Statistics Sector 1: Probability

Aims:

- Understand probability and be able to allocate probabilities using equally likely outcomes
- Identify mutually exclusive and independent events.
- Be able to understand set notation
- Use addition law and multiplication law

Introduction

An **event** is the outcome or set of outcomes of statistical experiment or trial. The **probability of an event A**, is denoted by $P(A)$ and is measured on a scale of 0 to 1. Zero represents impossibility and 1 represents certainty.

If a trial can result in N **equally likely outcomes,** of which $n(A)$ results in the event A, then

$$P(A) = \frac{n(A)}{N}$$

For example, if a fair die is thrown there are six equally likely outcomes so the probability of each outcome is $\frac{1}{6}$.

If it's not possible to assume equally likely outcomes, then $P(A)$ may be estimated by the **relative frequency of event A**,

$$\text{Relative Frequency } (A) = \frac{\text{Number of time event } A \text{ occurs}}{\text{Total number of trials}}$$

Example 1

A bookcase contains a mix of hardback and paperback books. The number of books of each type of three subject categories is shown in the table.

	Crime	**Romance**	**Travel**	**Total**
Paperback	43	12	18	73
Hardback	17	5	25	47
Total	60	17	43	120

A book is selected at random from the bookcase. Calculate the probability that the book is:

a) paperback $\frac{73}{120}$

b) romance $\frac{17}{120}$

c) crime hardback $\frac{17}{120}$

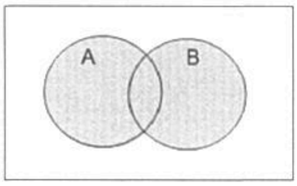
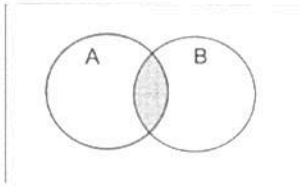
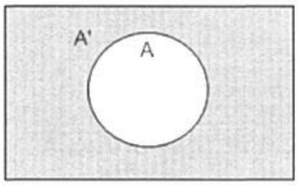
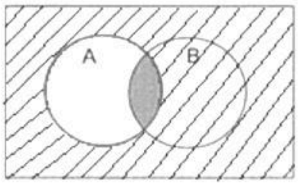
d) not a travel book $\frac{77}{120}$

e) crime or a hardback $\frac{90}{120}$

f) a romance book, given that it is a paperback. $\frac{12}{73}$

Set Notation

You are not required to use set notation in an exam but you must be able to understand it.

$A \cup B$	$A \cup B$ is the **union** of A and B, this denotes A or B or both events occurring.	
$A \cap B$	$A \cap B$ is the **intersection** of A and B, this denotes A and B both happening.	
A'	Given an event A then the **complementary event** is that A does not occur and is denoted by A'. Since either event A does or does not occur and the probability of exhaustive events must total 1 then: $P(A) + P(A') = 1$ or $P(A') = 1 - P(A)$	
$\mathrm{P}(B\|A)$	$\mathrm{P}(B\|A)$ denotes the **conditional probability** of B given A where: $P(B\|A) = \dfrac{P(A \cap B)}{P(A)}$	

Addition Law

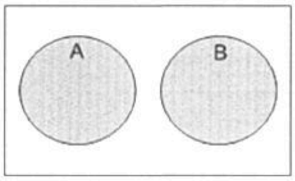
If A and B are any two events then the probability of at least one of A or B occurring is given:

$$\boxed{\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)}$$

This is given in the formula booklet.

Mutually Exclusive Events

Two events are said to be **mutually exclusive** if $P(A \cap B) = 0$. The Venn diagram shows the event $(A \cup B)$ when A and B are mutually exclusive. The addition law for mutually exclusive events simplifies to:



$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

This can extended to include any number of mutually exclusive events and is not given in the formula booklet. You may need to use this to show that events are/aren't mutually exclusive. When using the addition law always check first whether or not the events are mutually exclusive.

Exercise

A vehicle showroom contains 200 cars and 150 vans. A vehicle is selected at random.

C denotes the event 'the vehicle is a car'
V denotes the event 'the vehicle is a van'
R denotes the event 'the vehicle is red'
B denotes the event 'the vehicle is a black'

a) Name two of the events C, V, R and B that are mutually exclusive. C and V

b) Define, in the context of this question, the event:

i) V' THE EVENT 'THE VEHICLE IS NOT A VAN' IE IT IS A CAR.

ii) $C \cup R$

THE EVENT 'THE VEHICLE IS A CAR, OR RED (OR A RED CAR)

iii) $C \cap B'$

THE EVENT 'THE VEHICLE IS A CAR WHICH IS NOT BLACK'

iv) $V \cap (R \cup B)$

THE EVENT 'THE VEHICLE IS A RED OR BLACK VAN'

Example 2

On a Saturday Emma visits her local sports centre. The event that she goes swimming is denoted by S, and the event that she plays tennis is denoted by T. On each visit, Emma takes part in neither, or one or both of these activities.

a) Complete the table of probabilities.

	T	T'	Total
S	0.5	0.05	0.55
S'	0.20	0.25	0.45
Total	0.7	0.30	1.00

b) Hence or otherwise find the probability that on any given Saturday she goes swimming or plays tennis but not both.

$0.05 + 0.2 = 0.25$

c) Show that the events S and T are not mutually exclusive.

$P(S \cap T) = 0.5 \neq 0$

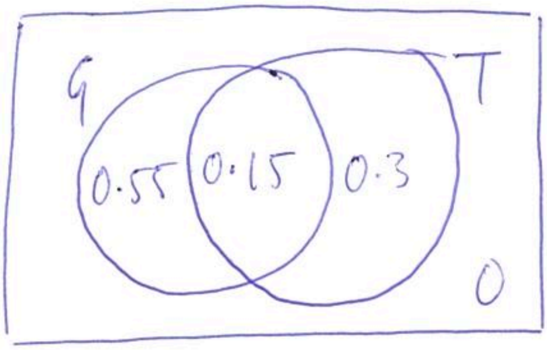
$\therefore$ S AND T NOT MUTUALLY EXCLUSIVE

Example 3

Andrew is a member of a tennis club that has a number of facilities including tennis courts and a gym. On any visit Andrew uses the tennis courts, the gym or both but no other facilities. The probability he uses the tennis courts, $P(T)$, is 0.45 and the probability that he uses both is 0.15. Calculate the probability that:

a) he doesn't use the gym
b) he uses the gym but not the tennis courts
c) he uses either the gym or the tennis courts but not both.

Draw a table or Venn diagram!



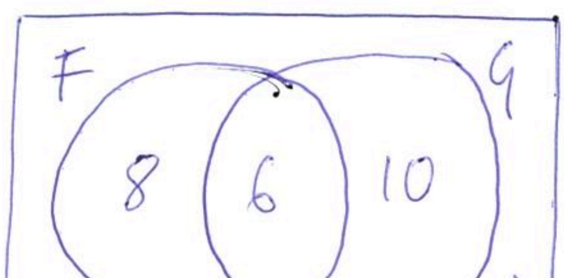
	G	G'	TOTAL
T	0.15	0.3	0.45
T'	0.55	0	0.55
TOTAL	0.7	0.3	1

a) 0.3
b) 0.55
c) 0.55 + 0.3 = 0.85

Example 4

There are 25 students in a certain tutor group at Philips College. There are 16 students in the tutor group studying German, 14 studying French and 6 students studying both French and German.

a) Find the probability that a randomly chosen student in the tutor group:
   i) studies French
   ii) studies French and German
   iii) studies French but not German
   iv) does not study French or German
b) Find the probability that a student studies French given he or she studies German.



a) i) $\frac{14}{25}$
ii) $\frac{6}{25}$
iii) $\frac{8}{25}$
iv) $\frac{1}{25}$

b) $\frac{6}{16}$

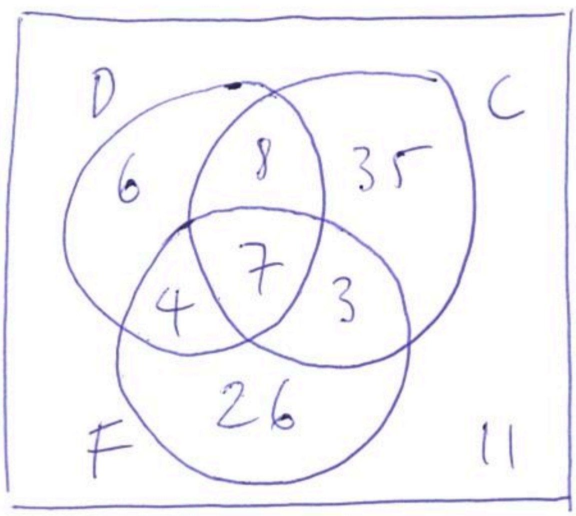
Example 5

A vet surveys 100 of her clients. She finds that

25 own dogs
53 own cats
40 own tropical fish
15 own dogs and cats
10 own cats and tropical fish
11 own dogs and tropical fish
7 own dogs, cats and tropical fish

a) Draw a Venn diagram to represent this information.
b) Find the probability that the client
   i) Owns dogs only
   ii) Does not own tropical fish
   iii) Does not own dogs, cats or tropical fish
c) Find the probability that the client owns exactly two types of the above pets given they own dogs.

a)



b) i) $\frac{6}{100}$

ii) $\frac{60}{100}$

iii) $\frac{11}{100}$

c) $\frac{4+8}{25} = \frac{12}{25}$

Multiplication Law
If A and B are any two events then:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B|A)$$
or
$$\mathrm{P}(A \cap B) = \mathrm{P}(B) \times \mathrm{P}(A|B)$$

where $\mathrm{P}(B|A)$ denotes the **conditional probability** of B given A. This is given in the formula booklet and can extended to include any number of events.

Independent Events

If the probability of A occurring is unaffected by B occurring or not then A and B are said to be **independent events**.

i.e.: A and B are independent events $\Leftrightarrow \mathrm{P}(A) = \mathrm{P}(A|B) \Leftrightarrow \mathrm{P}(B) = \mathrm{P}(B|A)$

Therefore, for independent events A and B the multiplication law simplifies to:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

This can be extended to include any number of independent events.
Always check whether events are independent before using the multiplication law.

Example 6

Sarah, Louise and Gary all have Maths on a Monday morning. The probabilities of them arriving late are, independently, 0.2, 0.15 and 0.3 respectively.

a) Calculate the probability that for a particular Monday:
   i) all three arrive late

$0.2 \times 0.15 \times 0.3 = 0.009$

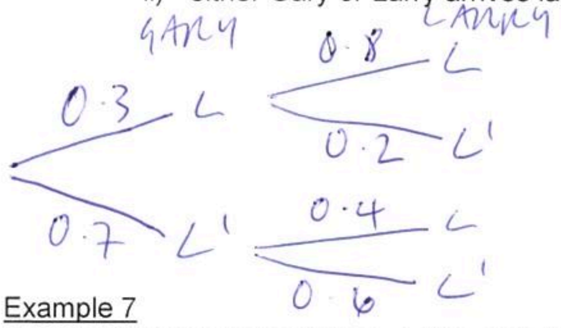
   ii) they all arrive on time

$0.8 \times 0.85 \times 0.7 = 0.476$

   iii) only Louise arrives late.

$0.8 \times 0.15 \times 0.7 = 0.084$

b) Gary's friend Larry is also in the same class. The probability that Larry arrives late is 0.8 when Gary arrives late and is 0.4 when Gary does not arrive late. Calculate the probability that:
   i) both Gary and Larry arrive late
   ii) either Gary or Larry arrives late but not both.

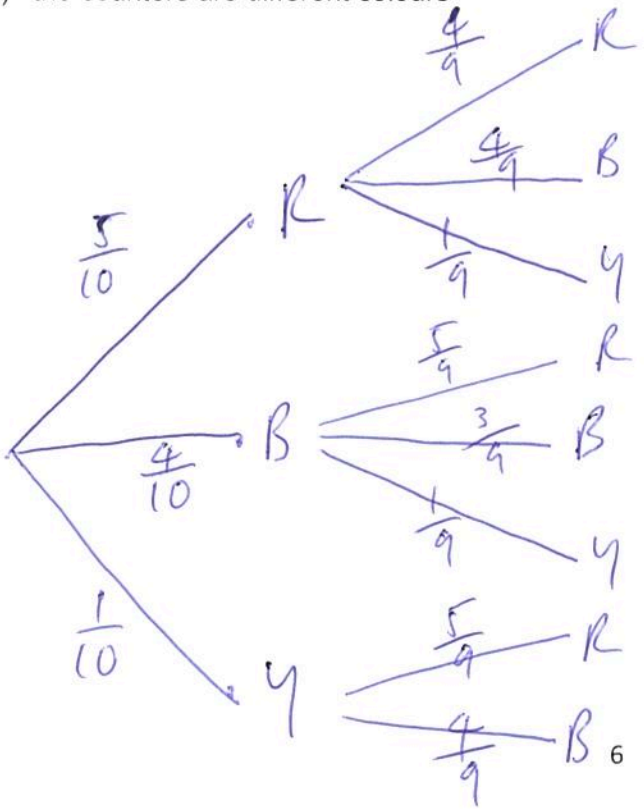


i) $0.3 \times 0.8 = 0.24$

ii) $0.7 \times 0.4 + 0.3 \times 0.2 = 0.34$

Example 7

A bag contains 5 red counters, 4 blue counters and 1 yellow counter. Two counters are selected at random without replacement. Calculate the probability that:
a) both counters are blue
b) the first counter is red and the second is yellow
c) exactly one of the counters is blue
d) the counters are different colours



a) $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$

b) $\frac{5}{10} \times \frac{1}{9} = \frac{1}{18}$

c) $\left(\frac{5}{10} \times \frac{4}{9}\right) + \left(\frac{4}{10} \times \frac{5}{9}\right) + \left(\frac{4}{10} \times \frac{1}{9}\right) + \left(\frac{1}{10} \times \frac{4}{9}\right) = \frac{8}{15}$

d) $1 - \left(\frac{5}{10} \times \frac{4}{9}\right) - \left(\frac{4}{10} \times \frac{3}{9}\right) = \frac{29}{45}$

Exam Style Questions

Example 8

A music shop contains a total 450 sale DVDs and Blu-rays. The genres of the DVDs and Blu-rays are shown in the table.

	Genre				
	Horror	Action	Comedy	Science Fiction	**Total**
DVD	44	67	42	80	233
Blu-ray	73	53	64	27	217
Total	117	120	106	107	450

a) A film is selected at random. Calculate the probability that the film was:

i) A DVD $\frac{233}{450}$

ii) Not action $\frac{330}{450}$

iii) Blu-ray or a comedy $\frac{259}{450}$

iv) a horror, given that it is a DVD $\frac{44}{233}$

v) a DVD, given that it is NOT an action film

$$\frac{166}{330}$$

b) Three DVDs are selected at random, without replacement. Calculate to three decimal places the probability that exactly one is a horror and exactly one is a comedy.

$$6 \times \frac{44}{233} \times \frac{42}{232} \times \frac{147}{231} = \frac{882}{6757} = 0.131$$

c) 7 discs are selected at random from the shop, with replacement. Calculate the probability that:

$X \sim B\left(7, \frac{233}{450}\right)$

i) All 7 are DVDs.

$P(X=7) = 0.00998$

ii) Exactly 3 are DVDs.

$P(X=3) = 0.263$

iii) At least 4 are DVDs.

$P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.46116... = 0.539$

d) State, giving a reason, whether or not the event of selecting a DVD is independent of the event of selecting a comedy.

$P(DVD) = \frac{233}{450} \approx 0.518$ $\quad P(DVD|C) = \frac{42}{106} \approx 0.396 \neq P(DVD)$

$\therefore$ SELECTING DVD IS NOT INDEPENDENT OF SELECTING COMEDY

Example 9

A college has 80 students in Year 12.

20 students study Biology
28 students study Chemistry
30 students study Physics
7 students study both Biology and Chemistry
11 students study both Chemistry and Physics
5 students study both Physics and Biology
3 students study all 3 of these subjects

(a) Draw a Venn diagram to represent this information.
(5)

A Year 12 student at the college is selected at random.

(b) Find the probability that the student studies Chemistry but not Biology or Physics.
(1)

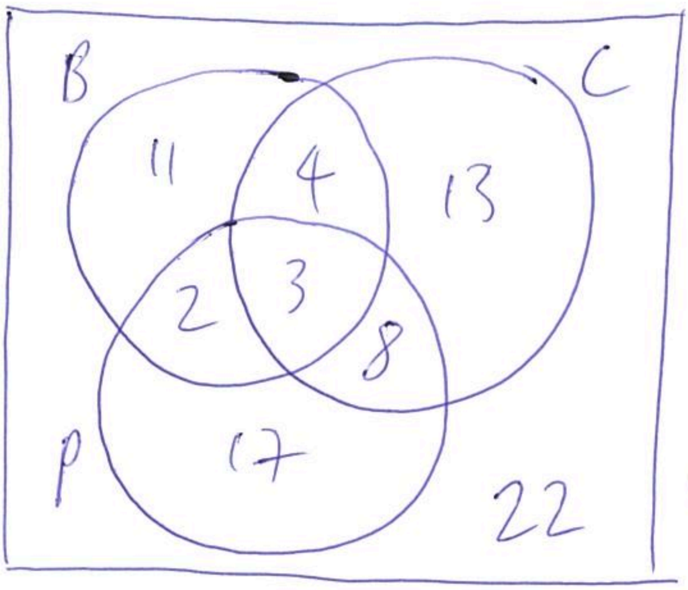
(c) Find the probability that the student studies Chemistry or Physics or both.
(2)

Given that the student studies Chemistry or Physics or both,

(d) find the probability that the student does not study Biology.
(2)

(e) Determine whether studying Biology and studying Chemistry are statistically independent.
(3)

a)



b) $\frac{13}{80}$

c) $\frac{47}{80}$

d) $\frac{17+8+13}{17+8+13+2+3+4} = \frac{38}{47}$

e) $P(B) = \frac{20}{80} = 0.25$

$P(B|C) = \frac{7}{28} = 0.25 = P(B)$

∴ STUDYING BIOLOGY AND CHEMISTRY ARE INDEPENDENT.

e) ALT:
COMPARE $P(C)$ WITH $P(C|B)$
OR $P(B \cap C)$ WITH $P(B) \times P(C)$

Example 10

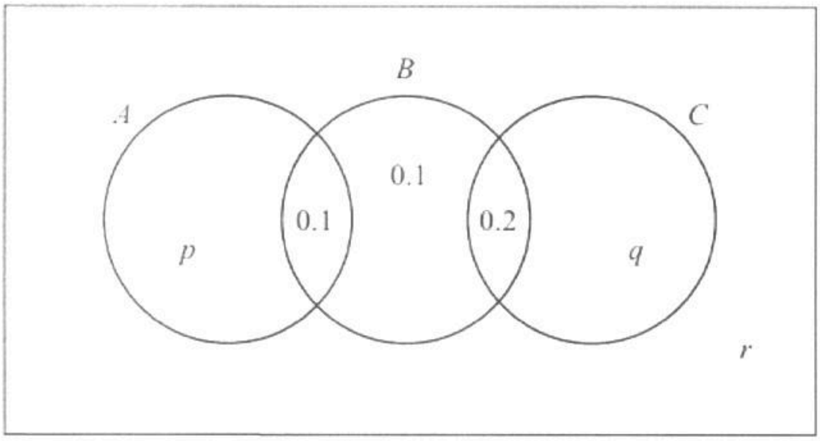


Figure 1

The Venn diagram in Figure 1 shows three events A, B and C and the probabilities associated with each region of B. The constants p, q and r each represent probabilities associated with the three separate regions outside B.

The events A and B are independent.

(a) Find the value of p.

(3)

Given that $P(B|C) = \frac{5}{11}$

(b) find the value of q and the value of r.

(4)

(c) Find $P(A \cup C|B)$.

(2)

a) $P(A \cap B) = P(A) \times P(B)$

$0.1 = P(A) \times 0.4 \Rightarrow P(A) = 0.25$

$p = 0.25 - 0.1 = \underline{0.15}$

b) $P(B \cap C) = P(C) \times P(B|C)$

$0.2 = P(C) \times \frac{5}{11} \Rightarrow P(C) = 0.44$

$q = 0.44 - 0.2 = \underline{0.24}$

$r = 1 - (0.15 + 0.1 + 0.1 + 0.2 + 0.24) = \underline{0.21}$

c) $P(A \cup C|B) = \frac{0.1 + 0.2}{0.4} = 0.75$

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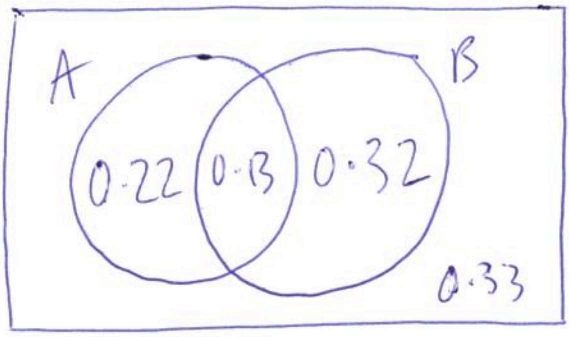
7. Given that

$$P(A) = 0.35\,,\quad P(B) = 0.45 \quad \text{and} \quad P(A \cap B) = 0.13$$

find



(a) $P(A \cup B)$ **(2)**

(b) $P(A' \mid B')$ **(2)**

The event C has $P(C) = 0.20$

The events A and C are mutually exclusive and the events B and C are independent.

(c) Find $P(B \cap C)$ **(2)**

(d) Draw a Venn diagram to illustrate the events A, B and C and the probabilities for each region. **(4)**

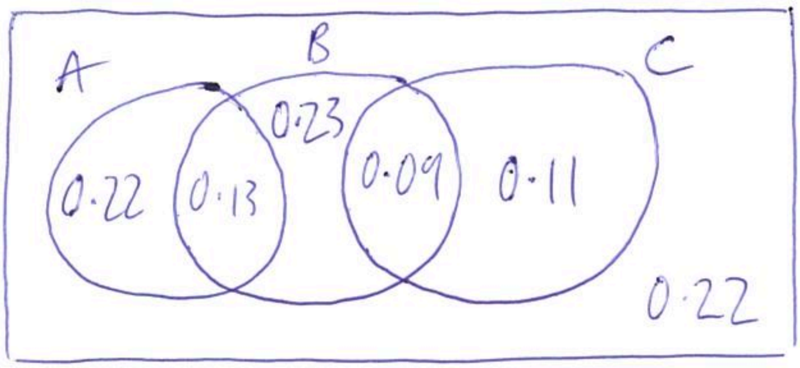
(e) Find $P([B \cup C]')$ **(2)**

a) $P(A \cup B) = 0.35 + 0.45 - 0.13 = 0.67$

b) $P(A' | B') = \dfrac{0.33}{0.33 + 0.22} = 0.6$

c) $P(B \cap C) = 0.45 \times 0.2 = 0.09$

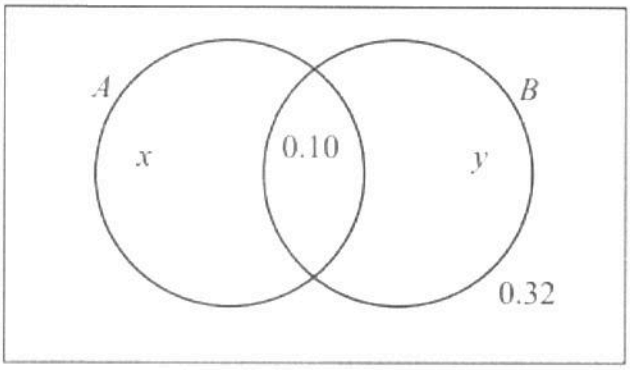
d)



e) $P([B \cup C]') = 0.22 + 0.22 = 0.44$

4. Events A and B are shown in the Venn diagram below

where x, y, 0.10 and 0.32 are probabilities.



(a) Find an expression in terms of x for

(i) $P(A)$

(ii) $P(B \mid A)$

(3)

(b) Find an expression in terms of x and y for $P(A \cup B)$

(1)

Given that $P(A) = 2P(B)$

(c) find the value of x and the value of y

(5)

a) i) $P(A) = x + 0.1$

ii) $P(B|A) = \dfrac{0.1}{x+0.1}$

b) $P(A \cup B) = x + 0.1 + y$

c) $x + 0.1 = 2(y + 0.1) \Rightarrow x - 2y = 0.1$

$x + 0.1 + y = 0.68 \Rightarrow x + y = 0.58$

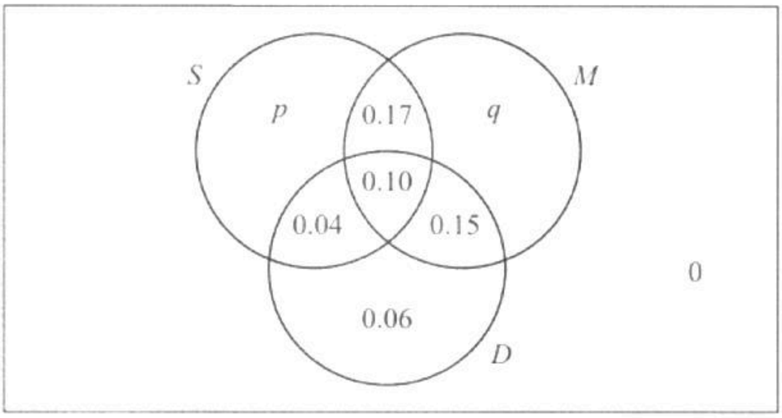
SOLVING SIMULTANEOUSLY: $x = 0.42$, $y = 0.16$

6. The Venn diagram below shows the probabilities of customers having various combinations of a starter, main course or dessert at Polly's restaurant.

S = the event a customer has a starter.

M = the event a customer has a main course.

D = the event a customer has a dessert.



Given that the events S and D are statistically independent

(a) find the value of p.

(4)

(b) Hence find the value of q.

(2)

(c) Find

(i) $\mathrm{P}(D \mid M \cap S)$

(ii) $\mathrm{P}(D \mid M \cap S')$

(4)

One evening 63 customers are booked into Polly's restaurant for an office party. Polly has asked for their starter and main course orders before they arrive.

Of these 63 customers

27 ordered a main course and a starter,

36 ordered a main course without a starter.

(d) Estimate the number of desserts that these 63 customers will have.

(2)

a) $P(S \cap D) = P(S) \times P(D)$

$0.14 = (p + 0.31) \times (0.35)$

$p = 0.09$

b) $q = 1 - (0.09 + 0.17 + 0.1 + 0.04 + 0.15 + 0.06)$

$= 0.39$

c) i) $P(D \mid M \cap S) = \dfrac{0.1}{0.1 + 0.17} = \dfrac{10}{27}$

ii) $P(D \mid M \cap S') = \dfrac{0.15}{0.15 + 0.39} = \dfrac{5}{18}$

d) $27 \times \dfrac{10}{27} + 36 \times \dfrac{5}{18} = 20$