO: $(m)^2 + (m+1)^2 = m^2 + m^2 + 2m + 1$

$= \underbrace{2m^2 + 2m}_{\text{THESE TWO TERMS ARE EVEN}} + \underbrace{1}_{\text{ODD TERM}}$

SO: SUM = EVEN + EVEN + ODD

SO THE SUM MUST BE ODD.

b Prove that the sum of the squares of two consecutive even numbers is always a multiple of four.

TWO CONSECUTIVE EVEN NUMBERS, $2m$ AND $2m+2$

SO: $(2m)^2 + (2m+2)^2 = 4m^2 + 4m^2 + 4m + 4m + 4$

$= 8m^2 + 8m + 4$

$= 4(2m^2 + 2m + 1)$

4 IS A FACTOR OF THIS EXPRESSION

SO THE SQUARES OF TWO CONSECUTIVE EVEN NUMBERS IS ALWAYS A MULTIPLE OF 4.

6 Show that the sum of four consecutive positive integers has both even factors and odd factors greater than one.

LET THE FOUR CONSECUTIVE POSITIVE INTEGERS BE, m, $m+1$, $m+2$ AND $m+3$

SO: $(m) + (m+1) + (m+2) + (m+3)$

$= 4m + 6$

$= 2(2m+3)$

2 IS EVEN $\qquad$ $2m+3$ IS ODD

HENCE THE SUM HAS BOTH ODD AND EVEN FACTORS, BOTH GREATER THAN ONE.

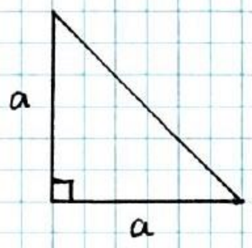
7 Prove that the square of the sum of any two positive numbers is greater than the sum of the squares of the numbers.

LET THE TWO NUMBERS BE m AND n

$$(m+n)^2 = m^2 + 2mn + n^2$$
$$= (m^2 + n^2) + 2mn$$

THIS IS $2mn$ MORE THAN THE SUM OF THE SQUARES, SINCE, IF m AND n ARE POSITIVE, THEN mn MUST ALSO BE POSITIVE.

8 Prove that the perimeter of an isosceles right-angled triangle is always greater than three times the length of one of the equal sides.



LET THE EQUAL SIDE BE LENGTH a

$$\therefore \text{ HYPOTENUSE } = \sqrt{a^2 + a^2}$$
$$= \sqrt{2a^2}$$
$$= \sqrt{2}\sqrt{a^2}$$
$$= a\sqrt{2}$$

THUS THE PERIMETER $= 2a + a\sqrt{2}$

NOW, SINCE $\sqrt{2} > 1$, $\Rightarrow 2a + a\sqrt{2} > 3a$

SO THE PERIMETER IS ALWAYS GREATER THAN THREE TIMES THE LENGTH OF ONE OF THE EQUAL SIDES.

9 a and b are two numbers such that a = b - 2 and the sum and product of a and b are equal. Prove that neither a nor b is an integer.

IF THE SUM = THE PRODUCT
THEN:
$$a + b = ab$$

NOW, $a = b - 2$

SO: $(b-2) + b = (b-2)b$

$\Rightarrow \quad 2b - 2 = b^2 - 2b$

$\Rightarrow \quad b^2 - 4b + 2 = 0$

NOW, $b^2 - 4ac = (-4)^2 - 4(1)(2)$
$= 16 - 8$
$= 8$ WHICH IS > 0 SO HAS TWO SOLUTIONS (REAL) BUT IS NOT A SQUARE NUMBER SO THE QUADRATIC DOES NOT FACTORISE.

HENCE b CANNOT BE AN INTEGER.

CONSEQUENTLY, SINCE $a = b - 2$, NEITHER CAN a.

10 If $(5y)^2$ is even for an integer y, prove that y must be even.

IF $(5y)^2$ IS EVEN, THEN 5y MUST BE EVEN.

SINCE 5 IS ODD, y MUST BE EVEN TO MAKE 5y EVEN.

Proof by Exhaustion

In this method, you list all the possible cases and test each one to see if the result you want to prove is true. All cases must be true for proof by exhaustion to work, since a single counter example would disprove the result.

To use proof by exhaustion:

- List a set of cases that exhausts all possibilities.
- Show the statement is true in each and every case.

Example

Prove, by exhaustion, that $p^2 + 1$ is not divisible by 3, where p is an integer and $6 \le p \le 10$.

p	**$p^2 + 1$**	**Divisible by 3?**
6	37	No
7	50	No
8	65	No
9	82	No
10	101	No

Write down every possible value of p and calculate p^2+1

Show that the statement is true in each case.

In all cases, $p^2 + 1$ is not divisible by 3.

Use Proof by Exhaustion to answer the following questions:

1 Prove that there is exactly one square number and exactly one cube number between 20 and 30.

NUMBER	SQUARE	CUBE
20	NO	NO
21	NO	NO
22	NO	NO
23	NO	NO
24	NO	NO
25	YES (5^2)	NO
26	NO	NO
27	NO	YES (3^3)
28	NO	NO
29	NO	NO
30	NO	NO

2 Prove that, for an integer x, $(x + 1)^3 \geq 3^x$ for $0 \leq x \leq 4$.

THERE ARE FIVE CASES:

$x = 0 \Rightarrow (0+1)^3 \geqslant 3^0 \Rightarrow 1 \geqslant 1$ TRUE

$x = 1 \Rightarrow (1+1)^3 \geqslant 3^1 \Rightarrow 8 \geqslant 3$ TRUE

$x = 2 \Rightarrow (2+1)^3 \geqslant 3^2 \Rightarrow 27 \geqslant 9$ TRUE

$x = 3 \Rightarrow (3+1)^3 \geqslant 3^3 \Rightarrow 64 \geqslant 27$ TRUE

$x = 4 \Rightarrow (4+1)^3 \geqslant 3^4 \Rightarrow 125 \geqslant 81$ TRUE

THUS THE STATEMENT IS TRUE.

3 Prove that no square numbers can have a last digit 2, 3, 7 or 8.

LET THE SQUARE NUMBER BE m^2

WHERE $m = 10p + k$ FOR INTEGERS p AND $k : 0 \leqslant k \leqslant 9$

{ 10p represents all multiples of 10, k represents the integers }

IF $k = 0 \Rightarrow m = 10p$ THEN $m^2 = (10p)^2 = 100p$ WHICH ENDS IN 0

IF $k = 1 \Rightarrow m = 10p + 1$ THEN $m^2 = (10p+1)^2 = 100p^2 + 20p + 1$ WHICH ENDS IN 1

IF $k = 2 \Rightarrow m = 10p + 2$ THEN $m^2 = (10p+2)^2 = 100p^2 + 40p + 4$ WHICH ENDS IN 4

IF $k = 3 \Rightarrow m = 10p + 3$ THEN $m^2 = (10p+3)^2 = 100p^2 + 60p + 9$ WHICH ENDS IN 9

IF $k = 4 \Rightarrow m = 10p + 4$ THEN $m^2 = (10p+4)^2 = 100p^2 + 80p + 16$ WHICH ENDS IN 6

IF $k = 5 \Rightarrow m = 10p + 5$ THEN $m^2 = (10p+5)^2 = 100p^2 + 100p + 25$ WHICH ENDS IN 5

IF $k = 6 \Rightarrow m = 10p + 6$ THEN $m^2 = (10p+6)^2 = 100p^2 + 120p + 36$ WHICH ENDS IN 6

IF $k = 7 \Rightarrow m = 10p + 7$ THEN $m^2 = (10p+7)^2 = 100p^2 + 140p + 49$ WHICH ENDS IN 7

IF $k = 8 \Rightarrow m = 10p + 8$ THEN $m^2 = (10p+8)^2 = 100p^2 + 160p + 64$ WHICH ENDS IN 4

IF $k = 9 \Rightarrow m = 10p + 9$ THEN $m^2 = (10p+9)^2 = 100p^2 + 180p + 81$ WHICH ENDS IN 1

HENCE ALL SQUARE NUMBERS END IN 0, 1, 4, 5, 6 OR 9.

SO THEY CANNOT HAVE A LAST DIGIT 2, 3, 7 OR 8.

Disproof by Counter Example

In this method, a statement can be disproved by finding just one example that does not fit the statement.

Example

Prove, by counter example, that the statement '$n^2 + n + 1$ is prime for all integers n' is false.

Let n = 4:

$$n^2 + n + 1 = 16 + 4 + 1 = 21 = 3 \times 7$$

Show that the statement is false for one value of n.

21 has factors 1, 3, 7 and 21, so is not prime.

This disproves the statement '$n^2 + n + 1$ is prime for all integers n'

Example

Bruce says "the difference between any two consecutive square numbers is always a prime number". Prove that Bruce is wrong.

Square numbers are 1^2, 2^2, 3^2, etc.

1^2 and 2^2 - 1 and 4 - difference = 3 - a prime number

2^2 and 3^2 - 4 and 9 - difference = 5 - a prime number

3^2 and 4^2 - 9 and 16 - difference = 7 - a prime number

Show that the statement is false for one value of n.

4^2 and 5^2 - 16 and 25 - difference = 9 - NOT a prime number, so Bruce is wrong.

Loads of examples are not a necessity if a counter example can be spotted earlier.

Give Counter Examples to disprove these statements:

1 The product of two prime numbers is always odd.

$2 \times 3 = 6$
WHICH DISPROVES THE STATEMENT

2 When you throw two six-sided dice, the total score shown is always greater than six.

$1 + 2 = 3$ WHICH IS < 6
AND DISPROVES THE STATEMENT

3 When you subtract one number from another, the answer is always less than the first number.

$3 - (-4) = 3 + 4 = 7$

$7 \not< 3$

WHICH DISPROVES THE STATEMENT

4 Five times any number is always greater than that number.

$5 \times (-2) = -10$

$-10 \not> -2$

WHICH DISPROVES THE STATEMENT

5 If $a > b$, then $a^b > b^a$.

EXAMPLE : $a = 4$ $b = 3$

$a > b$ $a^b > b^a$

$4 > 3$ BUT $4^3 = 64 \not> 3^4 = 81$

WHICH DISPROVES THE STATEMENT

6 The product of three consecutive integers is always divisible by four.

$1 \times 2 \times 3 = 6$ WHICH IS NOT DIVISIBLE BY 4

WHICH DISPROVES THE STATEMENT

Another method of proof is that by contradiction, possibly the trickiest of the proofs within this section.

Proof by Contradiction

To prove a statement by contradiction, you start by saying **"assume the statement … is not true"**. You then show that this would mean that something impossible would have to be true, which means that the initial assumption has to be wrong, so the original statement must be true.

Example

Prove the following statement: "If x^2 is even, then x must be even."

"If x^2 is even, then x must be even"

Assume that the statement is **not true**.

Therefore there must be an **odd number**, x, for which x^2 is even.

Let $x = 2k + 1$ (x is odd), where k is an integer.

So: $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Now, $2(2k^2 + 2k)$ is **even** due to the factor of 2, $\Rightarrow 2(2k^2 + 2k) + 1$ is **odd**

But this **is not possible** if the statement "x^2 is even" is true.

This is a **contradiction** of the assumption that there is an odd number x for which x^2 is even.

So if x^2 is **even**, then x must be **even**, hence, the original statement is **true**.

Example
Prove that $\sqrt{2}$ is irrational.

An irrational number is a real number that cannot be written as a fraction $\frac{a}{b}$ where a and b are both integers and $b \neq 0$.

Here, it can be assumed that a and b have no common factors because if they did, you could simplify the fraction to get a new a and b.

"$\sqrt{2}$ is irrational"

Start by assuming that the statement is **not true**, and that $\sqrt{2}$ can be written as a rational number $\frac{a}{b}$, with a and b both **non-zero integers**. Also assume that a and b **have no common factors**.

If $\sqrt{2} = \frac{a}{b}$ then $b\sqrt{2} = a$

Squaring both sides: $(b\sqrt{2})^2 = a^2$

$\Rightarrow 2b^2 = a^2$

So a^2 must be **even** due to the factor of 2.

Now, if a^2 is **even** then a must be **even**, by previous example.

So, let $a = 2k$, for some integer k.

Giving: $2b^2 = (2k)^2$

$\Rightarrow 2b^2 = 4k^2$

$\Rightarrow b^2 = 2k^2$

Like before, b^2 is therefore **even** due to the factor of 2.
And so if b^2 is **even** then so is b.

As both a and b are even, the assumption that there is a fraction, that, when fully simplified, can be written as $\frac{a}{b}$ has been **contradicted, as there must be common factors**.

Therefore, $\sqrt{2}$ **cannot** be written as a fraction $\frac{a}{b}$, so it is **irrational.**

Example
Prove by contradiction that there are infinitely many prime numbers.

"there are infinitely many prime numbers"
Assume that there are a **finite** number of primes, say n.
Label them: p_1, p_2 , p_3 , … , p_{n-1} , p_n – so p_n is the **largest** prime number.

Now, **multiply** all these together, and let this number equal P:

$$(p_1)(p_2)(p_3)(\ldots)(p_{n-1})(p_n) = P$$

Due to this definition, P is a **multiple** of **every** prime number.

Now, consider P + 1, **divide** P + 1 by p_1 we get:

$$\frac{P+1}{p_1} = \frac{p_1 p_2 p_3 \ldots p_{n-1} p_n + 1}{p_1}$$

$$= p_2 p_3 \ldots p_{n-1} p_n, \text{ remainder } 1$$

In fact dividing (P + 1) by any prime number gives a **remainder of 1**.

So (P + 1) **is not divisible by any** of the prime numbers, meaning that it must **also** be a prime number.

P + 1 is a **prime** number that is **larger** than p_n, which **contradicts** the assumption that p_n was the largest prime, so there must be **infinitely many prime numbers**.

Exercise

1 Prove, by contradiction, that there is no largest multiple of 3.

" there is no largest multiple of 3 "

Assume that there is a number x that is the largest multiple of 3.

This can then be written as : $x = 3k$, for some integer k.

Then : $x + 3 = 3k + 3$

$= 3(k+1)$ is also a multiple of 3 and is larger than x.

This contradicts the initial assumption.
So there cannot be a largest multiple of 3.

2 Prove that if x^2 is odd, then x must be odd.

"if x^2 is odd, then x must be odd"

Assume that there is some even number x for which x^2 is odd.

Since x is even it can be written as:

$x = 2n$ for some integer n.

Then: $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$
which is even due to the factor of 2.

This contradicts the assumption that x^2 is odd.

So if x^2 is odd, then x must also be odd.

3 (a) Prove that the product of a non-zero rational number and an irrational number is always irrational.

(a) "the product of a non-zero rational number and an irrational number is always irrational"

Assume there is a rational number $x \neq 0$ and an irrational number y such that xy is rational.

This means that x can be written as $x = \frac{a}{b}$ and xy as $xy = \frac{c}{d}$

where a, b, c and d are all non-zero integers.

So: $xy = \left(\frac{a}{b}\right)y = \frac{c}{d} \Rightarrow y = \frac{bc}{ad}$

Since bc and ad are both integers, this means that y is a rational number, which contradicts the assumption that y is irrational.

So the statement must be true.

(b) Disprove that the product of an irrational number and an irrational number is always irrational.

(b) To disprove the statement, find a counter-example.

Example: $\sqrt{2}$ is an irrational number

But: $(\sqrt{2})(\sqrt{2}) = 2$ which is rational,
So the statement is false.

4 Prove that there is no smallest positive rational number.

"there is no smallest positive rational number"

Assume that there is a smallest positive rational number, x.

Since x is rational, it can be written as $x = \frac{a}{b}$,

And, since it is positive, a and b are both positive integers (or both negative, in which case, the fraction can be simplified by dividing top and bottom by -1 to get a and b positive).

Then: $\frac{a}{b+1}$ is also a positive rational number, and is smaller than x.

Which contradicts the assumption that x is the smallest positive rational number.

So there cannot be a smallest positive rational number.

5 Prove that $1 + \sqrt{2}$ is irrational.

"$1 + \sqrt{2}$ is irrational"

Assume that $1 + \sqrt{2}$ is rational.

Therefore it can be written as $\frac{a}{b}$ where a and b are non-zero integers.

So: $1 + \sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2} = \frac{a}{b} - 1 \Rightarrow \sqrt{2} = \frac{a-b}{b}$

Now, since a and b are integers, $(a-b)$ is an integer, which means that $\sqrt{2}$ is rational, which is not true.

Therefore, $1 + \sqrt{2}$ must be irrational.

6 (a) Prove by contradiction that if x^2 is a multiple of 3, then x must be a multiple of 3.

(a) "if x^2 is a multiple of 3, then x must be a multiple of 3"

Suppose that there is a number x such that x^2 is a multiple of 3 but x is not.

If x is not a multiple of 3, then there are two cases to consider:

CASE 1: When $x = 3k+1$, for some integer k.

CASE 2: When $x = 3k+2$, for some integer k.

If $x = 3k+1$

then: $x^2 = (3k+1)^2$
$= 9k^2+6k+1$
$= 3(3k^2+2k)+1$

So: x^2 is not a multiple of 3.

If $x = 3k+2$

then: $x^2 = (3k+2)^2$
$= 9k^2+12k+4$
$= 3(3k^2+4k+1)+1$

So: x^2 is not a multiple of 3.

Therefore, by exhaustion, x^2 cannot be a multiple of 3 if x is not. Which contradicts the initial assumption.

So if x^2 is a multiple of 3, then x must also be a multiple of 3.

(b) Hence prove that $\sqrt{3}$ is irrational.

(b) "$\sqrt{3}$ is irrational"

Assume $\sqrt{3}$ is rational, so $\sqrt{3} = \frac{a}{b}$ for some non-zero integers a and b, that share no common factors.

So: $\sqrt{3} = \frac{a}{b} \Rightarrow b\sqrt{3} = a$

$\Rightarrow (b\sqrt{3})^2 = a^2$

$\Rightarrow 3b^2 = a^2$

$\Rightarrow a^2$ is a multiple of 3.

From part (a), if a^2 is a multiple of 3 then a is also a multiple of 3.

So: let $a = 3k$ for some integer k.

Then: $3b^2 = a^2 \Rightarrow 3b^2 = (3k)^2$

$\Rightarrow 3b^2 = 9k^2$

$\Rightarrow b^2 = 3k^2$

$\Rightarrow b^2$ is a multiple of 3

Again from part (a), this means that b is a multiple of 3.

But, it was assumed that a and b had no common factors.

This contradiction means that $\sqrt{3}$ cannot be written as an integer fraction, so $\sqrt{3}$ is irrational.

Proofs Can Appear In All Areas of Maths

From power laws and indices:

Worked Example
Show that the difference between 10^{18} and 6^{21} is a multiple of 2.

$$10^{18} - 6^{21} = (10 \times 10^{17}) - (6 \times 6^{20})$$
$$= (2 \times 5 \times 10^{17}) - (2 \times 3 \times 6^{20})$$
$$= 2[(5 \times 10^{17}) - (3 \times 6^{20})]$$

Which can be written as 2n where $n = [(5 \times 10^{17}) - (3 \times 6^{20})]$

So $10^{18} - 6^{21}$ is a multiple of 2.

To questions on mean, median, mode or range:

Worked Example
The range of a set of positive numbers is 5. Each number in the set is doubled. Show that the range of the new set of numbers also doubles.

Let the smallest value in the first set of numbers be n.

Then the largest value in this set is n+5 (as the range for this set is 5).

When the numbers are doubled, the smallest value in the new set is 2n and the largest value is 2(n+5) = 2n+10.

To find the new range, subtract the smallest value from the largest:
(2n + 10) - 2n = 10 = 2 x 5, which is double the original range.

Or questions where you have to use inequalities:

Worked Example
Anna says, "If $x > y$, then $x^2 > y^2$. Is she correct? Explain your answer.

Try some different values for x and y.
$x=2$, $y=1$: $x > y$ and $x^2 = 4 > 1 = y^2$

$x=5$, $y=2$: $x > y$ and $x^2 = 25 > 4 = y^2$

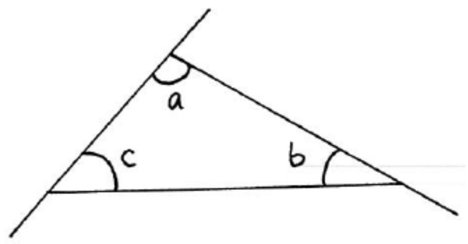
At first glance, it seems like she is correct.
But, trying: $x= -1$, $y= -2$: $x > y$ BUT $x^2 = 1 < 4 = y^2$

So Anna is wrong as the statement does not hold for all values of x and y.

Or even geometric proofs:

Worked Example
Prove that the sum of the exterior angles of a triangle is 360°.

First sketch a triangle with angles a, b and c.



Then the exterior angles are:
180° - a, 180° - b and 180° - c

So their sum is:

$$(180° - a) + (180° - b) + (180° - c) = 540° - (a + b + c)$$
$$= 540° - 180° \text{ (as the angles in a triangle add up to 180°)}$$
$$= 360°$$

Reasoning and Problem-Solving

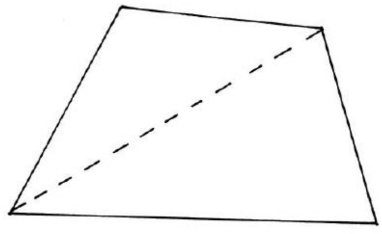
Strategy: 1 Decide which method of proof to use.
2 Follow the steps of the chosen method.
3 Write a clear conclusion that proves/disproves the statement.

Worked Example
Prove that the sum of the interior angles in any convex quadrilateral is 360°.

"A convex polygon is defined as a polygon with all its interior angles less than 180°. This means that all the vertices of the polygon will point outwards, away from the interior of the shape."

Use direct proof

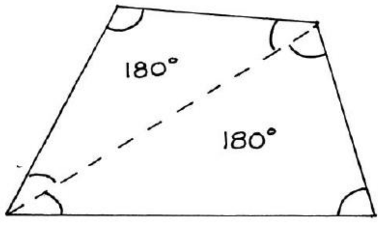
The sum of the interior angles of any quadrilateral can be found by breaking the quadrilateral into two triangles.



The sum of the interior angles of any triangle equals 180°, and each of the two triangles will contribute 180° to the total sum of all angles in the quadrilateral.

Write a clear conclusion

So the interior angle sum of a convex quadrilateral is the same as the sum of the interior angles of two triangles, which is 360°.



Worked Example

Lauren says that there are exactly three prime numbers between the numbers 15 and 21 (inclusive). Is she correct? Use a suitable method of proof to justify your answer.

NUMBER	PRIME
15	No → 15 = 1x15, 3x5
16	No → 16 = 1x16, 2x8, 4x4
17	Yes → 17 = 1x17
18	No → 18 = 1x18, 2x9, 3x6
19	Yes → 19 = 1x19
20	No → 20 = 1x20, 2x10, 4x5
21	No → 21 = 1x21, 3x7

Use proof by exhaustion to check all the numbers within the range of values

Write a clear conclusion

There are exactly two prime numbers between 15 and 21, so Lauren is wrong, there are not three.

Reasoning and Problem-Solving Questions

1 P is a prime number and Q is an odd number

(a) Sue says PQ is even, Liz says that PQ is odd and Graham says PQ could be either. Who is right? Use a suitable method of proof to justify your answer.

CASE 1 : P IS EVEN ⇒ P = 2 , Q IS ODD NUMBER
SO PQ = EVEN x ODD = EVEN

CASE 2 : P IS ODD , Q IS ODD NUMBER
SO PQ = ODD x ODD = ODD

SO GRAHAM IS RIGHT.

(b) Sue now says that P(Q + 1) is always even. Is she correct? Use a suitable method of proof to justify your answer.

CASE 1 : P IS EVEN ⇒ P = 2 , Q + 1 IS EVEN
SO P(Q+1) = EVEN x EVEN = EVEN

CASE 2 : P IS ODD , Q + 1 IS EVEN
SO P(Q+1) = ODD x EVEN = EVEN

SUE IS RIGHT.

2 Use a suitable method of proof to show whether the following statement is true or false.

'Any odd number between 90 and 100 is either a prime number or the product of only two prime numbers'

91	=	7 × 13		
93	=	3 × 31		
95	=	5 × 19		
97	IS	PRIME		
99	=	3 × 33		33 IS NOT PRIME

STATEMENT IS FALSE

3 Use a suitable method of proof to prove that the value of:

$9^n - 1$ is divisible by 8 for $1 \leq n \leq 6$

FOR $1 \leq n \leq 6$

$n = 1$	$\Rightarrow$	$9^1 - 1$	$= 8$	WHICH IS	8×1
$n = 2$	$\Rightarrow$	$9^2 - 1$	$= 80$		8×10
$n = 3$	$\Rightarrow$	$9^3 - 1$	$= 728$		8×91
$n = 4$	$\Rightarrow$	$9^4 - 1$	$= 6560$		8×820
$n = 5$	$\Rightarrow$	$9^5 - 1$	$= 59\,048$		8×7381
$n = 6$	$\Rightarrow$	$9^6 - 1$	$= 531\,440$		8×66430

SO THE VALUE OF $9^n - 1$ IS DIVISIBLE BY 8 FOR $1 \leq n \leq 6$

4 Is it true that 'all triangles are obtuse'?
Use a suitable method of proof to justify your answer.

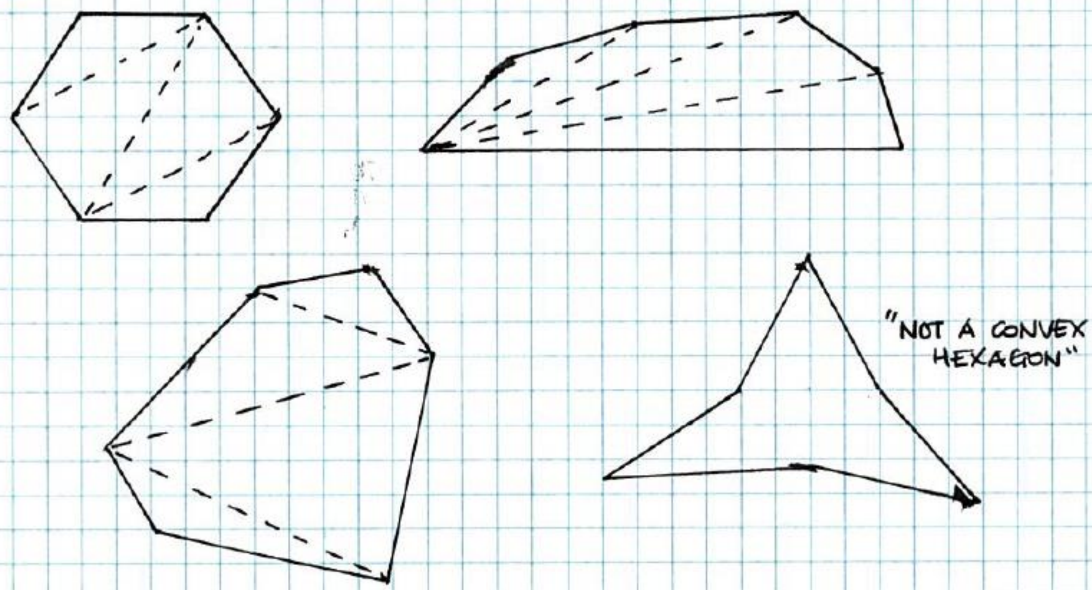
A RIGHT-ANGLED TRIANGLE IS NOT OBTUSE.

AN EQUILATERAL TRIANGLE IS NOT OBTUSE.

SO IT IS NOT TRUE 'ALL TRIANGLES ARE OBTUSE'

5 Prove that the sum of the interior angles of a convex hexagon is 720°.

"A convex hexagon is a 6-sided polygon in which all interior angles are less than 180°"



A CONVEX HEXAGON CAN BE SPLIT INTO 4 TRIANGLES.

SUM OF THE INTERIOR ANGLES OF A TRIANGLE = 180°

4 x 180° = 720°

SO THE SUM OF THE INTERIOR ANGLES OF A CONVEX HEXAGON = 720°

6 Martin says that 'all quadrilaterals with equal sides are squares'.
Use a suitable method of proof to show if his statement is true or false.

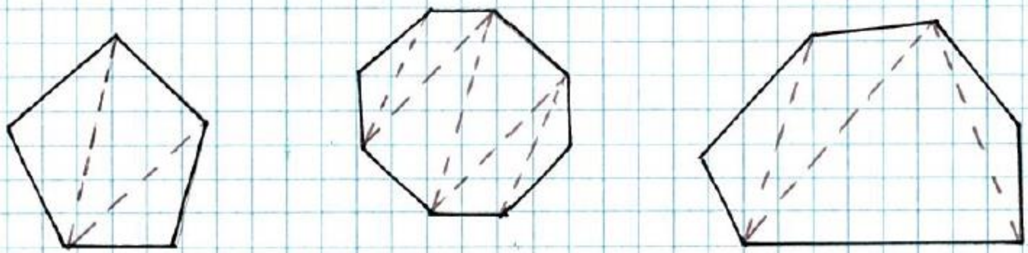
A RHOMBUS HAS EQUAL SIDES BUT IS NOT A SQUARE.

FALSE

All sides equal – not square.

7 Prove that the sum of the interior angles of a convex n-sided polygon is 180(n – 2)°

"A convex polygon is defined as a polygon with all its interior angles less than 180°. This means that all the vertices of the polygon will point outwards, away from the interior of the shape."



$n = 5$ $n = 8$ $n = 6$

A CONVEX n-SIDED POLYGON CAN BE SPLIT INTO $n-2$ TRIANGLES.
THE SUM OF THE INTERIOR ANGLES OF A TRIANGLE IS 180°.
SO, $(n-2) \times 180^\circ = 180(n-2)$

SO THE SUM OF THE INTERIOR ANGLES OF A CONVEX n-SIDED POLYGON IS $180(n-2)^\circ$.

8 Use a suitable method of proof to prove or disprove the statement:
"If m² = n² then m = n"

$(5)^2 = (-5)^2$ BUT $5 \neq -5$

FALSE

9 The hypotenuse of a right-angled triangle is (2s + a) cm and one other side is (2s – a) cm. Use a suitable method of proof to show that the square of the remaining side is a multiple of eight.

BY PYTHAGORAS' THEOREM:

$$(2s + a)^2 - (2s - a)^2$$
$$= \left((2s)^2 + 2as + 2as + a^2\right) - \left((2s)^2 - 2as - 2as + a^2\right)$$
$$= \cancel{4s^2} + 4as + \cancel{a^2} - \cancel{4s^2} + 4as - \cancel{a^2}$$
$$= 8as$$

SO THE SQUARE OF THE REMAINING SIDE IS A MULTIPLE OF 8.

10 Use a suitable method of proof to show that,
For 1 ≤ n ≤ 5,

$$\frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\cdots+\frac{1}{n\,x\,(n+1)}=\frac{n}{n+1}$$

FOR $n=1$: $\frac{1}{1\times2} = \frac{1}{2}$

FOR $n=2$: $\frac{1}{1\times2}+\frac{1}{2\times3} = \frac{1}{2}+\frac{1}{6} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}$

FOR $n=3$: $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4} = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{6+2+1}{12} = \frac{9}{12} = \frac{3}{4}$

FOR $n=4$: $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5} = \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$

$= \frac{30+10+5+3}{60} = \frac{48}{60} = \frac{4}{5}$

FOR $n=5$: $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}$

$= \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$

$= \frac{30+10+5+3+2}{60} = \frac{50}{60} = \frac{5}{6}$

SO, $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n\times(n+1)} = \frac{n}{n+1}$

FOR $1 \leq n \leq 5$

Extension Question

A teacher tells her class that any number is divisible by three if the sum of its digits is divisible by three. Use a suitable method to prove this result for two-digit numbers.

LET THE 2-DIGIT NUMBER BE X
WITH TENS DIGIT y AND UNITS DIGIT z

THEREFORE $X = 10y + z$

$= 9y + (y+z)$

LET THE SUM OF THE DIGITS BE DIVISIBLE BY 3
THEN $(y+z) = 3P$

SO: $X = 9y + 3P$

$= 3(3y+P)$

HENCE X IS A MULTIPLE OF 3