## Pure Sector 4: Parametric Equations

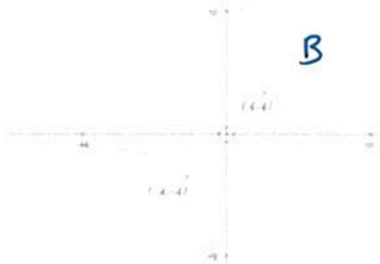
Aims
Understand and use the parametric equations of curves.
Be able to convert between Cartesian and parametric forms.
Be able to differentiate functions using the chain rule which are defined parametrically.
Be able to find stationary points, equations of tangent and normal for parametric equations.

| |
|---|
| **Parametric equations are when a relationship between two variables is defined relative to a parameter. Usually either t or θ.** |

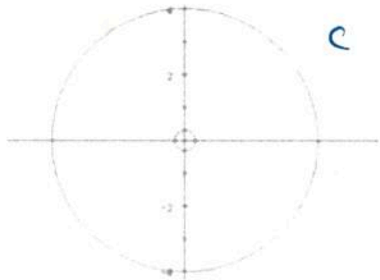
**Task** Match these functions to the graphs below.

| | |
|---|---|
| **A.** $x=2t^2$ , $y=4t$ | **B.** $x = 4t$, $y = 4/t$ |
| **C.** $x=4\cos\theta$, $y=4\sin\theta$ | **D.** $x=3\cos\theta$, $y=4\sin\theta$ |

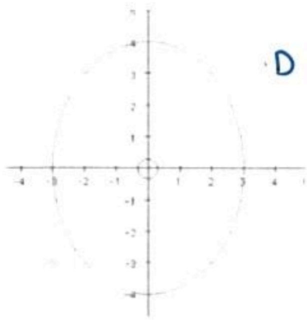
A



B



C



D

### Converting from parametric to Cartesian form

Example 1

Find a Cartesian equation for each of the following curves:

a) $x = 3p^2, y = 6p$, where $p$ is a parameter.

$p = y/6$

$x = 3(y/6)^2 \quad \therefore \quad x = \frac{y^2}{12}$

b) $x = 2q^2$, $y = q^3$, where $q$ is a parameter

$q^2 = x/2 \quad \therefore \quad y = (x/2)^{3/2}$

Example 2

Find a Cartesian equation for each of the following curves:

a) $x = 3\cos\theta$, $y = 2\sin\theta$, where $\theta$ is a parameter. USING $\cos^2\theta + \sin^2\theta \equiv 1$

$\cos\theta = x/3 \quad \sin\theta = y/2$

$(x/3)^2 + (y/2)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

b) $x = 3 + \sec\theta$, $y = 1 - \tan\theta$, where $\theta$ is a parameter. USING $1 + \tan^2\theta \equiv \sec^2\theta$

$\sec\theta = x - 3$

$\tan\theta = 1 - y$

$1 + (1-y)^2 = (x-3)^2$

**Differentiating Parametric Functions**

This comes from the **chain rule**

For a Parametric function:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 3

a) Find the gradient at the point $p$ where $t = -1$, on the curve given parametrically by:

$$x = t^2 - t \qquad y = t^3 - t^2$$

b) Hence find the equation of the normal at the point $p$.

(a) $dy/dt = 3t^2 - 2t$ $\quad dx/dt = 2t - 1$ $\quad dy/dx = \frac{3t^2 - 2t}{2t - 1}$

when $t = -1$, $dy/dx = \frac{3(-1)^2 - 2(-1)}{2(-1) - 1} = \frac{5}{-3}$

(b) when $t = -1$, $y = -2$, $x = 2$ $\quad (2, -2)$

$y + 2 = 3/5\,(x - 2)$

**Parametric Trigonometry**

Example 4

A curve is given parametrically by $x = 3 - \cos\theta, y = 2 + \sec\theta$ where $\theta$ is a parameter.

a) Show $\frac{dy}{dx} = \sec^2\theta$

b) Find the equation of the normal to the curve at $\theta = \frac{\pi}{3}$

$y = 2 + (\cos\theta)^{-1}$

$dy/d\theta = -(\cos\theta)^{-2}(-\sin\theta)$

(a) $dy/d\theta = \sin\theta\sec^2\theta$ $\quad dx/d\theta = \sin\theta$ $\quad dy/dx = \frac{\sin\theta\sec^2\theta}{\sin\theta} = \sec^2\theta$

(b) when $\theta = \pi/3$, $x = 3 - 1/2 = 5/2$, $y = 2 + 2 = 4$

$dy/dx = [\sec(\pi/3)]^2 = 4$

$y - 4 = -1/4\,(x - 5/2)$

4 A curve is defined by the parametric equations

$$x = 3e^t, \quad y = e^{2t} - e^{-2t}$$

(a) (i) Find the gradient of the curve at the point where $t = 0$. *(3 marks)*

(ii) Find an equation of the tangent to the curve at the point where $t = 0$. *(1 mark)*

(b) Show that the cartesian equation of the curve can be written in the form

$$y = \frac{x^2}{k} - \frac{k}{x^2}$$

where $k$ is an integer. *(2 marks)*

(a) (i) $dy/dx = \dfrac{2e^{2t} + 2e^{-2t}}{3e^t}$ when $t = 0$, $dy/dx = \dfrac{2e^0 + 2e^0}{3e^0} = 4/3$

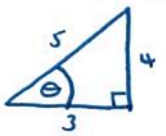
(ii) $x = 3, y = 0$ $\quad y = 4/3\,(x-3)$

(b) $x^2 = 9e^{2t}$ $\quad \therefore e^{2t} = x^2/9$

$y = x^2/9 - \dfrac{9}{x^2}$

6 (a) (i) Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. *(2 marks)*

$\sin 2\theta = 2\sin\theta\cos\theta$ $\quad \cos 2\theta = \cos^2\theta - \sin^2\theta$




$\sin\theta = 4/5$

(ii) Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{3}{5}$, show that $\sin 2\theta = \frac{24}{25}$ and find the value of $\cos 2\theta$. *(2 marks)*

$\sin 2\theta = 2(4/5)(3/5) = 24/25$ $\quad \cos 2\theta = (3/5)^2 - (4/5)^2 = -7/25$

(b) A curve has parametric equations

$$x = 3\sin 2\theta, \quad y = 4\cos 2\theta$$

(i) Find $\frac{dy}{dx}$ in terms of $\theta$. $\quad dy/dx = \dfrac{-8\sin 2\theta}{6\cos 2\theta} = \dfrac{-4\sin 2\theta}{3\cos 2\theta}$ *(3 marks)*

(ii) At the point $P$ on the curve, $\cos\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$. Find an equation of the tangent to the curve at the point $P$. *(3 marks)*

$x = 3(24/25) = 72/25$ $\quad y = 4(-7/25) = -28/25$

$dy/dx = \dfrac{-4(24/25)}{3(-7/25)} = \dfrac{32}{7}$

$y + \dfrac{28}{25} = \dfrac{32}{7}\left(x - \dfrac{72}{25}\right)$

5.

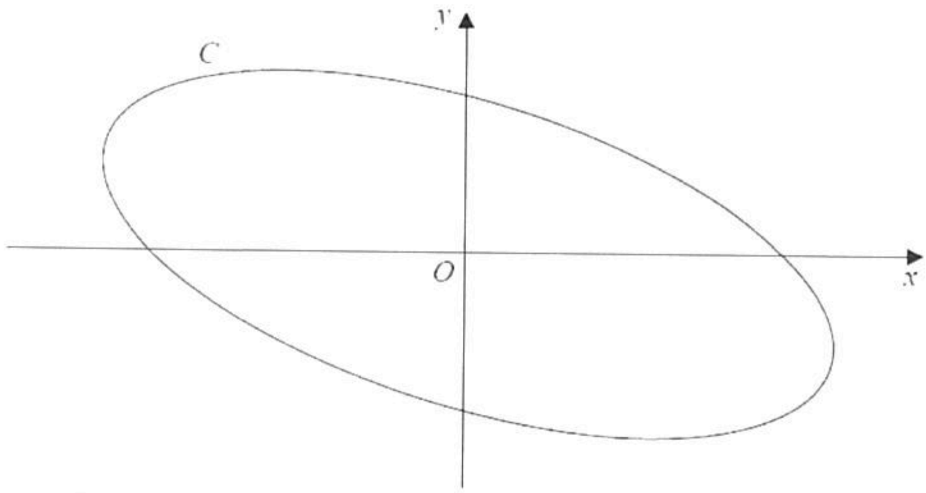



**Figure 3**

Figure 3 shows a sketch of the curve $C$ with parametric equations

$$x = 4\cos\left(t + \frac{\pi}{6}\right), \quad y = 2\sin t, \qquad 0 \leqslant t < 2\pi$$

(a) Show that

$$x + y = 2\sqrt{3}\cos t$$

**(3)**

(b) Show that a cartesian equation of $C$ is

$$(x + y)^2 + ay^2 = b$$

where $a$ and $b$ are integers to be determined.

**(2)**

$x + y =$

(a) $4\cos\left(t + \pi/6\right) + 2\sin t =$

$4\left[\cos t \cos \pi/6 - \sin t \sin \pi/6\right] + 2\sin t =$

$4\left[\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}\sin t\right] + 2\sin t =$

$2\sqrt{3}\cos t - 2\sin t + 2\sin t = 2\sqrt{3}\cos t$

(b) $(x+y)^2 = (2\sqrt{3}\cos t)^2 = 12\cos^2 t$

$12\cos^2 t + 4a\sin^2 t = b$

$\therefore a = 3$ and $b = 12$

$12\cos^2 t + 12\sin^2 t = 12$

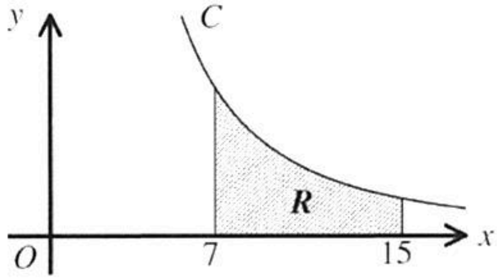
**Parametric Integration**

When working with parametric equations, you can use the chain rule so that the variable involved is the parameter.

$$Area = \int y(x)dx = \int y(t)\frac{dx}{dt}dt$$

Example 5

The figure shows the curve $C$, given parametrically by:




$$x = 4t - 1, \quad y = \frac{16}{t^2}, \quad t > 0$$

a) By converting the equation to Cartesian form find the area of the region $R$ enclosed by the curve, the $x$ axis, the lines $x = 7$ and $x = 15$.

b) Show that the area of $R$ can be written as

$$\int_{t_1}^{t_2} 64t^{-2}dt$$

Giving the values of $t_1$ and $t_2$.

c) Find the value of the integral and hence verify the answer to part a)

a) $x = 4t - 1$

$4t = x + 1$

$t = \frac{x+1}{4}$

$y = \frac{16}{\left(\frac{x+1}{4}\right)^2}$

$y = \frac{256}{(x+1)^2}$

$R = \int_7^{15} \frac{256}{(x+1)^2}\, dx$

$R = \int_7^{15} 256(x+1)^{-2}\, dx$

$R = 256\left[-(x+1)^{-1}\right]_7^{15}$

$R = -256\left(16^{-1} - 8^{-1}\right)$

$R = 16$

b) $\frac{dx}{dt} = 4$

Limits

$x = 7$ | $x = 15$

$7 = 4t - 1$ | $15 = 4t - 1$

$t = 2$ | $t = 4$

$R = \int_2^4 \left(\frac{16}{t^2}\right)(4)\, dt$

$R = \int_2^4 64t^{-2}\, dt$

c) $R = \left[-64t^{-1}\right]_2^4$

$R = -64\left(4^{-1} - 2^{-1}\right)$

$R = 16$

Example 6

The curve $C$, is given by the parametric equations

$$x = \ln t, \quad y = t + \sqrt{t}, \qquad 1 \le t \le 10$$

Find the area of the finite region bounded by $C$, the straight line with the equation $x = 1$ and $x = 2$ and the $x$ axis.

$$\frac{dx}{dt} = \frac{1}{t}$$

Limits $\quad x = 1 \quad 1 = \ln t$

$t = e^1 = e$

$x = 2 \quad t = e^2$

$$\text{Area} = \int_e^{e^2} \left(t + t^{1/2}\right)\left(\frac{1}{t}\right) dt$$

$$= \int_e^{e^2} 1 + t^{-1/2} \, dt$$

$$= \left[t + 2t^{1/2}\right]_e^{e^2}$$

$$= e^2 + 2(e^2)^{1/2} - \left(e + 2e^{1/2}\right)$$

$$= e^2 + e - 2e^{1/2}$$

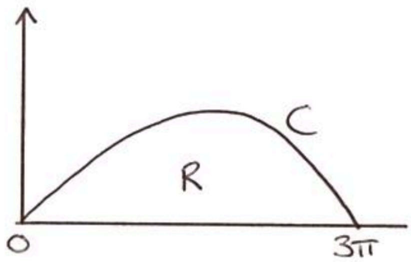
Note: You could find the cartesian equation and then integrate for this question.

Example 7

The curve $C$, is given by the parametric equations

$$x = 3t + \sin t, \quad y = 2\sin t, \qquad 0 \le t \le \pi$$

Find the area of the finite region $R$ bounded by $C$ and the $x$ axis.




$\frac{dx}{dt} = 3 + \cos t$

Limits $\quad x = 0 \quad t = 0$

$x = 3\pi \quad t = \pi$

$t = 0, \; x = 0, \; y = 0$

$t = \pi, \; x = 3\pi, \; y = 0$

$$\int_0^{\pi} (2\sin t)(3 + \cos t) \, dt$$

$$= \int_0^{\pi} 6\sin t + 2\sin t\cos t \, dt$$

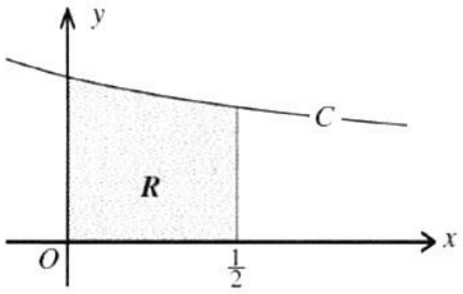
$$= \int_0^{\pi} 6\sin t + \sin 2t \, dt$$

$$= \left[-6\cos t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= -6\cos\pi - \frac{1}{2}\sin 2\pi - \left(-6\cos 0 - \frac{1}{2}\sin 0\right)$$

$$= 12$$

**Exam Style**




The figure above shows part of the curve $C$, with parametric equations

$$x = \cos 2\theta, \quad y = \sec\theta \ , \ 0 < \theta < \frac{\pi}{2}.$$

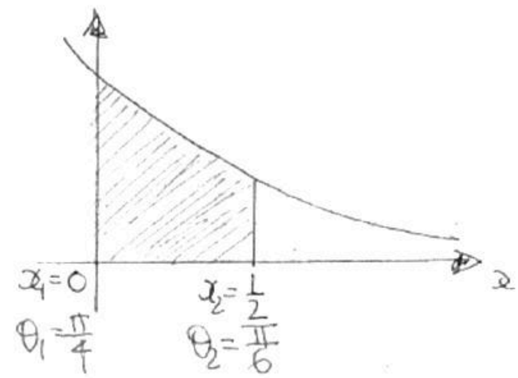
The finite region $R$ is bounded by $C$, the straight line with equation $x = \frac{1}{2}$ and the coordinate axes.

a) Show that the area of $R$ is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta \ d\theta. \qquad \textbf{(5)}$$

b) Evaluate the above integral to find an exact value for $R$. **(2)**

$7_6$ a)




$x = \cos 2\theta$ $\quad 0 < \theta < \frac{\pi}{2}$

$y = \sec\theta$

| | |
|---|---|
| $0 = \cos 2\theta$ | $\frac{1}{2} = \cos 2\theta$ |
| $2\theta = \frac{\pi}{2}$ | $2\theta = \frac{\pi}{3}$ |
| $\theta = \frac{\pi}{4}$ | $\theta = \frac{\pi}{6}$ |
| (ONLY SOLUTION IN RANGE) | (ONLY SOLUTION IN RANGE) |

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\,\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \underbrace{(\sec\theta)}_{y(\theta)}\underbrace{(-2\sin 2\theta)}_{\frac{dx}{d\theta}}\,d\theta$$

USE THE MINUS TO REVERSE THE LIMITS

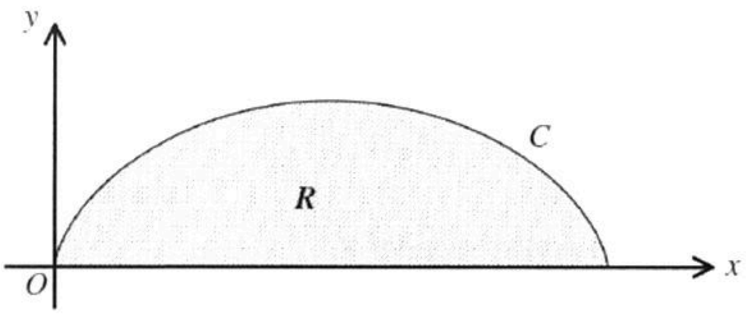
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos\theta} \times 2(2\sin\theta\cos\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta\,d\theta$$

AS REQUIRED

b) INTEGRATE... $= \left[-4\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 4\left[\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$$= 4\left[\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\right] = 4\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right]$$

$$= 2(\sqrt{3} - \sqrt{2}) = 2\sqrt{3} - 2\sqrt{2}$$




The figure above shows a cycloid $C$, whose parametric equations are

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta\ , \text{ for } 0 \le \theta \le 2\pi\ .$$

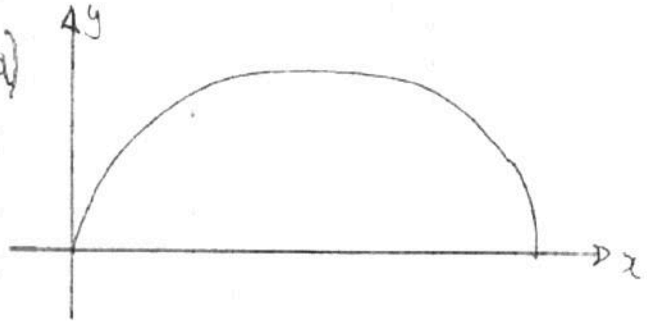
The finite region $R$ is bounded by $C$ and the $x$ axis.

a) Show, with full justification, that the area of $R$ is given by

$$\int_0^{2\pi} (1 - \cos\theta)^2 \; d\theta\ . \qquad (3)$$

b) Hence find the area of $R$. (7)

10. a)




$x = \theta - \sin\theta$

$y = 1 - \cos\theta$

$0 \leq \theta \leq 2\pi$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_0^{2\pi} \underbrace{(1-\cos\theta)}_{y(\theta)}\underbrace{(1-\cos\theta)}_{\frac{dx}{d\theta}}\,d\theta = \int_0^{2\pi} (1-\cos\theta)^2\,d\theta$$

AS REQUIRED

LIMITS $\theta = 0$ PRODUCES $x=0$, $y=0$ i.e. $(0,0)$

$\theta = 2\pi$ PRODUCES $x = 2\pi$, $y = 0$ i.e. $(2\pi, 0)$

b) $$\int_0^{2\pi} (1-\cos\theta)^2\,d\theta = \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \; d\theta$$

USING $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$= \int_0^{2\pi} 1 - 2\cos\theta + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \; d\theta$$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \left[\frac{3}{2}(2\pi) - \cancel{2\sin(2\pi)} + \cancel{\tfrac{1}{2}\sin 4\pi}\right] - \left[\cancel{0 - 2\sin 0 + \tfrac{1}{4}\sin 0}\right]$$

$$= 3\pi$$