Pure Sector 4: Parametric Equations

Aims

- Understand and use the parametric equations of curves.
- Be able to convert between Cartesian and parametric forms.
- Be able to differentiate functions using the chain rule which are defined parametrically.
- Be able to find stationary points, equations of tangent and normal for parametric equations.

Parametric equations are when a relationship between two variables is defined relative to a parameter. Usually either t or θ.

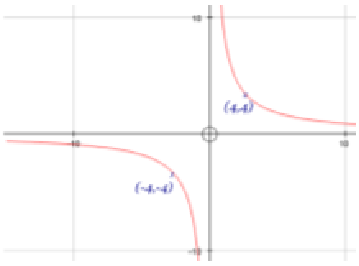
Match these functions to the correct graph.

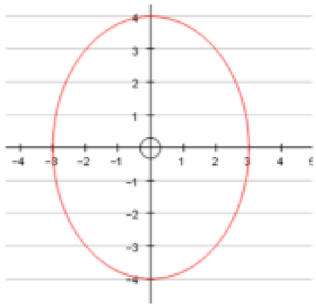
A $x = 2t^2, y = 4t$

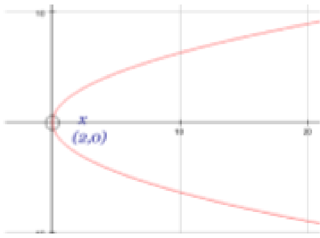
B $x = 4t, y = \frac{4}{t}$

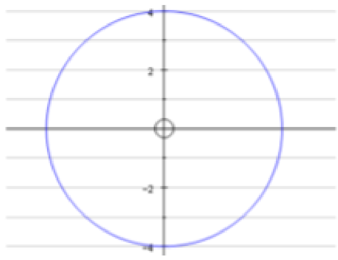
C $x = 4\cos\theta,\ y = 4\sin\theta$

D $x = 3\cos\theta,\ y = 4sin\theta$









Converting from Parametric to Cartesian form

<u>Example 1</u>

Find a Cartesian equation for each of the following curves:

a) $x = 3p^2, y = 6p$, where p is a parameter.

b) $x = 2q^2,\ y = q^3$, where q is a parameter

<u>Example 2</u>

Find a Cartesian equation for each of the following curves:

a) $x = 3\cos\theta,\ y = 2\sin\theta$, where θ is a parameter.

b) $x = 3 + \sec\theta,\ y = 1 - \tan\theta$, where θ is a parameter.

This comes from the **chain rule**

Differentiating Parametric Functions

For a parametric function:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

<u>Example 3</u>

a) Find the gradient at the point p where $t = -1$, on the curve given parametrically by:

$$x = t^2 - t \qquad\qquad y = t^3 - t^2$$

b) Hence find the equation of the normal at the point p.

Parametric Trigonometry

<u>Example 4</u>

A curve is given parametrically by $x = 3 - \cos\theta, y = 2 + \sec\theta$ where θ is a parameter.

a) Show $\frac{dy}{dx} = \sec^2\theta$

b) Find the equation of the normal to the curve at $\theta = \frac{\pi}{3}$

Exam Questions

4 A curve is defined by the parametric equations

$$x = 3e^t, \quad y = e^{2t} - e^{-2t}$$

(a) (i) Find the gradient of the curve at the point where $t = 0$. *(3 marks)*

(ii) Find an equation of the tangent to the curve at the point where $t = 0$. *(1 mark)*

(b) Show that the cartesian equation of the curve can be written in the form

$$y = \frac{x^2}{k} - \frac{k}{x^2}$$

where k is an integer. *(2 marks)*

6 (a) (i) Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. *(2 marks)*

(ii) Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{3}{5}$, show that $\sin 2\theta = \frac{24}{25}$ and find the value of $\cos 2\theta$. *(2 marks)*

(b) A curve has parametric equations

$$x = 3\sin 2\theta, \quad y = 4\cos 2\theta$$

(i) Find $\frac{dy}{dx}$ in terms of θ. *(3 marks)*

(ii) At the point P on the curve, $\cos\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$. Find an equation of the tangent to the curve at the point P. *(3 marks)*

5.

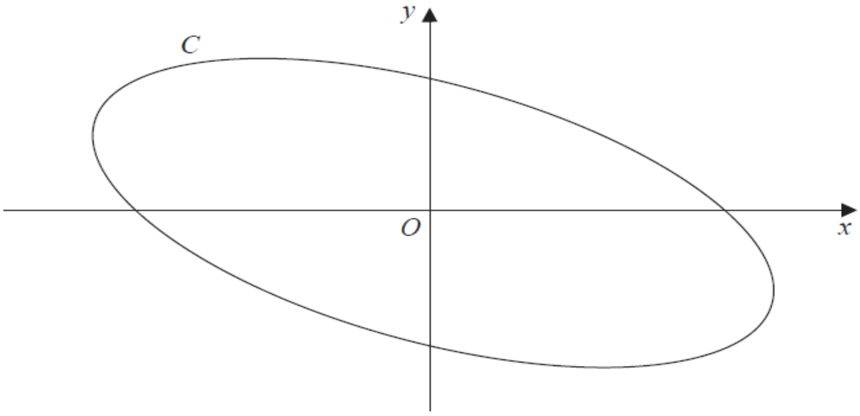


Figure 3

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 4\cos\left(t + \frac{\pi}{6}\right), \quad y = 2\sin t, \qquad 0 \leqslant t < 2\pi$$

(a) Show that

$$x + y = 2\sqrt{3}\cos t$$

(3)

(b) Show that a cartesian equation of C is

$$(x + y)^2 + ay^2 = b$$

where a and b are integers to be determined.

(2)

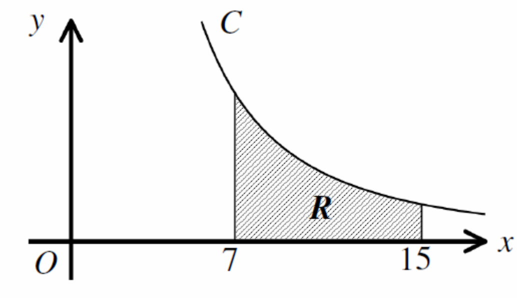
Parametric Integration

When working with parametric equations, you can use the chain rule so that the variable involved is the parameter.

$$Area = \int y(x)dx = \int y(t)\frac{dx}{dt}dt$$

Example 5

The figure shows the curve C, given parametrically by:



$$x = 4t - 1, \quad y = \frac{16}{t^2}, \quad t > 0$$

a) By converting the equation to Cartesian form find the area of the region R enclosed by the curve, the x axis, the lines $x = 7$ and $x = 15$.
b) Show that the area of R can be written as
$$\int_{t_1}^{t_2} 64t^{-2}dt$$
Giving the values of t_1 and t_2.
c) Find the value of the integral and hence verify the answer to part a)

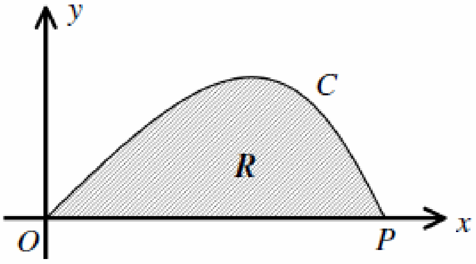
Example 6

The curve C, is given by the parametric equations

$$x = \ln t, \quad y = t + \sqrt{t}, \qquad 1 \leq t \leq 10$$

Find the area of the finite region bounded by C, the straight line with the equation $x = 1$ and $x = 2$ and the x axis.

Example 7

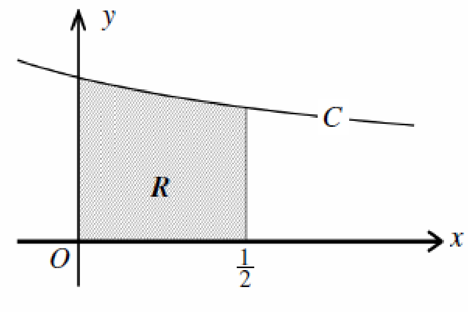


The curve C, is given by the parametric equations

$$x = 3t + \sin t, \quad y = 2\sin t, \qquad 0 \leq t \leq \pi$$

Find the area of the finite region R bounded by C and the x axis.

Exam Questions



The figure above shows part of the curve C, with parametric equations

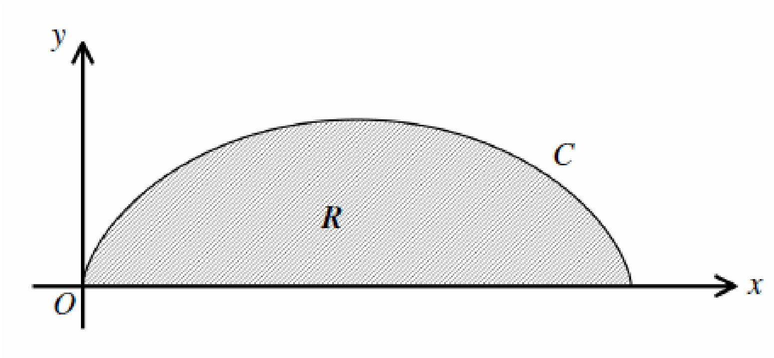
$$x = \cos 2\theta, \quad y = \sec\theta \ , \ 0 < \theta < \frac{\pi}{2}.$$

The finite region R is bounded by C, the straight line with equation $x = \frac{1}{2}$ and the coordinate axes.

a) Show that the area of R is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta \ d\theta. \qquad \textbf{(5)}$$

b) Evaluate the above integral to find an exact value for R. **(2)**



The figure above shows a cycloid C, whose parametric equations are

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \ , \text{ for } \ 0 \le \theta \le 2\pi .$$

The finite region R is bounded by C and the x axis.

a) Show, with full justification, that the area of R is given by

$$\int_0^{2\pi} (1-\cos\theta)^2 \ d\theta . \qquad \textbf{(3)}$$

b) Hence find the area of R. **(7)**