Pure Sector 3: Partial Fractions

Aims:

- To be able to simplify algebraic fractions.
- To be able to express algebraic fractions in Partial Fraction form.
- Utilise Partial Fractions to solve problems.

Simplifying Algebraic Fractions

Example 1

Simplify:

$$\frac{2x^2 - 8}{x^2 + 3x + 2}$$

$$= \frac{(2x-4)(x+2)}{(x+1)(x+2)}$$

$$= \frac{2x-4}{x+1}$$

Example 2

Simplify:

$$\frac{36 - 4x^2}{x^2 + x - 12}$$

$$= \frac{(-4x-12)(x-3)}{(x-3)(x+4)}$$

$$= \frac{-4x-12}{x+4}$$

$$= \frac{-4(x+3)}{x+4}$$

Improper Fractions

$Ax^2 + Bx + C$ is the **quotient**

D is the **remainder**

Example 3

Write $\frac{2x^3-3x^2+x-3}{x-2}$ in the form $\boldsymbol{Ax^2 + Bx + C + \frac{D}{x-2}}$

$$\frac{2x^3-3x^2+x-3}{x-2} = Ax^2 + Bx + C + \frac{D}{x-2}$$

$$2x^3-3x^2+x-3 = A(x^3-2x^2) + B(x^2-2x) + C(x-2) + D$$

$$= Ax^3 + (B-2A)x^2 + (C-2B)x + (D-2C)$$

$A=2$

$-3=B-2(2)$

$B=1$

$1=C-2(1)$

$C=3$

$-3=D-2(6)$

$D=9$

$$= 2x^2 + x + 3 + \frac{9}{x-2}$$

Example 4

Write $\frac{x^3-3x^2+1}{x^2-x-1}$ in the form $\boldsymbol{Ax + B + \frac{Cx+D}{x^2-x-1}}$

$$\frac{x^3-3x^2+1}{x^2-x-1} = Ax + B + \frac{Cx+D}{x^2-x-1}$$

$$x^3-3x^2+1 = A(x^3-x^2-x) + B(x^2-x-1) + Cx + D$$

$$= Ax^3 + (B-A)x^2 + (C-B-A)x + (D-B)$$

$A=1$

$B-1=-3$

$B=-2$

$0=C-(-2)-(-1)$

$0=C+3$

$C=-3$

$1=D-(-2)$

$1=D+2$

$D=-1$

$$= x-2 + \frac{-3x-1}{x^2-x-1}$$

Partial Fractions

Example 5

Write $\frac{3}{(2x+1)(x+2)}$ in the form $\frac{A}{(2x+1)} + \frac{B}{(x+2)}$

$$\frac{3}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$3 = A(x+2) + B(2x+1)$$

$x = -2$

$$3 = B(-3)$$

$$B = -1$$

$x = -\frac{1}{2}$

$$3 = A\left(2 - \tfrac{1}{2}\right)$$

$$3 = A\left(\tfrac{3}{2}\right)$$

$$A = 2$$

$$\frac{3}{(2x+1)(x+2)} = \frac{2}{2x+1} - \frac{1}{x+2}$$

Example 6

Express $\frac{3x+1}{2x^2-4x}$ in partial fractions.

$$\frac{3x+1}{x(2x-4)} = \frac{A}{x} + \frac{B}{2x-4}$$

$$3x+1 = A(2x-4) + Bx$$

$x = 0$

$$1 = -4A$$

$$A = -\frac{1}{4}$$

$x = 2$

$$7 = 2B$$

$$B = \frac{7}{2}$$

$$\frac{3x+1}{2x^2-4x} = \frac{-\frac{1}{4}}{x} + \frac{\frac{7}{2}}{2x-4}$$

$$= -\frac{1}{4x} + \frac{7}{4x-8}$$

Example 7

Express $\frac{x}{(2x+1)^2(x-1)}$ in the form $\frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(x-1)}$, where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are constants.

$$\frac{x}{(2x+1)^2(x-1)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$

$$x = A(2x+1)(x-1) + B(x-1) + C(2x+1)^2$$

$x = 1$
$1 = C(3)^2$
$C = \frac{1}{9}$

$x = -\frac{1}{2}$
$-\frac{1}{2} = B(-\frac{1}{2} - 1)$
$B(-\frac{3}{2}) = -\frac{1}{2}$
$B = \frac{1}{3}$

$x = 0$
$0 = A(1)(-1) + \frac{1}{3}(-1) + \frac{1}{9}(1)$
$0 = -A - \frac{1}{3} + \frac{1}{9}$
$A = \frac{1}{9} - \frac{1}{3}$
$A = -\frac{2}{9}$

$$= -\frac{2}{9(2x+1)} + \frac{1}{3(2x+1)^2} + \frac{1}{9(x-1)}$$

Example 8

Express in partial fractions

$$\frac{5x+3}{(2x+1)(x+1)^2}$$

$$\frac{5x+3}{(2x+1)(x+1)^2} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$5x+3 = A(x+1)^2 + B(2x+1)(x+1) + C(2x+1)$$

$x = -1$
$-2 = C(-1)$
$C = 2$

$x = -\frac{1}{2}$
$\frac{1}{2} = A(\frac{1}{4})$
$A = 2$

$x = \frac{-3}{5}$
$0 = 2(\frac{2}{5})^2 + B(\frac{1}{5})(\frac{2}{5}) + 2(\frac{1}{5})$
$0 = \frac{8}{25} + \frac{2}{5} + B(\frac{2}{25})$
$-\frac{18}{25} = B(\frac{2}{25})$
$B = 9$

$$= \frac{2}{2x+1} - \frac{9}{x+1} + \frac{2}{(x+1)^2}$$

General Forms:

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(ex+f)}$$

$$\frac{px+q}{(ax+b)(cx+d)^2} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Applications of Partial Fractions

<u>Example 9</u>

Show that $\int_1^2 \frac{2x+2}{2x+1}\,\mathrm{d}x = 1 + \frac{1}{2}\ln\frac{5}{3}$.

$$\frac{2x+2}{2x+1} = A + \frac{B}{2x+1}$$

$$2x+2 = A(2x+1) + B$$

$x = -\frac{1}{2}$

$$-1+2 = B$$
$$B = 1$$

$x = 0$

$$2 = A(1) + 1$$
$$A = 1$$

$$\frac{2x+2}{2x+1} = 1 + \frac{1}{2x+1}$$

$$\int_1^2 1 + \frac{1}{2x+1}\,dx = \int_1^2 1\,dx + \frac{1}{2}\int_1^2 \frac{2}{2x+1}\,dx$$

$$= \left[x + \tfrac{1}{2}(\ln 2x+1)\right]_1^2$$

$$= 2 + \tfrac{1}{2}\ln 5 - (1 + \tfrac{1}{2}\ln 3)$$

$$= 2 - 1 + \tfrac{1}{2}\ln 5 - \tfrac{1}{2}\ln 3$$

$$= 1 + \tfrac{1}{2}\ln\tfrac{5}{3}$$

<u>Example 10</u>

Expand $\frac{5-x}{(1+x)(1-2x)}$ up to the term in x^3.

$$\frac{5-x}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}$$

$$5-x = A(1-2x) + B(1+x)$$

$x = -1$

$$6 = 3A$$
$$A = 2$$

$x = \frac{1}{2}$

$$\frac{9}{2} = \frac{3}{2}B$$
$$B = 3$$

$$\frac{5-x}{(1+x)(1-2x)} = \frac{2}{1+x} + \frac{3}{1-2x}$$

$$= 2(1+x)^{-1} + 3(1-2x)^{-1}$$

$$= 2\left(1 - x + \frac{-1(-2)}{1.2}x^2 + \frac{-1(-2)(-3)}{1.2.3}x^3\right) + 3\left(1-(-2x) + \frac{-1(-2)}{1.2}(-2x)^2 + \frac{-1(-2)(-3)}{1.2.3}(-2x)^3\right)$$

$$= 2(1 - x + x^2 - x^3) + 3(1 + 2x + 4x^2 + 8x^3)$$

$$= 2 - 2x + 2x^2 - 2x^3 + 3 + 6x + 12x^2 + 24x^3$$

$$= 5 + 4x + 14x^2 + 22x^3$$

Example 11

Find the value of $\int_3^4 \frac{x+4}{(x+1)(x-2)}\,\mathbf{d}x$ leaving your answer in the form $p\ln 4 - q\ln 5$ where p and q are integers to be found.

$$\frac{x+4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$x+4 = A(x-2) + B(x+1)$$

$x=2$

$6=3B$

$B=2$

$x=-1$

$3=-3A$

$A=-1$

$$\frac{x+4}{(x+1)(x-2)} = -\frac{1}{x+1} + \frac{2}{x-2}$$

$$\int_3^4 -\frac{1}{x+1} + \frac{2}{x-2}\,dx$$

$$= -\int_3^4 \frac{1}{x+1}\,dx + 2\int_3^4 \frac{1}{x-2}\,dx$$

$$= -\left[\ln(x+1)\right]_3^4 + 2\left[\ln(x-2)\right]_3^4$$

$$= -(\ln 5 - \ln 4) + 2(\ln 2 - \ln 1)$$

$$= \ln 4 - \ln 5 + 2\ln 2 - 0$$

$$= \ln 4 - \ln 5 + \ln 4$$

$$= 2\ln 4 - \ln 5$$

Exam Questions

It is given that $\mathrm{f}(x) = \dfrac{19x-2}{(5-x)(1+6x)}$ can be expressed as $\dfrac{A}{5-x} + \dfrac{B}{1+6x}$, where A and B are integers.

(a) Find the values of A and B.

[3 marks]

(b) Hence show that $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x = k\ln 5$, where k is a rational number.

[6 marks]

a) $$\frac{19x-2}{(5-x)(1+6x)} = \frac{A}{5-x} + \frac{B}{1+6x}$$

$$19x-2 = A(1+6x) + B(5-x)$$

$x=5$

$95-2 = A(1+30)$

$93 = A(31)$

$A=3$

$x = -\frac{1}{6}$

$-\frac{19}{6} - 2 = B\left(5+\frac{1}{6}\right)$

$-\frac{31}{6} = B\left(\frac{31}{6}\right)$

$B=-1$

$$= \frac{3}{5-x} - \frac{1}{1+6x}$$

b) $$\int_0^4 \frac{3}{5-x} - \frac{1}{1+6x}\,dx$$

$$= -3\int_0^4 \frac{-1}{5-x}\,dx - \frac{1}{6}\int \frac{6}{1+6x}\,dx$$

$$= -3\left[\ln(5-x)\right]_0^4 - \frac{1}{6}\left[\ln(1+6x)\right]_0^4$$

$$= -3(\ln 1 - \ln 5) - \frac{1}{6}(\ln 25 - \ln 1)$$

$$= 3\ln 5 - \frac{1}{6}\ln 5^2$$

$$= 3\ln 5 - \frac{1}{3}\ln 5$$

$$= \frac{8}{3}\ln 5$$

(a) Express $\dfrac{19x-3}{(1+2x)(3-4x)}$ in the form $\dfrac{A}{1+2x}+\dfrac{B}{3-4x}$.

[3 marks]

(b) (i) Find the binomial expansion of $\dfrac{19x-3}{(1+2x)(3-4x)}$ up to and including the term in x^2.

[7 marks]

(ii) State the range of values of x for which this expansion is valid.

[1 mark]

a) $\dfrac{19x-3}{(1+2x)(3-4x)} = \dfrac{A}{1+2x} + \dfrac{B}{3-4x}$

$19x-3 = A(3-4x) + B(1+2x)$

$x = \frac{3}{4}$

$\frac{57}{4} - 3 = B(\frac{5}{2})$

$\frac{5}{2}B = \frac{45}{4}$

$B = \frac{9}{2}$

$x = -\frac{1}{2}$

$-\frac{19}{2} - 3 = A(3+2)$

$-\frac{25}{2} = 5A$

$A = -\frac{5}{2}$

$= -\dfrac{5}{2(1+2x)} + \dfrac{9}{2(3-4x)}$

b) i)

$= -\frac{5}{2}(1+2x)^{-1} + \frac{9}{2}(3-4x)^{-1}$

$= -\frac{5}{2}(1+2x)^{-1} + \frac{9}{2}(3)^{-1}(1-\frac{4}{3}x)^{-1}$

$= -\frac{5}{2}\left(1+(-1)(2x) + \frac{-1(-2)}{1 \cdot 2}(2x)^2\right) + \frac{3}{2}\left(1+(-1)(-\frac{4}{3}x) + \frac{(-1)(-2)}{1\cdot 2}(-\frac{4}{3}x)^2\right)$

$= -\frac{5}{2}(1-2x+4x^2) + \frac{3}{2}(1+\frac{4}{3}x+\frac{16}{9}x^2)$

$= -\frac{5}{2} + 5x - 10x^2 + \frac{3}{2} + \frac{12}{6}x + \frac{48}{18}x^2$

$= -1 + 7x - \frac{22}{3}x^2$

ii) For $(1+2x)^{-1}$,

$|2x| < 1$

$|x| \leq \frac{1}{2}$

for $(1-\frac{4}{3}x)^{-1}$

$|\frac{4}{3}x| < 1$

$|x| < \frac{3}{4}$

$\frac{3}{4} > \frac{1}{2}$

$\therefore |x| < \frac{1}{2}$ is valid

(a) Given that $\dfrac{4x^3-2x^2+16x-3}{2x^2-x+2}$ can be expressed as $Ax + \dfrac{B(4x-1)}{2x^2-x+2}$, find the values of the constants A and B.

[3 marks]

(b) The gradient of a curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^3-2x^2+16x-3}{2x^2-x+2}$$

The point $(-1, 2)$ lies on the curve. Find the equation of the curve.

[4 marks]

a) $\dfrac{4x^3-2x^2+16x-3}{2x^2-x+2} = Ax + \dfrac{B(4x-1)}{2x^2-x+2}$

$4x^3-2x^2+16x-3 = A(2x^3-x^2+2x) + B(4x-1)$

$= 2Ax^3 - Ax^2 + (2A+4B)x - B$

$4=2A \quad -2=-A \quad 2A+4B=16 \quad -3=-B$

$A=2 \quad A=2 \quad 4+12=16 \quad B=3$

$= 2x + \dfrac{3(4x-1)}{2x^2-x+2}$

b) $\displaystyle\int \frac{4x^3-2x^2+16x-3}{2x^2-x+2}\,dx$

$= \displaystyle\int 2x + \frac{3(4x-1)}{2x^2-x+2}\,dx$

$= \displaystyle\int 2x\,dx + 3\int \frac{4x-1}{2x^2-x+2}\,dx$

$y = x^2 + 3\ln(2x^2-x+2) + c$

at $(-1,2)$

$2 = (-1)^2 + 3\ln(2(-1)^2-(-1)+2) + c$

$2 = 1 + 3\ln(2+1+2) + c$

$\frac{1}{3} = \ln 5 + c$

$c = \frac{1}{3} - \ln 5$

$y = x^2 + \frac{1}{3} - \ln 5 + 3\ln(2x^2-x+2)$

5 (a) (i) Obtain the binomial expansion of $(1-x)^{-1}$ up to and including the term in x^2. *(2 marks)*

(ii) Hence, or otherwise, show that

$$\frac{1}{3-2x} \approx \frac{1}{3} + \frac{2}{9}x + \frac{4}{27}x^2$$

for small values of x. *(3 marks)*

(b) Obtain the binomial expansion of $\dfrac{1}{(1-x)^2}$ up to and including the term in x^2. *(2 marks)*

(c) Given that $\dfrac{2x^2-3}{(3-2x)(1-x)^2}$ can be written in the form $\dfrac{A}{(3-2x)} + \dfrac{B}{(1-x)} + \dfrac{C}{(1-x)^2}$,

find the values of A, B and C. *(5 marks)*

(d) Hence find the binomial expansion of $\dfrac{2x^2-3}{(3-2x)(1-x)^2}$ up to and including the term in x^2. *(3 marks)*

a) i) $(1-x)^{-1}$

$= 1 + (-1)(-x) + \frac{(-1)(-2)}{1.2}(-x)^2$

$= 1 + x + x^2$

ii) $\frac{1}{3-2x} = (3-2x)^{-1}$

$= \frac{1}{3}(1 - \frac{2}{3}x)^{-1}$

$\approx \frac{1}{3}(1 + (-1)(-\frac{2}{3}x) + \frac{(-1)(-2)}{1.2}(-\frac{2}{3}x)^2)$

$\approx \frac{1}{3}(1 + \frac{2}{3}x + \frac{4}{9}x^2$

$\approx \frac{1}{3} + \frac{2}{9}x + \frac{4}{27}x^2$

b) $\frac{1}{(1-x)^2} = (1-x)^{-2}$

$= 1 + (-2)(-x) + \frac{(-2)(-3)}{1.2}(-x)^2$

$= 1 + 2x + 3x^2$

c) $\frac{2x^2-3}{(3-2x)(1-x)^2} = \frac{A}{3-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$

$2x^2 - 3 = A(1-x)^2 + B(3-2x)(1-x) + C(3-2x)$

$x = 1$

$2 - 3 = C(3-2)$

$-1 = C(1)$

$C = -1$

$x = \frac{3}{2}$

$2(\frac{3}{2})^2 - 3 = A(-\frac{1}{2})^2$

$2(\frac{9}{4}) - 3 = \frac{1}{4}A$

$\frac{3}{2} = \frac{1}{4}A$

$A = 6$

$x = 0$

$-3 = 6 + B(3) - 1(3)$

$-3 = 6 - 3 + 3B$

$3B = -6$

$B = -2$

$= \frac{6}{3-2x} - \frac{2}{1-x} - \frac{1}{(1-x)^2}$

d) $6(3-2x)^{-1} - 2(1-x)^{-1} - (1-x)^{-2}$

$= 6(\frac{1}{3} + \frac{2}{9}x + \frac{4}{27}x^2) - 2(1 + x + x^2) - (1 + 2x + 3x^2)$

$= 2 + \frac{4}{3}x + \frac{8}{9}x^2 - 2 - 2x - 2x^2 - 1 - 2x - 3x^2$

$= -1 - \frac{8}{3}x - \frac{37}{9}x^2$