Polynomials

Aims:

- To be able to factorise polynomials and solve equations.
- To be able to divided a polynomial by a linear factor
- To be able to use the factor theorem.
- To be able to sketch polynomial graphs.

A polynomial is an expression involving different powers of x. A quadratic is a polynomial of degree 2 (as this is the highest power in the expression) and a cubic is a polynomial of degree 3.

Polynomial Terminolgy

Leading Coefficient

Degree 5

Constant Term

$$y = \underbrace{2x^5}_{\text{Leading Term}} - 3x^4 + 5x^3 - 4x^2 + 7x - 8$$

Other Coefficient

5th Order Polynomial

Polynomial of 5th Degree

Algebraic Division

Example 1

Be careful when using long division you need a column for **each** power of x.

Divide $x^3 + 7x^2 - 3x - 30$ by $x - 2$

Method 1

$$
\begin{array}{r}
x^2 + 9x + 15 \\
x-2 \overline{\smash{)}\, x^3 + 7x^2 - 3x - 30} \\
-\underline{x^3 - 2x^2 \qquad\qquad} \\
9x^2 - 3x - 30 \\
-\underline{9x^2 - 18x \qquad} \\
15x - 30 \\
-\underline{15x - 30} \\
0
\end{array}
$$

Method 2

$$\frac{x^3+7x^2-3x-30}{x-2}$$

$$= \frac{x^2(x-2) + 9x(x-2) + 15(x-2)}{x-2}$$

$$= x^2 + 9x + 15$$

method 3 :

	x^2	$9x$	$+15$
x	x^3	$9x^2$	$15x$
-2	$-2x^2$	$-18x$	-30

Example 2

Example 2

Divide $4x^3 + 13x + 100$ by $x + 3$

Method 1

$$
\begin{array}{r l}
 & 4x^2 - 12x + 49 \\
x+3 \,\big) & 4x^3 + 0x^2 + 13x + 100 \\
 & -\underline{(4x^3 + 12x^2)} \\
 & \quad -12x^2 + 13x + 100 \\
 & \quad -\underline{(-12x^2 - 36x)} \\
 & \qquad 49x + 100 \\
 & \qquad -\underline{(49x + 147)} \\
 & \qquad\quad -47
\end{array}
$$

Remainder $\rightarrow -47$

Method 2

$$\frac{4x^3+13x+100}{x+3}$$

$$= \frac{4x^2(x+3) - 12x(x+3) + 49(x+3) - 47}{x+3}$$

$$= 4x^2 - 12x + 49 - \frac{47}{x+3}$$

The Factor Theorem

We can use the Remainder Theorem to find the remainder without using long division. The Factor Theorem is just a special case of the Remainder Theorem where if the remainder is 0 we have a factor.

> If $(x - a)$ is a factor of the polynomial $p(x)$ then $p(a) = 0$.
>
> If $(bx - a)$ is a factor of the polynomial $p(x)$ then $p\left(\frac{a}{b}\right) = 0$.

<u>Example 3</u>

Show that $x - 2$ is a factor of $p(x) = x^3 + x^2 - 7x + 2$

Let $x = 2$ $\quad p(2) = 8 + 4 - 14 + 2 = 0$

$\therefore$ By the factor theorem, $(x-2)$ is a factor of $p(x)$

> You **must** include a statement to get full marks in the exam.

Example 4

Given that $(x-3)$ is a factor of $p(x) = x^3 + ax^2 + 3x + 18$ find the value of a.

$$p(3) = 0 \Rightarrow 3^3 + a(3)^2 + 3(3) + 18 = 0$$
$$27 + 9a + 9 + 18 = 0$$
$$9a = -54$$
$$a = -6$$

Example 5

The polynomial $g(x)$is defined by $g(x) = 3x^3 + cx^2 - 15x + d$. Given that $(3x+1)$ and $(x-4)$ are both factors of $g(x)$. Find the value of c and d.

$$p(-\tfrac{1}{3}) = 0 \Rightarrow 3(-\tfrac{1}{3})^3 + c(-\tfrac{1}{3})^2 - 15(-\tfrac{1}{3}) + d = 0$$
$$-\tfrac{1}{9} + \tfrac{c}{9} + 5 + d = 0$$
$$-1 + c + 45 + 9d = 0$$
$$c + 9d = -44$$

$$p(4) = 0 \Rightarrow 3(64) + 16c - 60 + d = 0$$
$$16c + d = -132$$

$$c = -8, \quad d = -4$$

Example 6

The polynomial $p(x)$ is given by $p(x) = x^3 - 13x - 12$

a) Use the Factor Theorem to show that $(x-4)$ is a factor of $p(x)$.

b) Express $p(x)$ as a product of three linear factors.

The polynomial $p(x)$ is given by $p(x) = x^3 - 13x - 12$

a) Use the Factor Theorem to show that $(x-4)$ is a factor of $p(x)$.

$p(4) = 4^3 - 13(4) - 12 = 64 - 52 - 12 = 0 \quad \therefore (x-4)$ is a factor of $p(x)$.

b) Express $p(x)$ as a product of three linear factors.

$$
\begin{array}{r}
x^2 + 4x + 3 \\
x-4 \,\big)\, \overline{x^3 + 0x^2 - 13x - 12} \\
-\; x^3 - 4x^2 \qquad\qquad \\
\hline
4x^2 - 13x - 12 \\
-\; 4x^2 - 16x \qquad \\
\hline
3x - 12 \\
-\; 3x - 12 \\
\hline
0
\end{array}
$$

$p(x) = (x-4)(x^2+4x+3)$

$= (x-4)(x+3)(x+1)$

Example 9

Example 7

The polynomial $p(x)$ is given by $p(x) = 9x^3 - 55x^2 + 78x - 8$

a) Find the value of $p(4)$.

b) Hence express $p(x)$ as the product of three linear factors.

c) Hence solve the equation $h(y) = 9y^6 - 55y^4 + 78y^2 - 8$.

a) $p(4) = 0$

b) By the factor theorem $(x-4)$ is a factor.

$\therefore\ p(x) = (x-4)(9x^2 - 19x + 2)$

$= (x-4)(9x-1)(x-2)$

c) $y^2 = x \Rightarrow p(y^2) = 9(y^2)^3 - 55(y^2)^2 + 78(y^2) - 8$

$p(y^2) = 9y^3 - 55y^4 + 78y^2 - 8$

$= h(y)$

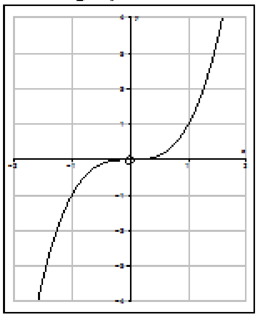
$\Rightarrow h(y) = (y^2-4)(9y^2-1)(y^2-2)$

$= (y+2)(y-2)(3y-1)(3y+1)(y-\sqrt{2})(y+\sqrt{2})$

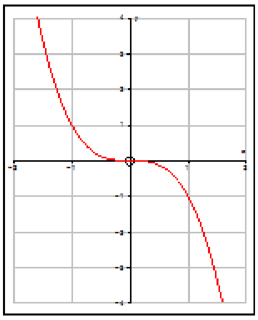
Sketching Cubic Graphs

You will have seen the graphs:

$y = x^3$

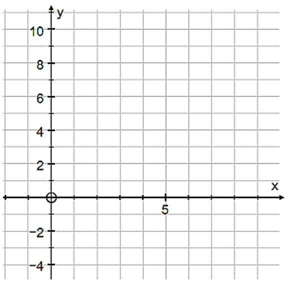


$y = -x^3$

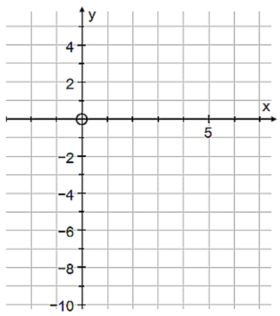


But most cubic graphs are shaped like this:

If the cubic has $+x^3$ the graph is this shape



If the cubic has $-x^3$ the graph is this shape



Example 8

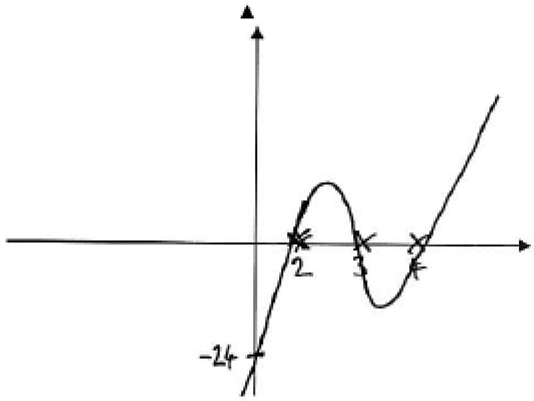
Sketch the graphs below, showing ALL the points where they cross the coordinate axes

a) $y = x^3 - 9x^2 + 26x - 24$
or
$y = (x-2)(x-3)(x-4)$

x Intercept $x = 2, 3, 4$

y Intercept $y = -24$

+ve shape

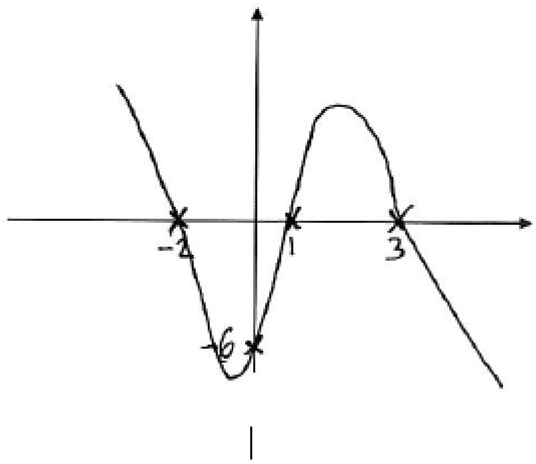


b) $y = -x^3 + 2x^2 + 11x - 6$
or
$y = (1-x)(x-3)(x+2)$

$y=0 \Rightarrow x = 1, 3, -2$

$x=0 \Rightarrow y = -6$

-ve shape



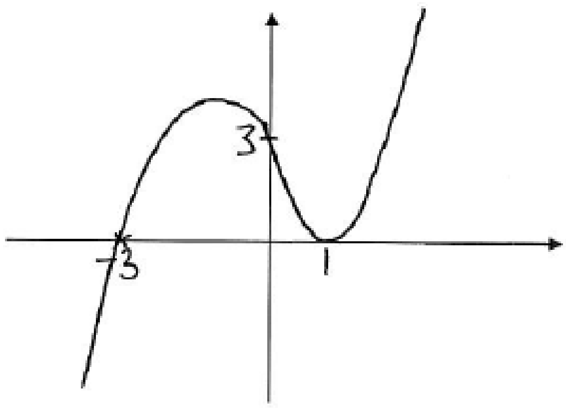
c) $y = x^3 + x^2 - 5x + 3$
(Hint! Factorise first...)

$y = (x-1)(x-1)(x+3)$

$y=0 \Rightarrow x = 1$ (twice!), -3

$x=0 \Rightarrow y = 3$

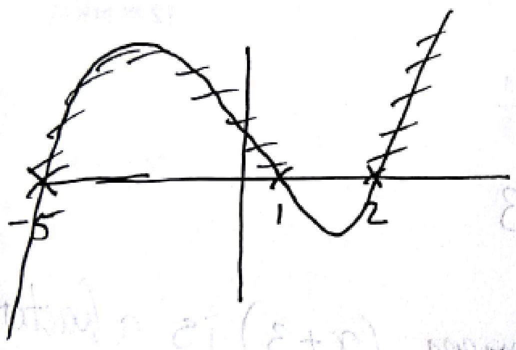
+ve shape.



Example 9

Given that the cubic has a factor of $x - 2$, solve the inequality $x^3 + 2x^2 - 13x + 10 > 0$

$$x^3 + 2x^2 - 13x + 10 = (x-2)(x^2+4x-5)$$
$$= (x-2)(x+5)(x-1)$$



$x > 2 \quad -5 < x < 1.$

Exam Question

The polynomial $f(x)$ is given by $f(x) = x^3 + 4x - 5$.

(i) Use the Factor Theorem to show that $x - 1$ is a factor of $f(x)$. *(2 marks)*

(ii) Express $f(x)$ in the form $(x-1)(x^2 + px + q)$, where p and q are integers. *(2 marks)*

(iii) Hence show that the equation $f(x) = 0$ has exactly one real root and state its value. *(3 marks)*

i) $f(1) = 1 + 4 - 5 = 0 \therefore (x-1)$ is a factor.

ii) $x^3 + 4x - 5 = (x-1)(x^2 + x + 5)$

iii) $x = 1$ is one root - need to show $x^2 + x + 5$ has no real roots: $a=1, b=1, c=5$

$$b^2 - 4ac = 1 - 4(1)(5) < 0$$

$\therefore$ quadratic factor has no real roots $\therefore x = 1$ is the only real root.

Sketching Cubic Graphs

$$p(x) = 2x^3 + 7x^2 + 2x - 3$$

(a) Use the factor theorem to prove that $x + 3$ is a factor of $p(x)$

[2 marks]

(b) Simplify the expression $\dfrac{2x^3 + 7x^2 + 2x - 3}{4x^2 - 1}$, $x \neq \pm\dfrac{1}{2}$

[4 marks]

a) $p(-3) = 2(-3)^3 + 7(-3)^2 + 2(-3) - 3$

$= -54 + 63 - 6 - 3$

$= 0$ $\therefore$ by the factor theorem $(x+3)$ is a factor.

b) $\dfrac{2x^3+7x^2+2x-3}{4x^2-1} = \dfrac{(x+3)(2x^2+x-1)}{(2x-1)(2x+1)}$

$= \dfrac{(x+3)\cancel{(2x-1)}(x+1)}{\cancel{(2x-1)}(2x+1)}$

$= \dfrac{(x+3)(x+1)}{(2x+1)}$

(a) The polynomial $f(x)$ is defined by $f(x) = 6x^3 - 11x^2 + 2x + 8$.

(i) Use the Factor Theorem to show that $(3x+2)$ is a factor of $f(x)$.

[2 marks]

(ii) Show that $f(x)$ has no other linear factors.

[4 marks]

(b) The polynomial $g(x)$ is defined by $g(x) = f(x) - (6x^2 - 2x - 4)$.

Given that $(3x+2)$ is a factor of $g(x)$, express $g(x)$ as a product of three linear factors.

[2 marks]

a) i) $f\left(-\frac{2}{3}\right) = 6\left(\frac{-2}{3}\right)^3 - 11\left(\frac{-2}{3}\right)^2 + 2\left(\frac{-2}{3}\right) + 8$

$= 0$ $\therefore$ $(3x+2)$ is a factor of $f(x)$ by the factor theorem.

ii) $6x^3 - 11x^2 + 2x + 8 = (3x+2)(2x^2 - 5x + 4)$

"$b^2 - 4ac$" $= (-5)^2 - 4(2)(4)$

$= 25 - 32 < 0$ $\therefore$ No real sol^ns

$\Rightarrow$ $(3x+2)$ is the only linear factor.

b) $g(x) = (3x+2)(2x^2 - 5x + 4) - (6x^2 - 2x - 4)$

$= (3x+2)(2x^2 - 5x + 4) - (3x+2)(2x - 2)$

$= (3x+2)(2x^2 - 5x + 4 - 2x + 2)$

$= (3x+2)(2x^2 - 7x + 6)$

$= (3x+2)(2x - 3)(x - 2)$

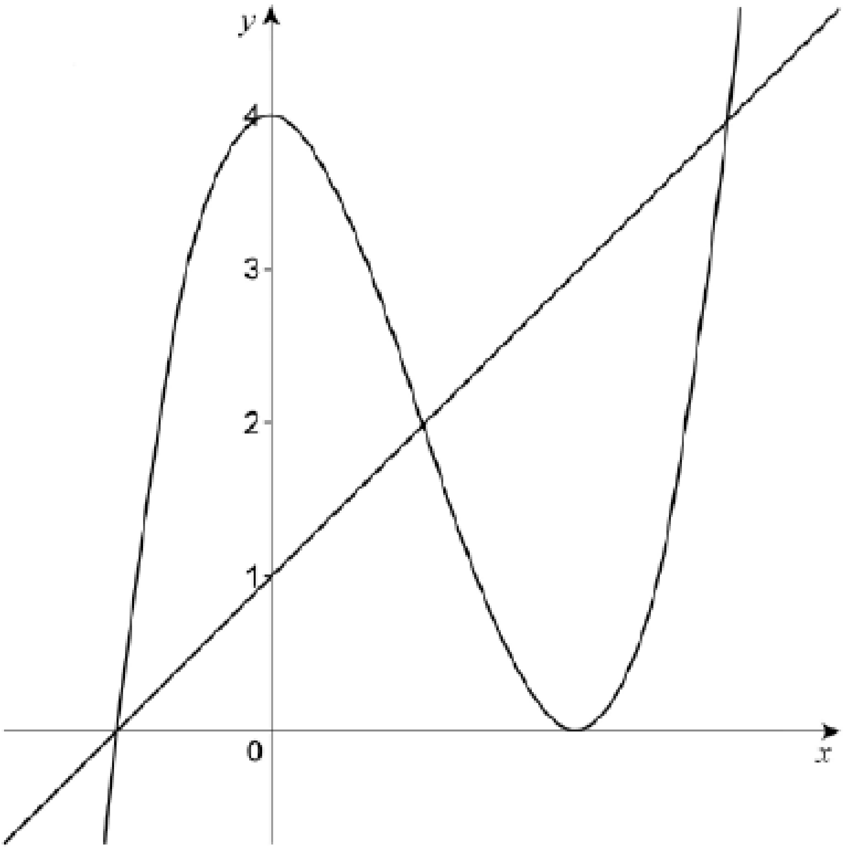
c) Ans = $1 - 2x^{-1}$

c) $h(x) = \dfrac{(3x+2)(2x-3)(x-2)}{6x^3 - 5x^2 - 6x} = \dfrac{(3x+2)(2x-3)(x-2)}{x(6x^2 - 5x - 6)}$

$= \dfrac{\cancel{(3x+2)}\cancel{(2x-3)}(x-2)}{x\cancel{(3x+2)}\cancel{(2x-3)}} = \dfrac{x-2}{x}$

I know, it's messy

The graphs of $y = x^3 - ax^2 + b$ and $y = cx + d$ are shown on the diagram.



One of the points of intersection of the two graphs is (3, 4).

Find the values of a, b, c and d.

Find the values of a, b, c and d.

$y = x^3 - ax^2 + b \quad y = cx + d$

$\Rightarrow \quad x^3 - ax^2 + b = cx + d$

$x^3 - ax^2 - cx + b - d = 0$ (###)

$y = cx + d$ from graph, $d = 1$; $4 = 3c + 1$

$c = 1$

$y = x^3 - ax^2 + b$ from graph $b = 4$

$\therefore$ (###) $\Rightarrow \quad x^3 - ax^2 - x + 3 = 0$

$x = 3$ satisfies $\Rightarrow 27 - 9a - 3 + 3 = 0$

$27 = 9a$

$a = 3$