Pure Sector 3: Polynomials

Aims:

- To be able to factorise polynomials and solve equations.
- To be able to divided a polynomial by a linear factor
- To be able to use the factor theorem.
- To be able to sketch polynomial graphs.

A polynomial is an expression involving different powers of x. A quadratic is a polynomial of degree 2 (as this is the highest power in the expression) and a cubic is a polynomial of degree 3.

Algebraic Division

Be careful when using long division you need a column for **each** power of x.

Example 1

Divide $x^3 + 7x^2 - 3x - 30$ by $x - 2$

Method 1	Method 2

Example 2

Divide $4x^3 + 2x^2 - 16x + 3$ by $2x - 5$

Method 1	Method 2

The Factor Theorem

We can use the Remainder Theorem to find the remainder without using long division. The Factor Theorem is just a special case of the Remainder Theorem where if the remainder is 0 we have a factor.

If $(x-a)$ is a factor of the polynomial $p(x)$ then $p(a)=0$.

If $(bx-a)$ is a factor of the polynomial $p(x)$ then $p\left(\frac{a}{b}\right)=0$.

Example 3

Show that $x-2$ is a factor of $p(x)=x^3+x^2-7x+2$

You **must** include a statement to get full marks in the exam.

Example 4

Given that $(x-3)$ is a factor of $p(x)=x^3+ax^2+3x+18$ find the value of a.

Example 5

The polynomial $g(x)$is defined by $g(x)=3x^3+cx^2-15x+d$. Given that $(3x+1)$ and $(x-4)$ are both factors of $g(x)$. Find the value of c and d.

Factorising Cubics

Example 6

The polynomial $p(x)$is given by $p(x) = x^3 - 13x - 12$

a) Use the Factor Theorem to show that $(x - 4)$ is a factor of $p(x)$.
b) Express $p(x)$ as a product of three linear factors.

Example 7

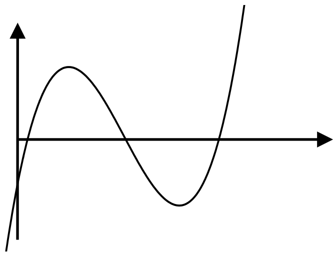
The polynomial $p(x)$ is given by $p(x) = 9x^3 - 55x^2 + 78x - 8$

a) Find the value of $p(4)$.
b) Hence express $p(x)$ as the product of three linear factors.
c) Hence factorise the equation $h(y) = 9y^6 - 55y^4 + 78y^2 - 8$.

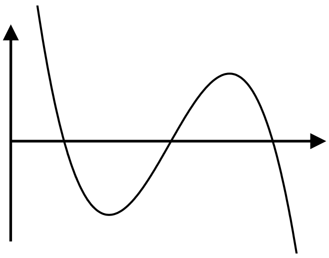
Sketching Cubic Graphs

Most cubic graphs are shaped like this:

If the cubic has $+x^3$ the graph is this shape



If the cubic has $-x^3$ the graph is this shape



Example 8

Sketch the graphs below, showing ALL the points where they cross the coordinate axes

a) $y = x^3 - 9x^2 + 26x - 24$
or
$y = (x-2)(x-3)(x-4)$

b) $y = -x^3 + 2x^2 + 11x - 6$
or
$y = (1-x)(x-3)(x+2)$

c) $y = x^3 + x^2 - 5x + 3$
(Hint! Factorise first…)

Example 9

Given that the cubic has a factor of $x - 2$, solve the inequality $x^3 + 2x^2 - 13x + 10 > 0$

Exam Questions

The polynomial $f(x)$ is given by $f(x) = x^3 + 4x - 5$.

(i) Use the Factor Theorem to show that $x - 1$ is a factor of $f(x)$. *(2 marks)*

(ii) Express $f(x)$ in the form $(x - 1)(x^2 + px + q)$, where p and q are integers. *(2 marks)*

(iii) Hence show that the equation $f(x) = 0$ has exactly one real root and state its value. *(3 marks)*

$$p(x) = 2x^3 + 7x^2 + 2x - 3$$

(a) Use the factor theorem to prove that $x + 3$ is a factor of $p(x)$

[2 marks]

(b) Simplify the expression $\dfrac{2x^3 + 7x^2 + 2x - 3}{4x^2 - 1}$, $x \neq \pm \dfrac{1}{2}$

[4 marks]

(a) The polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = 6x^3 - 11x^2 + 2x + 8$.

(i) Use the Factor Theorem to show that $(3x+2)$ is a factor of $\mathrm{f}(x)$.

[2 marks]

(ii) Show that $\mathrm{f}(x)$ has no other linear factors.

[4 marks]

(b) The polynomial $\mathrm{g}(x)$ is defined by $\mathrm{g}(x) = \mathrm{f}(x) - (6x^2 - 2x - 4)$.

Given that $(3x+2)$ is a factor of $\mathrm{g}(x)$, express $\mathrm{g}(x)$ as a product of three linear factors.

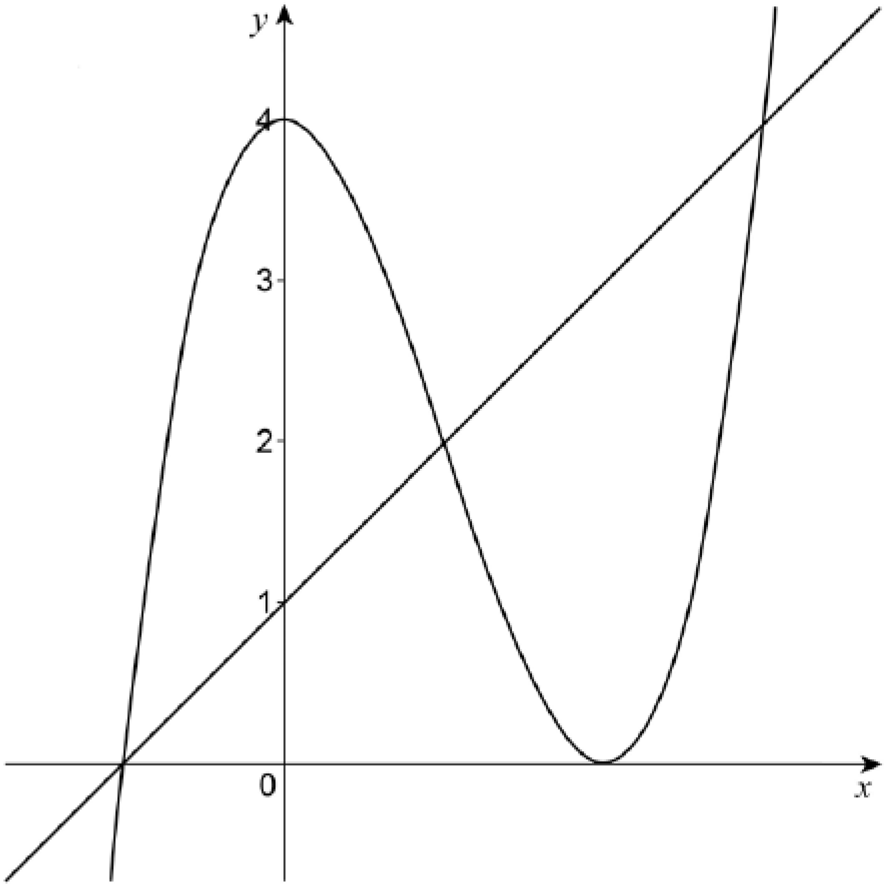
[2 marks]

(c) The function h is defined by $\mathrm{h}(x) = \dfrac{\mathrm{g}(x)}{6x^3 - 5x^2 - 6x}$.

Show that $\mathrm{h}(x)$ can be simplified to the form $p + qx^n$ where p, q and n are integers.

[2 marks]

The graphs of $y = x^3 - ax^2 + b$ and $y = cx + d$ are shown on the diagram.



One of the points of intersection of the two graphs is (3, 4).

Find the values of a, b, c and d.