Proof

Remember:

Even Number	$2n$
Odd Number	$2n+1$ or $2n-1$
Multiples of other numbers	Multiples of 5, $5n$ etc
Consecutive numbers	$n, n+1, n+2$ etc
Non-consecutive numbers	n and m or $2n$ and $2m$ etc
Rational number	$\frac{a}{b}$
Irrational number	n

5 Prove that, for integer values of n such that $0 \leq n < 4$

$$2^{n+2} > 3^n$$

[2 marks]

1 Celia states that $n^2 + 2n + 10$ is always odd when n is a prime number.

Prove that Celia's statement is false. **[2]**

Q	Marking instructions	AO	Marks	Typical solution
5	Selects and begins to use a suitable method of proof. **Exhaustion:** Must check at least two correct values for n in the range $0 \le n < 4$ and make at least two correct comparisons. Comparisons are implied by ticks/crosses or use of true/false **Direct proof:** Takes logs to any base of both sides and uses a law of logs correctly once **Contradiction:** Must be clear they are attempting contradiction with "$0 \le n < 4$ and $2^{n+2} \le 3^n$" assumed explicitly. Condone strict inequality	3.1a	M1	<table><tr><td>n</td><td>2^{n+2}</td><td>3^n</td><td></td></tr><tr><td>0</td><td>4</td><td>1</td><td>4>1</td></tr><tr><td>1</td><td>8</td><td>3</td><td>8>3</td></tr><tr><td>2</td><td>16</td><td>9</td><td>16>9</td></tr><tr><td>3</td><td>32</td><td>27</td><td>32>27</td></tr></table> Hence $2^{n+2} > 3^n$ for integer values of n such that $0 \le n < 4$
	Completes a reasoned mathematical argument, proving $2^{n+2} > 3^n$ when n is an integer and $0 \le n < 4$. Must include a fully correct concluding statement which refers to 'integer' or lists the four integers If using direct proof or contradiction they must use the laws of logs correctly to remove n from the exponent. Condone use of equality if direct proof used	2.1	R1	
	Total		**2**	

Question			Answer	Marks	AOs	Guidance	
1			When $n = 2$, $2^2 + 2 \times 2 + 10$ $(= 18)$ which is not odd So the statement must be false (counterexample)	M1 A1 [2]	2.1 2.2a	Use of $n = 2$ seen Complete argument must include a clear conclusion that statement false	

13. (a) Prove that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$

(4)

(b) Prove, by counter example, that this is not true for all values of a and b.

(1)

3. (a) "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."

Disprove this statement by means of a counter example.

(2)

Question	Scheme	Marks	AOs
13 (a)	States $(2a-b)^2 \geqslant 0$	M1	2.1
	$4a^2+b^2 \geqslant 4ab$	A1	1.1b
	(As $a>0, b>0$) $\dfrac{4a^2}{ab}+\dfrac{b^2}{ab} \geqslant \dfrac{4ab}{ab}$	M1	2.2a
	Hence $\dfrac{4a}{b}+\dfrac{b}{a} \geqslant 4$ * CSO	A1*	1.1b
		(4)	
(b)	$a=5, b=-1 \Rightarrow \dfrac{4a}{b}+\dfrac{b}{a}=-20-\dfrac{1}{5}$ which is less than 4	B1	2.4
		(1)	

(a) (condone the use of $>$ for the first three marks)

M1: For the key step in stating that $(2a-b)^2 \geqslant 0$

A1: Reaches $4a^2+b^2 \geqslant 4ab$

M1: Divides each term by $ab \Rightarrow \dfrac{4a^2}{ab}+\dfrac{b^2}{ab} \geqslant \dfrac{4ab}{ab}$

A1*: Fully correct proof with steps in the correct order and gives the reasons why this is true:

- when you square any (real) number it is always greater than or equal to zero
- dividing by ab does not change the inequality as $a>0$ and $b>0$

(b)

B1: Provides a counter example and shows it is not true.
This requires values, a calculation or embedded values(see scheme) and a conclusion. The conclusion must be in words eg the result does not hold or not true
Allow 0 to be used as long as they explain or show that it is undefined so the statement is not true.

Question	Scheme	Marks	AOs
3	Statement: "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."		
(a)	E.g. $m=\sqrt{3}, n=\sqrt{12}$	M1	1.1b
	$\{mn=\}$ $(\sqrt{3})(\sqrt{12})=6$ $\Rightarrow$ statement untrue **or** 6 is not irrational or 6 is rational	A1	2.4
		(2)	

(a)	
M1:	States or uses any pair of ***different*** numbers that will disprove the statement. E.g. $\sqrt{3}, \sqrt{12}$; $\sqrt{2}, \sqrt{8}$; $\sqrt{5}, -\sqrt{5}$; $\dfrac{1}{\pi}, 2\pi$; $3e, \dfrac{4}{5e}$;
A1:	Uses correct reasoning to disprove the given statement, with a correct conclusion
Note:	Writing $(3e)\left(\dfrac{4}{5e}\right)=\dfrac{12}{5} \Rightarrow$ untrue is sufficient for M1A1

16. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

Question	Scheme	Marks	AOs
16	Sets up the contradiction and factorises: There are positive integers p and q such that $(2p+q)(2p-q)=25$	M1	2.1
	If true then $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ or $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ **Award for deducing either of the above statements**	M1	2.2a
	Solutions are $p=6.5, q=12$ or $p=2.5, q=0$ Award for one of these	A1	1.1b
	This is a contradiction as there are no integer solutions hence there are no positive integers p and q such that $4p^2-q^2=25$	A1	2.1
		(4)	

M1: For the key step in setting up the contradiction and factorising

M1: For deducing that for p and q to be integers then either $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ or $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ must be true.

Award for deducing either of the above statements.

You can ignore any reference to $\begin{matrix}2p+q=1\\2p-q=25\end{matrix}$ as this could not occur for positive p and q.

A1: For correctly solving one of the given statements,

For $\begin{matrix}2p+q=25\\2p-q=1\end{matrix}$ candidates only really need to proceed as far as $p=6.5$ to show the contradiction.

For $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ candidates only really need to find either p or q to show the contradiction.

Alt for $\begin{matrix}2p+q=5\\2p-q=5\end{matrix}$ candidates could state that $2p+q\neq 2p-q$ if p,q are positive integers.

A1: For a complete and rigorous argument with both possibilities and a correct conclusion.

16. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)

4 Prove by contradiction that there is no greatest multiple of 5. **[3]**

Question	Scheme	Marks	AOs
16	NB any natural number can be expressed in the form: $3k$, $3k+1$, $3k+2$ or equivalent e.g. $3k-1$, $3k$, $3k+1$		
	Attempts to square **any two** distinct cases of the above	M1	3.1a
	Achieves accurate results and makes a valid comment for **any two** of the possible three cases: E.g. $(3k)^2 = 9k^2 \left(= 3\times 3k^2\right)$ is a multiple of 3 $(3k+1)^2 = 9k^2+6k+1 = 3\times\left(3k^2+2k\right)+1$ is one more than a multiple of 3 $(3k+2)^2 = 9k^2+12k+4 = 3\times\left(3k^2+4k+1\right)+1$ $\left(\text{or } (3k-1)^2 = 9k^2-6k+1 = 3\times\left(3k^2-2k\right)+1\right)$ is one more than a multiple of 3	A1 **M1 on EPEN**	1.1b
	Attempts to square in all 3 distinct cases. E.g. attempts to square $3k$, $3k+1$, $3k+2$ or e.g. $3k-1$, $3k$, $3k+1$	M1 **A1 on EPEN**	2.1
	Achieves accurate results for all three cases and gives a minimal conclusion (allow tick, QED etc.)	A1	2.4
		(4)	

M1: Makes the key step of attempting to write the natural numbers in any 2 of the 3 distinct forms or equivalent expressions, as shown in the mark scheme, and attempts to square these expressions.

A1(M1 on EPEN): Successfully shows for 2 cases that the squares are either a multiple of 3 or 1 more than a multiple of 3 using algebra. This must be made explicit e.g. reaches $3\times\left(3k^2+2k\right)+1$ and makes a statement that this is one more than a multiple of 3 but also allow other rigorous arguments that reason why $9k^2+6k+1$ is one more than a multiple of 3 e.g. "$9k^2$ is a multiple of 3 and $6k$ is a multiple of 3 so $9k^2+6k+1$ is one more than a multiple of 3"

M1(A1 on EPEN): Recognises that all natural numbers can be written in one of the 3 distinct forms or equivalent expressions, as shown in the mark scheme, and attempts to square in all 3 cases.

A1: Successfully shows for all 3 cases that the squares are either a multiple of 3 or 1 more than a multiple of 3 using algebra and makes a conclusion

4			Assume that there is a greatest multiple of 5 ie $N = 5k$	**B1***	**2.1**	Assumption for contradiction	Some indication that they are starting with the greatest multiple of 5
			$N+5 = 5k+5 = 5(k+1)$	**M1**	**2.1**	Add on 5, or a multiple of 5	Or any equiv operation that would result in a larger multiple of 5 M0 if just numerical example
			This is a multiple of 5, and $N+5 > N$ which contradicts the assumption Hence there is no greatest multiple of 5	**A1d*** **[3]**	**2.4**	Statement denying assumption	Need justification about why it is a multiple of 5, why it is greater, as well as 'contradiction' or clear equiv such as 'initial assumption is incorrect'

15. Given $n \in \mathbb{N}$, prove that $n^3 + 2$ is not divisible by 8

(4)

Logical approach	States that if n is odd, n^3 is odd	M1	2.1
	so n^3+2 is odd and therefore cannot be divisible by 8	A1	2.2a
	States that if n is even, n^3 is a multiple of 8	M1	2.1
	so n^3+2 cannot be a multiple of 8 So (Given $n \in \mathbb{N}$), n^3+2 is not divisible by 8	A1	2.2a
		(4)	
			4 marks

First M1: States the result of cubing an odd or an even number
First A1: Followed by the result of adding two and gives a valid reason why it is not divisible by 8.
So for odd numbers accept for example

"odd number + 2 is still odd and odd numbers are not divisible by 8"

"n^3+2 is odd and cannot be divided by 8 **exactly**"

and for even numbers accept

"a multiple of 8 add 2 is not a multiple of 8, so n^3+2 is not divisible by 8"

"if n^3 is a multiple of 8 then n^3+2 cannot be divisible by 8

Second M1: States the result of cubing an odd and an even number
Second A1: Both valid reasons must be given followed by a concluding statement.

Question	Scheme	Marks	AOs
15 Algebraic approach	(If n is even,) $n=2k$ and $n^3+2=(2k)^3+2=8k^3+2$	M1	2.1
	Eg.'This is 2 more than a multiple of 8, hence not divisible by 8' Or 'as $8k^3$ is divisible by 8 , $8k^3+2$ isn't'	A1	2.2a
	(If n is odd,) $n=2k+1$ and $n^3+2=(2k+1)^3+2$	M1	2.1
	$=\underline{\underline{8k^3+12k^2+6k+3}}$ which is an even number add 3, therefore odd. Hence it is not divisible by 8 So (given $n \in \mathbb{N}$,) n^3+2 is not divisible by 8	A1	2.2a
		(4)	
Alt algebraic approach	(If n is even,) $n=2k$ and $\frac{n^3+2}{8}=\frac{(2k)^3+2}{8}=\frac{8k^3+2}{8}$	M1	2.1
	$=k^3+\frac{1}{4}$ oe which is not a whole number and hence not divisible by 8	A1	2.2a
	(If n is odd,) $n=2k+1$ and $\frac{n^3+2}{8}=\frac{(2k+1)^3+2}{8}$	M1	2.1
	$=\frac{8k^3+12k^2+6k+3}{8}$ ** The numerator is odd as $\underline{\underline{8k^3+12k^2+6k+3}}$ is an even number +3 hence not divisible by 8 So (Given $n \in \mathbb{N}$,) n^3+2 is not divisible by 8	A1	2.2a
		(4)	

Notes
Correct expressions are required for the M's. There is no need to state "**If n is even,**" $n=2k$ and "**If n is odd,** $n=2k+1$" for the two M's as the expressions encompass all numbers. However the concluding statement must attempt to show that it has been proven for all $n \in \mathbb{N}$ Some students will use $2k-1$ for odd numbers There is no requirement to change the variable. They may use $2n$ and $2n \pm 1$ Reasons must be correct. Don't accept $8k^3+2$ cannot be divided by 8 for example. (It can!) Also **"$=\frac{8k^3+12k^2+6k+3}{8}=k^3+\frac{3}{2}k^2+\frac{3}{4}k+\frac{3}{8}$ which is not whole number" is too vague so A0

10. (i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

(ii) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of $(x - 9)$."
State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

(i)
M1: Awarded for setting up the proof for either the even or odd numbers.
A1: Concludes correctly with a reason why n^2+2 cannot be divisible by 4 for either n odd or even.
dM1: Awarded for setting up the proof for both even and odd numbers
A1: Fully correct proof with valid explanation and conclusion for all n

Example of an algebraic proof

For $n=2m,\quad n^2+2=4m^2+2$	M1	2.1
Concludes that this number is not divisible by 4 (as the explanation is trivial)	A1	1.1b
For $n=2m+1,\quad n^2+2=(2m+1)^2+2=\ldots$ FYI $\left(4m^2+4m+3\right)$	dM1	2.1
Correct working and concludes that this is a number in the 4 times table add 3 so cannot be divisible by 4 or writes $4\left(m^2+m\right)+3$ ....AND states ......hence true for all	A1*	2.4
	(4)	

Example of a very similar algebraic proof

For $n=2m,\quad \dfrac{4m^2+2}{4}=m^2+\dfrac{1}{2}$	M1	2.1
Concludes that this is not divisible by 4 due to the $\dfrac{1}{2}$ (A suitable reason is required)	A1	1.1b
For $n=2m+1,\quad \dfrac{n^2+2}{4}=\dfrac{4m^2+4m+3}{4}=m^2+m+\dfrac{3}{4}$	dM1	2.1
Concludes that this is not divisible by 4 due to the $\dfrac{3}{4}$ ...AND states ...... hence for all n, n^2+2 is not divisible by 4	A1*	2.4
	(4)	

Example of a proof via logic

When n is odd, "odd $\times$odd" = odd	M1	2.1
so n^2+2 is odd , so (when n is odd) n^2+2 cannot be divisible by 4	A1	1.1b
When n is even, it is a multiple of 2, so "even $\times$even" is a multiple of 4	dM1	2.1
Concludes that when n is even n^2+2 cannot be divisible by 4 because n^2 is divisible by 4.....AND STATES ....trues for all n.	A1*	2.4
	(4)	

Example of proof via contradiction

Sets up the contradiction ‘Assume that n^2+2 is divisible by 4 $\Rightarrow n^2+2=4k$ ’	M1	2.1
$\Rightarrow n^2=4k-2=2(2k-1)$ and concludes even Note that the M mark (for setting up the contradiction must have been awarded)	A1	1.1b
States that n^2 is even, then n is even and hence n^2 is a multiple of 4	dM1	2.1
Explains that if n^2 is a multiple of 4 then n^2+2 cannot be a multiple of 4 and hence divisible by 4 Hence there is a contradiction and concludes Hence true for all n.	A1*	2.4
	(4)	

10 Prove by contradiction that $\sqrt[3]{2}$ is an irrational number.

[7 marks]

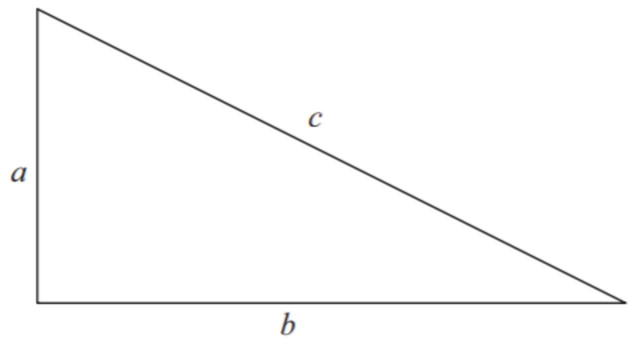
9 Prove that the sum of a rational number and an irrational number is always irrational.

[5 marks]

Q	Marking Instructions	AO	Marks	Typical Solution
10	Begins proof by contradiction, assumes that $\sqrt[3]{2}$ is rational OE	AO3.1a	M1	Assume $\sqrt[3]{2}$ is rational
	Uses language and notation correctly to state initial assumptions	AO2.5	B1	$\sqrt[3]{2} = \frac{a}{b}$, a and b have no common factors
	Manipulates fraction including cubing.	AO1.1a	M1	$\Rightarrow \sqrt[3]{2}b = a$ $\Rightarrow 2b^3 = a^3$
	Deduces a is even	AO2.2a	R1	$\therefore$ a is even
	Deduces b is even	AO2.2a	R1	let $a = 2d$ then $2b^3 = 8d^3$ $\Rightarrow b^3 = 4d^3$ $\therefore$ b is even
	Explains why there is a contradiction	AO2.4	E1	Hence, a and b have a common factor of 2. This is a contradiction.
	Completes rigorous argument to show that $\sqrt[3]{2}$ is irrational	AO2.1	R1	$\therefore$ the assumption that $\sqrt[3]{2}$ is rational must be incorrect and it is proved that $\sqrt[3]{2}$ is an irrational number
	Total		**7**	

Q	Marking instructions	AO	Mark	Typical solution
9	Begins proof by contradiction. This may be evidenced by: stating assumption at the start "the sum is rational" Or Sight of "contradiction" later as part of argument.	3.1a	M1	Assume m is rational and n is irrational and their sum is rational.
	Forms an equation of the form rational + irrational = rational with the rationals written algebraically $\frac{a}{b} + n = \frac{c}{d}$ n must clearly be irrational and not written as an algebraic fraction and not a specific value.	2.5	M1	Then $\frac{a}{b} + n = \frac{c}{d}$ Where a, b, c and d are all integers. $n = \frac{c}{d} - \frac{a}{b}$ $= \frac{bc - ad}{bd}$
	Manipulates their equation to show that n is rational	1.1b	A1	$\therefore$ *n* is rational, which is a contradiction.
	Explains or demonstrates why there is a contradiction	2.4	E1	So the original statement is false and the sum of a rational and irrational must be irrational.
	Completes rigorous argument to prove the required result including correct initial assumptions Where a, b, c and d are all integers.	2.1	R1	
	Total		**5**	

6 The three sides of a right-angled triangle have lengths a, b and c, where $a, b, c \in \mathbb{Z}$



6 (a) State an example where a, b and c are all even.

[1 mark]

6 (b) Prove that it is **not** possible for all of a, b and c to be odd.

[3 marks]

Q	Marking instructions	AO	Mark	Typical solution
6(a)	States an appropriate even Pythagorean triple	2.2a	B1	$a=6$ $b=8$ $c=10$
6(b)	Begins an appropriate method of proof assuming at least two sides are odd eg states 'assume a, b odd' or defines a, b (or c) algebraically with different unknowns	3.1a	B1	Assume a and b are odd so a = 2m + 1 and b = 2n + 1 $(2m+1)^2 + (2n+1)^2$ $= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ $= 2(2m^2 + 2m + 2n^2 + 2n + 1)$ which is even, so c^2 is even, so c is even. Therefore it is not possible for all three to be odd.
	Uses Pythagoras' theorem with at least two odd sides either in words or algebraically	1.1a	M1	
	Completes rigorous argument to prove the required result **CSO**	2.1	R1	
	Total		**4**	