Pure Sector 4: Implicit Differentiation

Implicit functions are functions in x and y that aren't written $y = f(x)$ or $x = f(y)$

$x^2 + y^2 = 4 \qquad x + y = 10$
$\sin(x+y) = y \; xy = 10$

In general, to differentiate a function of y with respect to x we differentiate it with respect to y and multiply by $\frac{dy}{dx}$

Differentiate these functions with respect to x

$x^2 + y^2 = 4$	$x + y = 10$
$2x + 2y\,dy/dx = 0$ $(2y)\,dy/dx = -2x$ $dy/dx = \frac{-2x}{2y} = \frac{-x}{y}$	$1 + dy/dx = 0$ $dy/dx = -1$
$\sin(x+y) = y$	$xy = 10$
$(\cos(x+y))(1 + dy/dx) = dy/dx$ $\cos(x+y) + dy/dx\cos(x+y) = dy/dx$ $\cos(x+y) = dy/dx\,(1 - \cos(x+y))$ $dy/dx = \frac{\cos(x+y)}{1-\cos(x+y)}$	$y + x\,dy/dx = 0$ $dy/dx = \frac{-y}{x}$

Example 1

Find $\frac{dy}{dx}$ when $x^3 + 3y^2 = x$

$3x^2 + 6y\,dy/dx = 1$

$6y\,dy/dx = 1 - 3x^2$

$dy/dx = \frac{1-3x^2}{6y}$

Example 2

Find the gradient of the curve $3x^2 + xy - y^3 = 20$ at the point $(4,2)$.

$6x + (y + x\,dy/dx) - 3y^2\,dy/dx = 0$

$dy/dx\,(x - 3y^2) = -6x - y$

$dy/dx = \frac{-6x - y}{x - 3y^2}$

At $(4,2)$ $dy/dx = \frac{-6(4) - (2)}{4 - 3(2)^2} = \frac{-26}{-8}$

$= 13/4$

Example 3

Find the equation of the normal to the curve $x^3 + 2x^2y = y^3 + 15$ at the point $(2,1)$. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

$3x^2 + (4xy + 2x^2\,dy/dx) = 3y^2\,dy/dx$

$3x^2 + 4xy = dy/dx\,(3y^2 - 2x^2)$

$dy/dx = \frac{3x^2 + 4xy}{3y^2 - 2x^2}$

At $(2,1)$ $dy/dx = \frac{3(2)^2 + 4(2)(1)}{3(1)^2 - 2(2)^2} = \frac{12 + 8}{3 - 8} = \frac{20}{-5} = -4$

$y - 1 = 1/4\,(x - 2)$

$4y - 4 = x - 2 \quad \therefore \quad x - 4y + 2 = 0$

Example 4

Find the coordinates of the two stationary points of the curve $x^2 + 4xy + 2y^2 + 18 = 0$

$2x + (4y + 4x\, dy/dx) + 4y\, dy/dx = 0$

$dy/dx\,(4x + 4y) = -2x - 4y$

$dy/dx = \dfrac{-2x - 4y}{4x + 4y}$

Stationary when $dy/dx = 0$

$0 = -2x - 4y$

$2x = -4y$

$y = \dfrac{-x}{2}$

Substitute $y = -x/2$ into curve

$x^2 + 4x\left(-\frac{x}{2}\right) + 2\left(\frac{-x}{2}\right)^2 + 18 = 0$

$x^2 - 2x^2 + \frac{x^2}{2} + 18 = 0$

$\frac{-x^2}{2} = 18 \quad \therefore \quad x^2 = 36$

$x = \pm 6$

Stat. points $= (6, -3)$ and $(-6, 3)$

Example 5

A curve is defined by the equation

$$2x^2y + 2x + 4y - \cos(\pi y) = 17$$

a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

$4xy + 2x^2\, dy/dx + 2 + 4\, dy/dx + \pi \sin(\pi y)\, dy/dx = 0$

$dy/dx\,(2x^2 + 4 + \pi \sin(\pi y)) = -4xy - 2$

$dy/dx = \dfrac{-4xy - 2}{2x^2 + 4 + \pi \sin(\pi y)}$

The point P with coordinates $\left(3, \frac{1}{2}\right)$ lies on C.

The normal to C at P meets the x-axis at the point A.

(b) Find the x coordinate of A, giving your answer in the form $\dfrac{a\pi + b}{c\pi + d}$, where a, b, c and d are integers to be determined.

At P, $dy/dx = \dfrac{-4(3)(\frac{1}{2}) - 2}{2(3)^2 + 4 + \pi \sin(\pi/2)} = \dfrac{-8}{22 + \pi}$

Normal gradient $= \dfrac{22 + \pi}{8}$

$y - \frac{1}{2} = \left(\frac{22 + \pi}{8}\right)(x - 3)$

When $y = 0$, $-1/2 = \left(\frac{22 + \pi}{8}\right)(x - 3)$

$-4 = 22x + \pi x - 66 - 3\pi$

$x = \dfrac{62 + 3\pi}{22 + \pi}$ $\qquad a = 3;\ b = 62;\ c = 1;\ d = 22$

Exam Questions

6 A curve has equation $x^3y + \cos(\pi y) = 7$.

(a) Find the exact value of the x-coordinate at the point on the curve where $y = 1$. *(2 marks)*

(b) Find the gradient of the curve at the point where $y = 1$. *(5 marks)*

(a) $x^3(1) + \cos(\pi) = 7$

$x^3 - 1 = 7$

$x^3 = 8 \quad \therefore \quad x = 2$

(b) $3x^2y + x^3 \, dy/dx + (-\pi \sin(\pi y)) \, dy/dx = 0$

$$dy/dx = \frac{-3x^2y}{x^3 - \pi\sin(\pi y)}$$

At $(2,1)$ $$dy/dx = \frac{-3(2)^2(1)}{(2)^3 - \pi\sin(\pi)} = \frac{-12}{8} = -3/2$$

5 A curve is defined by the equation

$$y^2 - xy + 3x^2 - 5 = 0$$

(a) Find the y-coordinates of the two points on the curve where $x = 1$. *(3 marks)*

(b) (i) Show that $\dfrac{dy}{dx} = \dfrac{y - 6x}{2y - x}$. *(6 marks)*

(ii) Find the gradient of the curve at each of the points where $x = 1$. *(2 marks)*

(iii) Show that, at the two stationary points on the curve, $33x^2 - 5 = 0$. *(3 marks)*

(a) $y^2 - (1)y + 3(1)^2 - 5 = 0$

$y^2 - y - 2 = 0$

$(y - 2)(y + 1) = 0 \quad \therefore \; y = 2$ and $y = -1$

(b) (i) $2y \, dy/dx - (y + x \, dy/dx) + 6x = 0$

$dy/dx \, (2y - x) = y - 6x$

$$dy/dx = \frac{y - 6x}{2y - x}$$

(ii) At $(1,2)$, $$dy/dx = \frac{2-6}{4-1} = \frac{-4}{3}$$

At $(1,-1)$, $$dy/dx = \frac{-1-6}{-2-1} = \frac{-7}{-3} = 7/3$$

(c) Stationary points when $dy/dx = 0 \quad \therefore \; y - 6x = 0 \quad \therefore \; y = 6x$

$(6x)^2 - x(6x) + 3x^2 - 5 = 0$

$36x^2 - 6x^2 + 3x^2 - 5 = 0 \quad \therefore \quad 33x^2 - 5 = 0$

(a) $8x - 3y^2 \, dy/dx - \left(4y + 4x \, dy/dx\right) + 2^y \ln 2 \, dy/dx = 0$

The curve C has equation

$$4x^2 - y^3 - 4xy + 2^y = 0$$

$dy/dx \left(2^y \ln 2 - 3y^2 - 4x\right) = 4y - 8x$

$dy/dx = \dfrac{4y - 8x}{2^y \ln 2 - 3y^2 - 4x}$

The point P with coordinates $(-2, 4)$ lies on C.

(a) Find the exact value of $\frac{dy}{dx}$ at the point P.

At $(-2, 4)$ $dy/dx = \dfrac{16 - (-16)}{(6)16 \ln 2 - 48 + 8}$

$= \dfrac{32}{16 \ln 2 - 40} = \dfrac{4}{2 \ln 2 - 5}$

The normal to C at P meets the y-axis at the point A.

(b) Find the y coordinate of A, giving your answer in the form $p + q \ln 2$, where p and q are constants to be determined.

(3)

$y - 4 = \dfrac{-2\ln 2 + 5}{4}(x + 2)$

when $x = 0$

$y = \dfrac{-2 \ln 2 + 5}{2} + 4$

$y = \dfrac{13}{2} - \ln 2$ $\quad \therefore p = 13/2$ and $q = -1$

6 A curve is defined by the equation $2y + e^{2x}y^2 = x^2 + C$, where C is a constant.

The point $P\left(1, \frac{1}{e}\right)$ lies on the curve.

$2\left(\frac{1}{e}\right) + e^2\left(\frac{1}{e}\right)^2 = 1 + C \quad \therefore C = \frac{2}{e}$

(a) Find the exact value of C. *(1 mark)*

(b) Find an expression for $\frac{dy}{dx}$ in terms of x and y. *(7 marks)*

(c) Verify that $P\left(1, \frac{1}{e}\right)$ is a stationary point on the curve. *(2 marks)*

(b) $2 \, dy/dx + \left(2e^{2x}y^2 + e^{2x} 2y \, dy/dx\right) = 2x$

$dy/dx \left(2 + 2e^{2x}y\right) = 2x - 2e^{2x}y^2$

$dy/dx = \dfrac{2x - 2e^{2x}y^2}{2 + 2e^{2x}y} = \dfrac{x - e^{2x}y^2}{1 + e^{2x}y}$

(c) At P $dy/dx = \dfrac{1 - e^2\left(\frac{1}{e}\right)^2}{\cancel{2}\, 1 + e^2\left(\frac{1}{e}\right)} = 0$

$\therefore$ stationary point.

6 A curve is defined by the equation $\sin 2x + \cos y = \sqrt{3}$.

(i) Verify that the point P $\left(\frac{1}{6}\pi, \frac{1}{6}\pi\right)$ lies on the curve. [1]

$\sin \pi/3 + \cos \pi/6 = \sqrt{3}$ $\quad \therefore$ P lies on curve.

(ii) Find $\frac{dy}{dx}$ in terms of x and y.

Hence find the gradient of the curve at the point P. [5]

$2 \cos 2x - \sin y \left(dy/dx\right) = 0$

$dy/dx = \dfrac{2 \cos 2x}{\sin y}$

At P, $dy/dx = \dfrac{2 \cos \pi/3}{\sin \pi/6} = \dfrac{1}{1/2} = 2.$