Pure Sector 2: Differentiation 2

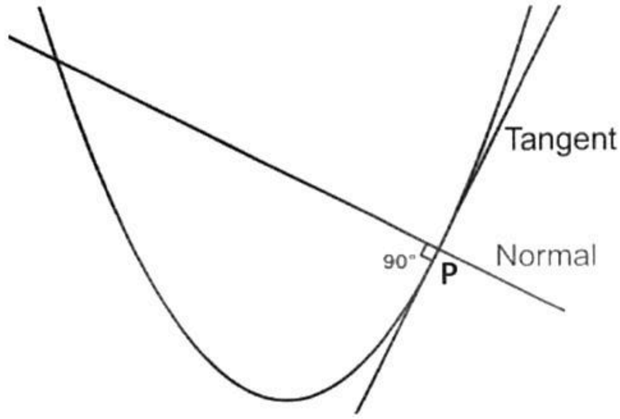
Aims

- Apply differentiation to find tangents and normal, maxima and minima and stationary points
- Identify where functions are increasing or decreasing.
- Understand and use the second derivative as the rate of change of gradient;

The equations of the tangent and normal to a curve

As mentioned before, $\frac{dy}{dx}$ is an expression for the gradient of a line or curve.
We can use this expression to find the gradient of the tangent or normal line at certain co-ordinates.

The gradient and point P (the point where the tangent touches the curve) can then be used to find the equation of the line for the tangent and/or the normal.



Key Points

- The tangent to a curve touches the curve once (at the point P on the diagram).
- The normal to a curve at a point is the line at right angles to the tangent that passes through the point where the tangent touches the curve (P).
- The gradient at a point on a curve (found using $\frac{dy}{dx}$ and the co-ordinates of P) is the same as the gradient of the tangent at that point.
- When two lines are perpendicular (at right angles to each other) the product of their gradients is -1. This can be used to help you find the gradient of the normal line, if required.

General Method – finding the equation of the line for the tangent

- Differentiate the original function to find the gradient function.
- Substitute the x co-ordinate of the point P into the gradient function to find the gradient of the tangent at P.
- Use the co-ordinates of P and the gradient of the tangent with $y - y_1 = m(x - x_1)$ to find the equation of the line for the tangent.

Example 1

Find the equation of the tangent to the curve $y = x^2 + 2x + 1$ at the point where $x = 3$.

Step 1- Differentiate the function	$\frac{dy}{dx} = 2x + 2$
Step 2- Substitute the x co-ordinate into the derivative to find the gradient of the tangent	$x = 3$ $\frac{dy}{dx} = 2(3) + 2$ $= 8$
Step 3- Find the corresponding y value (if it is not given) using the original equation	$y = (3)^2 + 2(3) + 1$ $y = 16$
Step 4- Substitute into the straight line equation and simplify if required.	$y - 16 = 8(x - 3)$

Example 2

a) Find the gradient of the tangent of the curve $y = x^3 - 6x + 5$ at the point where $x = 2$

$\frac{dy}{dx} = 3x^2 - 6$

$x = 2$

$\frac{dy}{dx} = 3(2)^2 - 6$

$= 6$

b) A curve has equation $y = \frac{1}{x^3} + 48x$ and $P(1, 49)$ is a point on the curve. Find the equation of the tangent to the curve at P.

$\frac{dy}{dx} = -3x^{-4} + 48$

$x = 1$

$\frac{dy}{dx} = -3(1)^{-4} + 48$

$= 45$

$y - 49 = 45(x - 1)$

General Method – finding the equation of the line for the normal

- Differentiate the original function to find the gradient function.
- Substitute the x co-ordinate of the point P into the gradient function to find the gradient of the tangent at P.
- Use the rules of perpendicular lines to find the gradient of the normal at P.
- Use the co-ordinates of P and the gradient of the normal to find the equation of the line for the normal.

Example 3

Find the equation of the normal to the curve $y = 6 - 3x - 4x^2 - x^3$ at the point where $x = -1$.

Step 1- Differentiate the function	$\frac{dy}{dx} = -3 - 8x - 3x^2$
Step 2a- Substitute in to the derivative to find the gradient of the tangent	$x = -1$ $\frac{dy}{dx} = -3 - 8(-1) - 3(-1)^2$ $= 2$
Step 2b- Find the gradient of the normal	$-1 \div 2 = -\frac{1}{2}$
Step 3- Find the corresponding y value (if it is not given) using the original equation	$y = 6 - 3(-1) - 4(-1)^2 - (-1)^3$ $y = 6$
Step 4- Substitute into the straight line equation and simplify if required.	$y - 6 = -\frac{1}{2}(x + 1)$

Example 4

Find the gradient of the normal of the curve $y = x^2 - 2\sqrt{x} - 4$ at the point where $x = 4$.

$\frac{dy}{dx} = 2x - x^{-\frac{1}{2}}$

$x = 4$

$\frac{dy}{dx} = 2(4) - (4)^{-\frac{1}{2}}$

$= 8 - \frac{1}{2}$

$= \frac{15}{2}$

$-1 \div \frac{15}{2} = -\frac{2}{15}$

$y = 4^2 - 2\sqrt{4} - 4$

$= 16 - 4 - 4$

$= 8$

$y - 8 = -\frac{2}{15}(x - 4)$

Exam Questions
AQA January 2005

2 A curve has equation $y = x^5 - 6x^3 - 3x + 25$.

(a) Find $\frac{dy}{dx}$. *(3 marks)*

(b) The point P on the curve has coordinates $(2, 3)$.

(i) Show that the gradient of the curve at P is 5. *(2 marks)*

(ii) Hence find an equation of the normal to the curve at P, expressing your answer in the form $ax + by = c$, where a, b and c are integers. *(3 marks)*

a) $\frac{dy}{dx} = 5x^4 - 18x^2 - 3$

b) i) $x = 2$

$\frac{dy}{dx} = 5(2)^4 - 18(2)^2 - 3$
$= 80 - 75$
$= 5$

ii) $-1 \div 5 = -\frac{1}{5}$
$y - 3 = -\frac{1}{5}(x - 2)$
$-5y + 15 = x - 2$
$x + 5y = 17$

Edexcel C1 January 2013

11. The curve C has equation

$$y = 2x - 8\sqrt{x} + 5, \quad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving each term in its simplest form. $\frac{dy}{dx} = 2 - 4x^{-\frac{1}{2}}$ **(3)**

The point P on C has x-coordinate equal to $\frac{1}{4}$

(b) Find the equation of the tangent to C at the point P, giving your answer in the form $y = ax + b$, where a and b are constants. **(4)**

The tangent to C at the point Q is parallel to the line with equation $2x - 3y + 18 = 0$

(c) Find the coordinates of Q. **(5)**

b) at $x = \frac{1}{4}$

$\frac{dy}{dx} = 2 - 4(\frac{1}{4})^{-\frac{1}{2}}$
$= 2 - \frac{4}{\sqrt{\frac{1}{4}}}$
$= 2 - \frac{4}{\frac{1}{2}}$
$= 2 - 8$
$= -6$

$y = 2(\frac{1}{4}) - 8\sqrt{\frac{1}{4}} + 5$
$= \frac{1}{2} - 4 + 5$
$= \frac{3}{2}$

$y - \frac{3}{2} = -6(x - \frac{1}{4})$
$y - \frac{3}{2} = \frac{6}{4} - 6x$
$y = \frac{3}{2} + \frac{3}{2} - 6x$
$y = -6x + 3$

In some questions, like the one above, you are asked to find the point on the curve where either the gradient of the tangent is given, or another line equation is given that is parallel to the tangent at the required point.

The method is similar, but this time we **set the expression for $\frac{dy}{dx}$ equal to the given gradient value**, and solve this equation to find the x and y co-ordinates of the point where the tangent touches the curve.

There will then be sufficient information to find the equation of the tangent.

c) $2x + 18 = 3y$
$y = \frac{2}{3}x + 6$
$\frac{dy}{dx} = \frac{2}{3}$

$\frac{2}{3} = 2 - 4x^{-\frac{1}{2}}$
$4x^{-\frac{1}{2}} = \frac{4}{3}$
$x^{-\frac{1}{2}} = \frac{1}{3}$

$\sqrt{x} = 3$
$x = 9$

$y = 2(9) - 8\sqrt{9} + 5$
$= 18 - 24 + 5$
$y = -1$
$Q(9, -1)$

Harder Examples (involving exponentials, trigonometric functions, logarithms and functions raised to a power)

Recap Example

Differentiate the following:

a) $y = e^{3x-5}$

$\frac{dy}{dx} = 3e^{3x-5}$

b) $f(x) = \cos 5x$

$f'(x) = -5\sin 5x$

c) $g(x) = \ln(6 - 2x^2)$

$g'(x) = \frac{4x}{6-2x^2}$

d) $y = (x^3 - 9)^6$

$\frac{dy}{dx} = 3x^2(6)(x^3-9)^5$

$= 18x^2(x^3-9)^5$

You will meet questions where the equation of the curve is not a polynomial. These seem harder but the key is that the methods to find the equation of the tangent or normal line are the same!

What is different is that you must use the rules from the Differentiation 1 handout when finding $\frac{dy}{dx}$.

Example 5

The function $f(x) = 5\sin x - 6x + 7$ is defined for all real values of x. Find the equation of the tangent to the curve $y = f(x)$ at the point $x = 0$.

$f'(x) = 5\cos x - 6$

$f'(0) = 5 - 6$

$= -1$

$y = 5\sin 0 - 6(0) + 7$

$y = 7$

$y - 7 = -1(x)$

$y - 7 = -x$

Example 6

Find the equation of the normal to the curve $y = \ln(3x - 2)$ at the point where $x = 1$, giving your answer in the form $ax + by + c = 0$

$\frac{dy}{dx} = \frac{3}{3x-2}$

$x = 1$

$\frac{dy}{dx} = \frac{3}{3(1)-2}$

$= \frac{3}{1}$

$= 3$

$y = \ln(3(1)-2)$

$= \ln 1$

$= 0$

$y = 3(x-1)$

$3x - y - 3 = 0$

Example 7

Find the equation of the tangent to the curve $y = e^{3-x}$ at the point where $x = 3$, giving your answer in the form $y = mx + c$

$\frac{dy}{dx} = -e^{3-x}$

$x = 3$

$\frac{dy}{dx} = -e^{0}$

$= -1$

$y = e^{3-3}$

$y = e^{0}$

$= 1$

$y - 1 = -(x-3)$

$y - 1 = -x + 3$

$y = -x + 4$

Example 8

A curve has equation $= \sqrt{x^2 + 3x}$, $x \geq 0$. Find the equation of the tangent to the curve at point (1, 2).

$y = (x^2 + 3x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(2x+3)(x^2+3x)^{-\frac{1}{2}}$

$y - 2 = \frac{5}{4}(x-1)$

at $x = 1$

$\frac{dy}{dx} = \frac{1}{2}(2(1)+3)(1^2 + 3(1))^{-\frac{1}{2}}$

$= \frac{1}{2}(5)(4)^{-\frac{1}{2}}$

$= \frac{1}{2}(5)(\frac{1}{2})$

$= \frac{5}{4}$

Example 9

Find the x co-ordinates of the points on the curve $y = (2x - 3)^2 - 12x$ where the tangents at these points are parallel to the line $y = 12x - 20$.

$y = 12x - 20$

$m = 12$

$\frac{dy}{dx} = 2(2)(2x-3) - 12$

$= 4(2x-3) - 12$

$= 8x - 24$

where $\frac{dy}{dx} = 12$

$12 = 8x - 24$

$36 = 8x$

$x = \frac{9}{2}$

Exam Questions

Edexcel C3 Jan 2013

1. The curve C has equation

$$y = (2x-3)^5$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(a) the value of w. $-32 = (2w-3)^5$

$-2 = 2w-3$

$1 = 2w \quad w = \frac{1}{2}$ **(2)**

(b) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants. **(5)**

$x = \frac{1}{2}$

$\frac{dy}{dx} = 10(2x-3)^4 \quad \frac{dy}{dx} = 10(1-3)^4 = 10(-2)^4 = 160$

$y+32 = 160(x-\frac{1}{2})$

$y+32 = 160x - 80$

$y = 160x - 112$

Edexcel C3 June 2010

2. A curve C has equation

$$y = \frac{3}{(5-3x)^2}, \quad x \neq \frac{5}{3}$$

$y = 3(5-3x)^2$

$y = 3(5-3(2))^2$

$y = 3(-1)^2$

$y = 3$

$\frac{dy}{dx} = -18(5-3x)$

$= 54x - 90$

$x = 2$

$= 108 - 90$

$= 18$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers. **(7)**

$y - 3 = 18(x-2) \quad y - 3 = 18x - 36$

$18x - y - 33 = 0$

Edexcel C3 June 2008

1. The point P lies on the curve with equation

$$y = 4e^{2x+1}.$$

The y-coordinate of P is 8.

(*a*) Find, in terms of ln 2, the x-coordinate of P. **(2)**

(*b*) Find the equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found. **(4)**

a) $8 = 4e^{2x+1}$

$2 = e^{2x+1}$

$\ln 2 = 2x+1$

$2x = \ln 2 - 1$

$x = \frac{1}{2}\ln 2 - \frac{1}{2}$

$\frac{dy}{dx} = 8e^{2x+1}$

$x = \frac{1}{2}\ln 2 - \frac{1}{2}$

$= 8e^{2(\frac{1}{2}\ln 2 - \frac{1}{2})+1}$

$= 8e^{\ln 2}$

$= 16$

$y - 8 = 16(x - (\frac{1}{2}\ln 2 - \frac{1}{2}))$

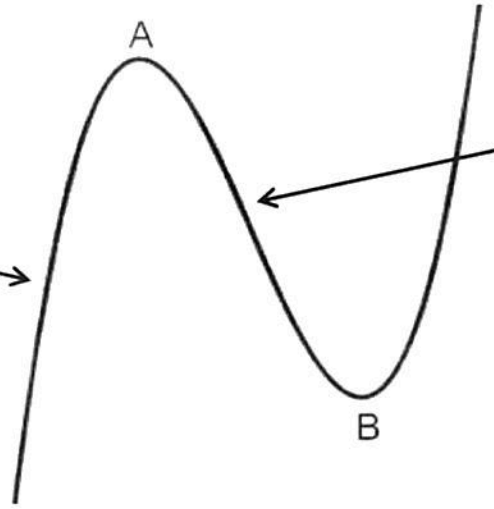
$y - 8 = -8\ln 2 + 8 + 16x$

$y = 16x - 8\ln 2 + 16$

Increasing and decreasing functions

A function is increasing if the gradient is positive

$$\frac{dy}{dx} > 0$$



A function is decreasing if the gradient is negative

$$\frac{dy}{dx} < 0$$

Example 10

Find the range of values of x for which the function $f(x) = x^2 - 8x - 7$ is decreasing

$f'(x) = 2x - 8$

where $f'(x) < 0$

$0 > 2x - 8$

$8 > 2x$

$x < 4$

Example 11

A curve C has equation $y = \frac{x^3+\sqrt{x}}{x}$ $x > 0$. Is the curve increasing or decreasing at $x = 4$?

$y = x^{-1}(x^3 + x^{\frac{1}{2}})$

$= x^2 + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 2x - \frac{1}{2}x^{-\frac{3}{2}}$

$x = 4$

$= 8 - \frac{1}{2}(8)^{-\frac{3}{2}}$

$= 8 - \frac{1}{2}\left(\frac{1}{4}\right) = 8 - \frac{1}{8}$

$= \frac{63}{8}$

$\frac{63}{8} > 0$ $\therefore$ increasing at $x = 4$

Example 12

Prove that the function $f(x) = 2 - x^3$ is never increasing.

$f'(x) = -3x^2$

x^2 is always positive $\therefore -3x^2$ is negative

$-3x^2 < 0$ $\therefore$ decreasing always

Example 13

The function $f(x) = 5\sin x - 6x + 7$ is defined for all real values of x.

Prove $f(x)$ is a decreasing function

$f'(x) = 5\cos x - 6$

$\cos x < |1|$

$5\cos x < |5|$

$|5| < 6$

$\therefore |{\pm}5| - 6$ is negative and decreasing

Example 14

Find the range of values of x for which $g(x) = x - 3\ln x$ is an increasing function

$g'(x) = 1 - \frac{3}{x}$

where $g'(x) > 0$

$1 - \frac{3}{x} > 0$

$1 > \frac{3}{x}$

$x > 3$

The Second Derivative

If you differentiate a function you get the (first) derivative, if you **differentiate again** you get the second derivative.

$$f(x) \to f'(x) \to f''(x)$$

or

$$y \to \frac{dy}{dx} \to \frac{d^2y}{dx^2}$$

where y is a function of x

The second derivative is the rate of change of the first derivative with respect to x.

Example 15

Find the first and second derivatives of the following functions, simplify where possible.

a) $y = 5x^3 + 2x^2 + \frac{1}{2}x - 7$

$\frac{dy}{dx} = 15x^2 + 4x + \frac{1}{2}$ $\quad \frac{d^2y}{dx^2} = 30x + 4$

b) $f(x) = 5\sqrt{x}(x + 2x^2)$

$F(x) = 5x^{\frac{1}{2}}(x + 2x^2)$

$= 5(x^{\frac{3}{2}} + 2x^{\frac{5}{2}})$

$= 5x^{\frac{3}{2}} + 10x^{\frac{5}{2}}$

$f'(x) = \frac{15}{2}x^{\frac{1}{2}} + 25x^{\frac{3}{2}}$ $\quad f''(x) = \frac{15}{4}x^{-\frac{1}{2}} + \frac{75}{2}x^{\frac{1}{2}}$

c) $y = \frac{4}{x^4} - \frac{1}{3x^3} + \frac{7}{2x}$

$y = 4x^{-4} - \frac{1}{3}x^{-3} + \frac{7}{2}x^{-1}$

$\frac{dy}{dx} = -16x^{-5} + x^{-4} - \frac{7}{2}x^{-2}$ $\quad \frac{d^2y}{dx^2} = 80x^{-6} - 4x^{-5} + 7x^{-3}$

d) $y = 3x - e^{5-2x}$

$\frac{dy}{dx} = 3 + 2e^{5-2x}$

$\frac{d^2y}{dx^2} = -4e^{5-2x}$

e) $h(x) = \ln(3x - 5)$

$h'(x) = \frac{3}{3x-5}$

$h'(x) = 3(3x-5)^{-1}$

$h''(x) = -3(3)(3x-5)^{-2}$

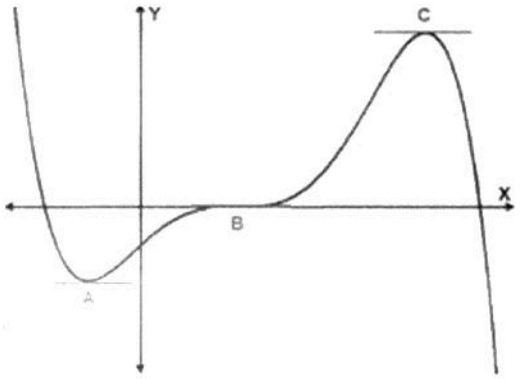
$= -9(3x-5)^{-2}$

f) $y = 5\cos 4x$

$\frac{dy}{dx} = -20\sin 4x$

$\frac{d^2y}{dx^2} = -80\cos 4x$

Stationary points and classification of their nature



At the points A and B and C the gradient of the function is zero, i.e. $\frac{dy}{dx} = 0$.

The points where $\frac{dy}{dx} = 0$ are called **stationary or turning points**.

The tangents to the curve at points A and B and C are horizontal lines parallel to the x axis.

A function could have zero, one or more stationary points.

To find the co-ordinates of a stationary point, you must:

- Find $\frac{dy}{dx}$ (or $f'(x)$, depending on the notation you are using)
- State that **"At stationary points, $\frac{dy}{dx} = 0$ (or $f'(x) = 0$)"**, set $\frac{dy}{dx} = 0$ and **solve for** x. These solutions are the x co-ordinates of any stationary points
- Substitute these values of x into the $original\ equation$ to find the y co-ordinates.

Example 16

Find the coordinates of any stationary points on the curve $f(x) = x^3 + 3x^2 - 9x - 10$.

$f'(x) = 3x^2 + 6x - 9$

At stationary points, $f'(x) = 0$

$0 = 3x^2 + 6x - 9$

$0 = (3x - 3)(x + 3)$

$x = +1, -3$

$F(1) = 1^3 + 3(1^2) - 9(1) - 10$

$= 1 + 3 - 9 - 10$

$= 4 - 19$

$= -15$

$(1, -15)$

$F(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 10$

$= -27 + 27 + 27 - 10$

$= 17$

$(-3, 17)$

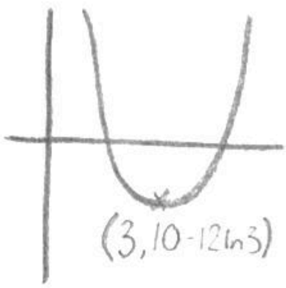
Example 17

A curve has the equation $y = \frac{2}{3}x^2 - 12\ln x + 4$

Find the co-ordinates of the stationary point of the curve, giving your answer in exact form.

$\frac{dy}{dx} = \frac{4}{3}x - \frac{12}{x}$

Extension: Sketch the curve, marking on the stationary point



At stationary points, $\frac{dy}{dx} = 0$

$0 = \frac{4}{3}x - \frac{12}{x}$

$\frac{12}{x} = \frac{4}{3}x$

$36 = 4x^2$

$36 = (2x)^2$ | $9 = x^2$

$6 = 2x$ | $x = 3$

$x = 3$

$y = \frac{2}{3}(3^2) - 12\ln 3 + 4$

$= \frac{18}{3} - 12\ln 3 + 4$

$= 6 + 4 - 12\ln 3$

$= 10 - 12\ln 3$

$(3, 10 - 12\ln 3)$

There are 3 types of stationary point: **maximum** (e.g.C), **minimum** (e.g.A) and **points of inflection** (e.g.B)

At a minimum point
The gradient changes as x increases from negative to zero (at A) to positive.
This means that the gradient is increasing
so $\frac{d^2y}{dx^2} > 0$

At a maximum point
The gradient changes as x increases from positive to zero (at C) to negative.
This means that the gradient is decreasing
so $\frac{d^2y}{dx^2} < 0$

NB
We will be covering points of inflection in the Differentiation 3 handout.

Example 18
A curve has equation $f(x) = x^3 + 3x^2 - 9x - 21$
a) Find the coordinates of the 2 stationary points.
b) Determine the nature of each stationary point.

a) $f'(x) = 3x^2 + 6x - 9$

at stationary points, $f'(x) = 0$

$0 = 3x^2 + 6x - 9$
$0 = x^2 + 2x - 3$
$0 = (x+3)(x-1)$
$x = -3, 1$

$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 21$
$= -27 + 27 + 27 - 21$
$= 6$ $(-3, 6)$

$f(1) = 1^3 + 3(1^2) - 9(1) - 21$
$= 1 + 3 - 9 - 21$
$= -26$
$(1, -26)$

b) $f''(x) = 6x + 6$

$f''(-3) = 6(-3) + 6$
$= -18 + 6$
$= -12 < 0 \therefore$ maximum point

$f''(1) = 6(1) + 6$
$= 12 > 0 \therefore$ minimum point

Example 19
A curve has equation $y = e^{2x} - 8x$
a) Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of x
b) The curve has a single turning point, P.
i) Find the x co-ordinates of P in an exact form, and show that its y co-ordinate is $4(1 - \ln 4)$
ii) Find the value of $\frac{d^2y}{dx^2}$ at P, and hence deduce the nature of the turning point.

a) $\frac{dy}{dx} = 2e^{2x} - 8$

$\frac{d^2y}{dx^2} = 4e^{2x}$

b) i) at turning point, $\frac{dy}{dx} = 0$

$0 = 2e^{2x} - 8$
$2e^{2x} = 8$
$e^{2x} = 4$
$2x = \ln 4$
$x = \frac{1}{2}\ln 4$
$x = \ln 2$

$y = e^{2\ln 2} - 8\ln 2$
$y = e^{\ln 4} - 4(2\ln 2)$
$y = 4 - 4\ln 4$
$y = 4(1 - \ln 4)$

ii) At P, $x = \ln 2$

$\frac{d^2y}{dx^2} = 4e^{2\ln 2}$
$= 4e^{\ln 4}$
$= 4(4)$
$= 16 > 0 \therefore$ minimum point

Exam Questions

Edexcel C2 January 2013

8. The curve C has equation $y = 6 - 3x - \frac{4}{x^3}$, $x \neq 0$

(a) Use calculus to show that the curve has a turning point P when $x = \sqrt{2}$ **(4)**

$\frac{dy}{dx} = -3 + 12x^{-4}$ at $x = \sqrt{2}$ $\frac{dy}{dx} = -3 + 12(\sqrt{2})^{-4}$ $= -3 + \frac{12}{4}$ $= -3 + 3 = 0$ ∴ turning point

(b) Find the x-coordinate of the other turning point Q on the curve. **(1)**

$\frac{dy}{dx} = 0$ $0 = -3 + \frac{12}{x^4}$ $3 = \frac{12}{x^4}$ $x^4 = 4$ $x = \pm\sqrt{2}$ ∴ Q, $x = -\sqrt{2}$

(c) Find $\frac{d^2y}{dx^2}$. $\frac{d^2y}{dx^2} = -48x^{-5}$ **(1)**

(d) Hence or otherwise, state with justification, the nature of each of these turning points P and Q. **(3)**

at P $\frac{d^2y}{dx^2} = -48(\sqrt{2})^{-5} = \frac{-48}{4\sqrt{2}} = -\frac{12}{\sqrt{2}} < 0$ ∴ Maximum

at Q $\frac{d^2y}{dx^2} = -48(-\sqrt{2})^{-5} = \frac{-48}{-4\sqrt{2}} = \frac{12}{\sqrt{2}} > 0$ ∴ minimum

OCR C1 June 2014

8 A curve has equation $y = 3x^3 - 7x + \frac{2}{x}$.

(i) Verify that the curve has a stationary point when $x = 1$. [5]

$\frac{dy}{dx} = 9x^2 - 7 - 2x^{-2}$ $x = 1$ $\frac{dy}{dx} = 9(1)^2 - 7 - 2(1)^{-2} = 9 - 9 = 0$ ∴ Stationary

(ii) Determine the nature of this stationary point. [2]

$\frac{d^2y}{dx^2} = 18x + 4x^{-3}$ $x = 1$ $\frac{d^2y}{dx^2} = 18(1) + 4(1)^{-3} = 22 > 0$ ∴ minimum

(iii) The tangent to the curve at this stationary point meets the y-axis at the point Q. Find the coordinates of Q. [2]

$x = 1$ $y = 3(1)^3 - 7(1) - \frac{2}{1} = 3 - 7 - 2 = -6$

$y + 6 = 0(x - 1)$
$y = -6$
$Q(0, -6)$

AQA C3 June 2006

5 (a) A curve has equation $y = e^{2x} - 10e^x + 12x$.

(i) Find $\frac{dy}{dx}$. $\frac{dy}{dx} = 2e^{2x} - 10e^x + 12$ *(2 marks)*

(ii) Find $\frac{d^2y}{dx^2}$. $\frac{d^2y}{dx^2} = 4e^{2x} - 10e^x$ *(1 mark)*

(b) The points P and Q are the stationary points of the curve.

(i) Show that the x-coordinates of P and Q are given by the solutions of the equation

$$e^{2x} - 5e^x + 6 = 0$$ *(1 mark)*

At SP, $\frac{dy}{dx} = 0$ $0 = 2e^{2x} - 10e^x + 12$
$0 = e^{2x} - 5e^x + 6$

(ii) By using the substitution $z = e^x$, or otherwise, show that the x-coordinates of P and Q are $\ln 2$ and $\ln 3$. *(3 marks)*

$z^2 - 5z + 6 = 0$ $(z - 2)(z - 3) = 0$ $z = 2, 3$ $e^x = 2$, $x = \ln 2$ $e^x = 3$, $x = \ln 3$

(iii) Find the y-coordinates of P and Q, giving each of your answers in the form $m + 12 \ln n$, where m and n are integers. *(3 marks)*

(iv) Using the answer to part (a)(ii), determine the nature of each stationary point. *(3 marks)*

iii) At P,
$y = e^{2\ln 2} - 10e^{\ln 2} + 12\ln 2$
$= e^{\ln 4} - 10(2) + 12\ln 2$
$= 4 - 20 + 12\ln 2$
$= -16 + 12\ln 2$

At Q,
$y = e^{2\ln 3} - 10\ln 3 + 12\ln 3$
$y = e^{\ln 9} - 10(3) + 12\ln 3$
$y = 9 - 30 + 12\ln 3$
$y = -21 + 12\ln 3$

iv) $x = \ln 2$
$\frac{d^2y}{dx^2} = 4e^{2\ln 2} - 10e^{\ln 2}$
$= 4e^{\ln 4} - 20$
$= 16 - 20$
$= -4 < 0$ ∴ maximum

$x = \ln 3$
$\frac{d^2y}{dx^2} = 4e^{2\ln 3} - 10e^{\ln 3}$
$= 4e^{\ln 9} - 10(3)$
$= 36 - 30$
$= 6 > 0$ ∴ minimum

Edexcel C2 June 2013

9. The curve with equation

$$y = x^2 - 32\sqrt{(x)} + 20, \quad x > 0$$

has a stationary point P.

a) $\frac{dy}{dx} = 2x - 16x^{-\frac{1}{2}}$

Use calculus

ASP, $\frac{dy}{dx} = 0$ $\quad 0 = 2x - \frac{16}{x^{\frac{1}{2}}}$

(a) to find the coordinates of P,

$\frac{16}{x^{\frac{1}{2}}} = 2x$

$8 = x^{\frac{3}{2}} \quad x = 4$

$y = 16 - 32(2) + 20$

$= 16 - 64 + 20$

$= -28$

$P(4, -28)$

(6)

(b) to determine the nature of the stationary point P.

(3)

b) $\frac{d^2y}{dx^2} = 2 + 8x^{-\frac{3}{2}}$

at P

$\frac{d^2y}{dx^2} = 2 + 8(4)^{-\frac{3}{2}}$

$= 2 + \frac{8}{8} = 3 > 0 \therefore$ minimum

AQA C2 June 2013

6 A curve has the equation

$$y = \frac{12 + x^2\sqrt{x}}{x}, \quad x > 0$$

(a) Express $\frac{12 + x^2\sqrt{x}}{x}$ in the form $12x^p + x^q$. *(3 marks)*

$x^{-1}(12 + x^{\frac{5}{2}})$

$= 12x^{-1} + x^{\frac{3}{2}}$

(b) (i) Hence find $\frac{dy}{dx}$. *(2 marks)*

$\frac{dy}{dx} = -12x^{-2} + \frac{3}{2}x^{\frac{1}{2}}$

(ii) Find an equation of the normal to the curve at the point on the curve where $x = 4$. *(4 marks)*

$x = 4$

$\frac{dy}{dx} = -12(4)^{-2} + \frac{3}{2}(4)^{\frac{1}{2}} = -\frac{12}{16} + \frac{3}{2}(2) = 3 - \frac{3}{4} = \frac{9}{4}$ $\quad \therefore m_{nor} = -\frac{4}{9}$

$y = \frac{12 + 16(2)}{4}$

$= \frac{44}{4} = 11$

$y - 11 = -\frac{4}{9}(x - 4)$

(iii) The curve has a stationary point P. Show that the x-coordinate of P can be written in the form 2^k, where k is a rational number. *(3 marks)*

At SP, $\frac{dy}{dx} = 0$

$0 = -12x^{-2} + \frac{3}{2}x^{\frac{1}{2}}$

$\frac{12}{x^2} = \frac{3}{2}x^{\frac{1}{2}} \quad 8 = x^{\frac{5}{2}}$

$x^{\frac{5}{2}} = 2^3 \quad x^5 = 2^6 \quad x = 2^{\frac{6}{5}}$

OCR C4 June 2013

4 The equation of a curve is $y = \cos 2x + 2\sin x$. Find $\frac{dy}{dx}$ and hence find the coordinates of the stationary points on the curve for $0 < x < \pi$. [6]

$\frac{dy}{dx} = -2\sin 2x + 2\cos x$

At SP, $\frac{dy}{dx} = 0$

$0 = -2\sin 2x + 2\cos x$

$0 = -2(\sin x\cos x + \sin x\cos x) + 2\cos x$

$0 = -2(2\sin x\cos x) + 2\cos x$

$4\sin x\cos x = 2\cos x$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

at $\frac{\pi}{6}$

$y = \cos\frac{\pi}{3} + 2\sin\frac{\pi}{6}$

$y = \frac{1}{2} + 1$

$= \frac{3}{2}$

$(\frac{\pi}{6}, \frac{3}{2})$

at $\frac{5\pi}{6}$

$y = \cos\frac{5\pi}{3} + 2\sin\frac{5\pi}{6}$

$y = \frac{1}{2} + 1$

$y = \frac{3}{2}$

$(\frac{5\pi}{6}, \frac{3}{2})$

Optimisation Problems

Some problems involve finding the maximum or minimum amount of a quantity (such as volume or surface area of a shape).

These are solved using a similar approach to stationary points.

Example 20

On a journey, the average speed of a car is ms^{-1}. For $v \geq 5$, the cost per km, C pence, of the journey is modelled by

$$C = \frac{160}{v} + \frac{v^2}{100}$$

Step 1: Differentiate the expression to find $\frac{dC}{dv}$ and set this equal to zero ($\frac{dC}{dv} = 0$ when C is either a maximum or a minimum)

$$\frac{dC}{dv} = -160v^{-2} + \frac{v}{50} \qquad 0 = -160v^{-2} + \frac{v}{50}$$

Step 2: Solve this to find the value of v that gives you the maximum or minimum value of C

$$\frac{160}{v^2} = \frac{v}{50}$$

$$v^3 = 8000 \quad v = 20$$

Step 3: To find the maximum or minimum value of C, substitute the value of v that you've found in step 2 into the original expression for C.

$$C = \frac{160}{20} + \frac{(20)^2}{100}$$

$$= 8 + \frac{400}{100} = 8 + 4 = 12$$

Step 4: To determine whether the quantity is a maximum or minimum (or to prove that it's one of them), find the **second derivative** and substitute your value of v from step 2 into the resulting second derivative expression.

If the **answer to the second derivative is > 0**, the value of C found is a **minimum**.

If the **answer to the second derivative is < 0**, the value of C found is a **maximum**.

$$\frac{d^2C}{dv^2} = 320v^{-3} + \frac{1}{50}$$

$$= \frac{320}{20^3} + \frac{1}{50} = \frac{320}{8000} + \frac{1}{50} = \frac{3}{50} > 0 \therefore \text{minimum}$$

Extension: Find the minimum cost of a 250 km journey.

$$12 \times 250 = 3000p$$

Exam Questions

Edexcel C2 January 2011

10. The volume V cm^3 of a box, of height x cm, is given by

$$V = 4x(5-x)^2, \quad 0 < x < 5$$

(a) Find $\frac{dV}{dx}$. $\frac{dV}{dx} = -8x(5-x) + 4(5-x)^2$
$= 8x^2 - 40x + 4(5-x)^2$ (4)

(b) Hence find the maximum volume of the box. (4)

$\frac{dV}{dx} = 0$ $\quad 0 = 8x^2 - 40x + 4(25 - 10x + x^2)$
$0 = 8x^2 - 40x + 100 - 40x + 4x^2$ $\quad 0 = 12x^2 - 80x + 100$ $\quad 0 = 6x^2 - 40x + 50$
$0 = 3x^2 - 20x + 25$

(c) Use calculus to justify that the volume that you found in part (b) is a maximum. (2)

$\frac{d^2V}{dx^2} = 16x - 40 - 8(5-x)$

at $x = \frac{5}{3}$
$\frac{d^2V}{dx^2} = 16\left(\frac{5}{3}\right) - 40 - 40 + 8\left(\frac{5}{3}\right)$
$= \frac{80}{3} - 80 + \frac{40}{3} = -40 < 0 \therefore$ max

$0 = (x-5)(3x-5)$
$x = 5, \frac{5}{3}$
$\therefore x = \frac{5}{3}$

AQA C1 January 2013

2 A bird flies from a tree. At time t seconds, the bird's height, y metres, above the horizontal ground is given by

$$y = \tfrac{1}{8}t^4 - t^2 + 5, \quad 0 \leqslant t \leqslant 4$$

(a) Find $\frac{dy}{dt}$. *(2 marks)*

$\frac{dy}{dt} = \frac{1}{2}t^3 - 2t$

(b) (i) Find the rate of change of height of the bird in metres per second when $t = 1$. *(2 marks)*

$t=1$ $\quad \frac{dy}{dt} = \frac{1}{2}(1)^3 - 2(1) = \frac{1}{2} - 2 = -\frac{3}{2}$ ms^{-1}

(ii) Determine, with a reason, whether the bird's height above the horizontal ground is increasing or decreasing when $t = 1$. *(1 mark)*

$-\frac{3}{2} < 0 \therefore$ decreasing

(c) (i) Find the value of $\frac{d^2y}{dt^2}$ when $t = 2$. *(2 marks)*

$\frac{d^2y}{dt^2} = \frac{3}{2}t^2 - 2$ $\quad t=2$ $\quad \frac{d^2y}{dt^2} = \frac{3}{2}(4) - 2 = 4$

(ii) Given that y has a stationary value when $t = 2$, state whether this is a maximum value or a minimum value. $4 > 0 \therefore$ minimum value *(1 mark)*

AQA C1 January 2006

7 The volume, V m^3, of water in a tank at time t seconds is given by

$$V = \tfrac{1}{3}t^6 - 2t^4 + 3t^2, \quad \text{for } t \geqslant 0$$

(a) Find:

(i) $\frac{dV}{dt}$; $= 2t^5 - 8t^3 + 6t$ *(3 marks)*

(ii) $\frac{d^2V}{dt^2}$. $= 10t^4 - 24t^2 + 6$ *(2 marks)*

(b) Find the rate of change of the volume of water in the tank, in m^3 s^{-1}, when $t = 2$. *(2 marks)*

$t=2$ $\quad \frac{dV}{dt} = 2(2)^5 - 8(2)^3 + 6(2) = 64 - 64 + 12 = 12$ m^{3}s^{-1}

(c) (i) Verify that V has a stationary value when $t = 1$. *(2 marks)*

$t=1$ $\quad \frac{dV}{dt} = 2(1) - 8(1) + 6 = 8 - 8 = 0 \therefore$ stationary

(ii) Determine whether this is a maximum or minimum value. *(2 marks)*

$t=1$ $\quad \frac{d^2V}{dt^2} = 10(1)^4 - 24(1)^2 + 6$
$= 16 - 24$
$= -8 < 0 \therefore$ maximum

Harder Optimisation Questions – forming the expression of related quantities

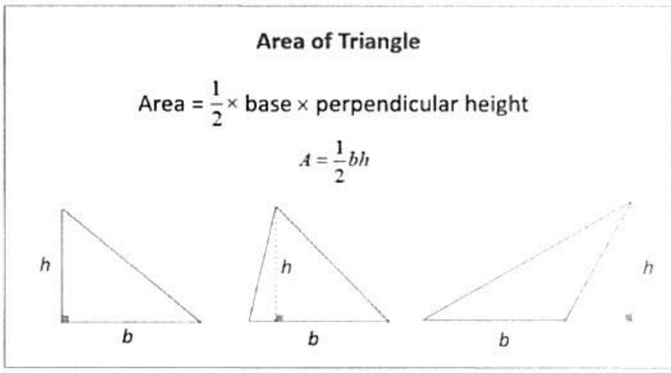
Sometimes you are not given the expression of the quantity that is at a maximum or minimum value. However, you are given information that you can utilise, alongside your knowledge and understanding of the quantity (such as volumes of certain shapes) to form the required expression.
You can then proceed with the previous method.

Example 21

In the right-angled triangle ABC the lengths of AB and BC vary so that sum is always $6\ cm$.

a) If the length of AB is x cm, write down, in terms of x, the length of BC.

b) Find the maximum area of the triangle.



$BC = 6 - x$

$A = \frac{1}{2}x(6-x)$

$= \frac{1}{2}(6x - x^2) = 3x - \frac{1}{2}x^2$

$\frac{dA}{dx} = 3 - x$

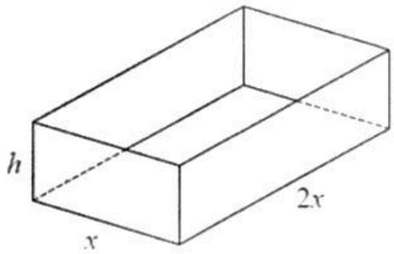
Maximum $\therefore \frac{dA}{dx} = 0$

$0 = 3 - x$

$x = 3$

Example 22

The diagram shows a container in the shape of an open-topped cuboid made from a thin metal sheet. The base of the container measures x cm by $2x$ cm, the height of the container is h cm and its volume is 288 cm³.



a) Show that the area of metal sheet used to make the container, A cm², is given by

$$A = 2x^2 + \frac{864}{x}$$

b) Use calculus to find the minimum value of A.

c) Prove that your value of A is a minimum.

$$Volume = length \times width \times height$$
$$Surface\ Area = Sum\ of\ areas\ of\ each\ face$$

a) $288 = 2x^2h$

$h = \frac{288}{2x^2}$

$A = 2x^2 + 2hx + 4hx$

$A = 2x^2 + 6hx$

$A = 2x^2 + 6\left(\frac{288}{2x^2}\right)x$

$A = 2x^2 + \frac{864}{x}$

b) $\frac{dA}{dx} = 4x - 864x^{-2}$

Minimum $\therefore \frac{dA}{dx} = 0$

$0 = 4x - 864x^{-2}$

$\frac{864}{x^2} = 4x \quad 216 = x^3 \quad x = 6$

c) $\frac{d^2A}{dx^2} = 4 + 1728x^{-3}$

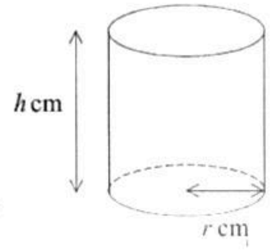
at $x = 6$

$= 4 + \frac{1728}{216}$

$= 12 \quad < 0 \therefore$ minimum

Example 23

The diagram shows a closed cylindrical container of radius r cm and height h cm. The external surface area of the container is $54\pi\ \text{cm}^2$ and the volume of the container is $V\ \text{cm}^3$.



Use calculus to find the stationary value of V, and determine whether this stationary value is a maximum or a minimum value.

$$Volume\ of\ Cylinder = \pi r^2 h$$

$$Surface\ area\ (closed) = 2\pi r^2 + 2\pi rh$$

$$54\pi = 2\pi r^2 + 2\pi rh$$

$$54 = 2r^2 + 2rh$$

$$27 = r^2 + rh$$

$$27 - r^2 = rh$$

$$h = \frac{27 - r^2}{r}$$

$$V = \pi r^2 \left(\frac{27 - r^2}{r}\right)$$

$$V = \pi r(27 - r^2)$$

$$V = \pi(27r - r^3)$$

$$\frac{dV}{dr} = \pi(27 - 3r^2)$$

Stationary $\therefore \frac{dV}{dr} = 0$

$$0 = \pi(27 - 3r^2)$$

$$27 = 3r^2$$

$$r^2 = 9$$

$$r = 3$$

$$V = \pi(9)\left(\frac{27 - 9}{3}\right)$$

$$V = 9\pi(6)$$

$$V = 54\pi$$

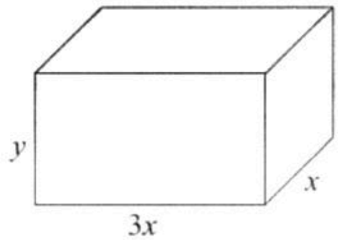
$$\frac{d^2V}{dr^2} = \pi(-6r)$$

at $r = 3$

$= -18\pi < 0 \therefore$ maximum value

Exam Questions
AQA C1 June 2012

4 The diagram shows a solid cuboid with sides of lengths x cm, $3x$ cm and y cm.



The total surface area of the cuboid is $32\,\text{cm}^2$.

a) i) $32 = 2(xy + 3x^2 + 3xy)$

$32 = 2xy + 6x^2 + 6xy$

$32 = 6x^2 + 8xy$

$16 = 3x^2 + 4xy$

ii) $V = 3x^2y$ $\quad 4xy = 16 - 3x^2$

$V = 3x^2\left(\frac{16-3x^2}{4x}\right)$ $\quad y = \frac{16-3x^2}{4x}$

$V = 12x - 9x^3$

(a) (i) Show that $3x^2 + 4xy = 16$. *(2 marks)*

(ii) Hence show that the volume, $V\,\text{cm}^3$, of the cuboid is given by

$\frac{dV}{dx} = 12 - \frac{27}{4}x^2$

$$V = 12x - \frac{9x^3}{4}$$ *(2 marks)*

(b) Find $\frac{dV}{dx}$. *(2 marks)*

(c) (i) Verify that a stationary value of V occurs when $x = \frac{4}{3}$. *(2 marks)*

$x = \frac{4}{3}$ $\quad \frac{dV}{dx} = 12 - \frac{27}{4}\left(\frac{4}{3}\right)^2 = 12 - 12 = 0 \therefore$ Stationary

(ii) Find $\frac{d^2V}{dx^2}$ and hence determine whether V has a maximum value or a minimum value when $x = \frac{4}{3}$. *(2 marks)*

$\frac{d^2V}{dx^2} = -\frac{54}{4}x$ $\quad$ at $x = \frac{4}{3}$

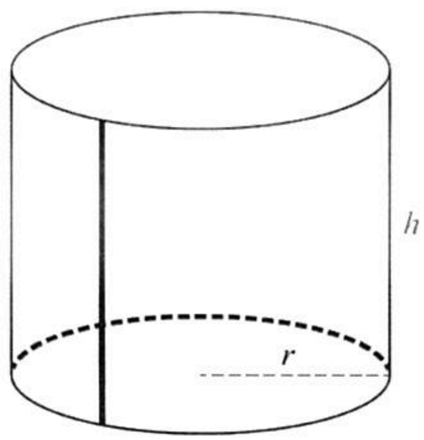
$= -\frac{27}{2}x$ $\quad \frac{d^2V}{dx^2} = -\frac{27}{2}\left(\frac{4}{3}\right)$

$= -\frac{108}{6} = -18 < 0 \therefore$ Maximum

AQA AS Paper 2 June 2018

11 Rakti makes open-topped cylindrical planters out of thin sheets of galvanised steel.

She bends a rectangle of steel to make an open cylinder and welds the joint. She then welds this cylinder to the circumference of a circular base.



$8000 = \pi r^2 h$

$h = \frac{8000}{\pi r^2}$ $L = \frac{8000}{\pi r^2} + 2\pi r$

$\frac{dL}{dr} = -\frac{16000}{\pi} r^{-3} + 2\pi$

The planter must have a capacity of 8000 cm^3

Welding is time consuming, so Rakti wants the total length of weld to be a minimum.

Calculate the radius, r, and height, h, of a planter which requires the minimum total length of weld.

Fully justify your answers, giving them to an appropriate degree of accuracy.

[9 marks]

minimum $\therefore \frac{dL}{dr} = 0$

$0 = -\frac{16000}{\pi} r^{-3} + 2\pi$ $\frac{16000}{\pi r^3} = 2\pi$ $r^3 = \frac{16000}{2\pi^2}$ $r = 9.32388\text{cm}$

$r = 9.32\text{cm}$

$8000 = \pi (9.32)^2 h$

$h = 29.29183775\text{cm}$

$= 29.3\text{cm}$

Edexcel C2 January 2012

8.

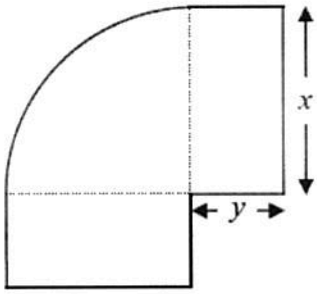


Figure 3

Figure 3 shows a flowerbed. Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to y metres.

Given that the area of the flowerbed is 4 m^2,

(a) show that

$$y = \frac{16 - \pi x^2}{8x}$$

(3)

a) $4 = 2xy + \frac{1}{4}\pi x^2$

$16 = 8xy + \pi x^2$

$8xy = 16 - \pi x^2$

$y = \frac{16 - \pi x^2}{8x}$

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation

$$P = \frac{8}{x} + 2x$$

(3)

b) $P = 2x + 4y + \frac{1}{4}\pi 2x$

$= 2x + 4y + \frac{1}{2}\pi x$

$= 2x + 4\left(\frac{16 - \pi x^2}{8x}\right) + \frac{\pi x}{2}$

$= 2x + \frac{8}{x} - \frac{\pi x^2}{2x} + \frac{\pi x}{2}$

$= \frac{8}{x} + 2x - \frac{\pi x}{2} + \frac{\pi x}{2}$

$= \frac{8}{x} + 2x$

(c) Use calculus to find the minimum value of P.

(5)

(d) Find the width of each rectangle when the perimeter is a minimum. Give your answer to the nearest centimetre.

(2)

c) $\frac{dP}{dx} = -8x^{-2} + 2$

minimum $\cdot$ $\frac{dP}{dx} = 0$

$0 = -8x^{-2} + 2$

$8x^{-2} = 2$

$\frac{1}{x^2} = \frac{1}{4}$ $x = 2$

d) $4 = 2(2)y + \frac{1}{4}\pi(2)^2$

$4 = 4y + \pi$

$4 - \pi = 4y$

$y = \frac{4 - \pi}{4}$

$y = 0.2146\text{m}$

$= 21.46\text{cm}$

$= 21\text{cm}$

Edexcel C2 June 2014

10.

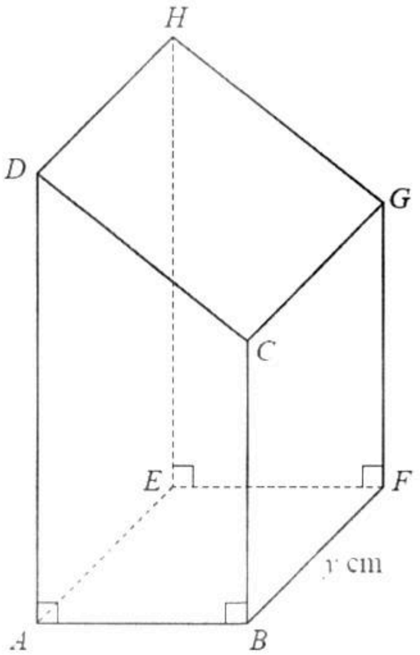


Figure 4

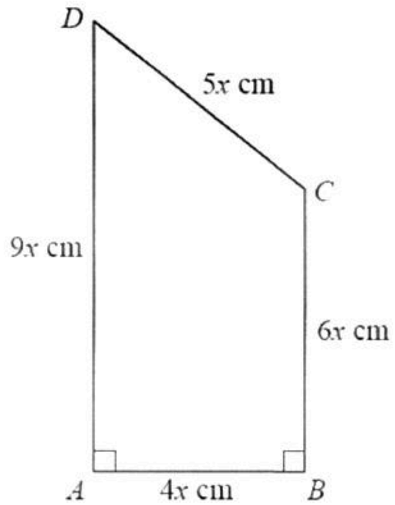


Figure 5

Figure 4 shows a closed letter box *ABFEHGCD*, which is made to be attached to a wall of a house.

The letter box is a right prism of length y cm as shown in Figure 4. The base *ABFE* of the prism is a rectangle. The total surface area of the six faces of the prism is S cm^2.

The cross section *ABCD* of the letter box is a trapezium with edges of lengths $DA = 9x$ cm, $AB = 4x$ cm, $BC = 6x$ cm and $CD = 5x$ cm as shown in Figure 5.

The angle $DAB = 90°$ and the angle $ABC = 90°$.

The volume of the letter box is 9600 cm^3.

(a) Show that

$$y = \frac{320}{x^2}$$

(2)

a) $9600 = \frac{1}{2}(6x+9x)4xy$

$9600 = \frac{1}{2}(15x)4xy$

$9600 = 30x^2y$

$y = \frac{320}{x^2}$

(b) Hence show that the surface area of the letter box, S cm^2, is given by

$$S = 60x^2 + \frac{7680}{x}$$

(4)

b) $S = \frac{1}{2}(15x)(4x)(2) + 6xy + 9xy + 5xy + 4xy$

$S = 60x^2 + 24xy$

$S = 60x^2 + 24x\left(\frac{320}{x^2}\right)$

$S = 60x^2 + \frac{7680}{x}$

(c) Use calculus to find the minimum value of S.

(6)

(d) Justify, by further differentiation, that the value of S you have found is a minimum.

(2)

c) $\frac{dS}{dx} = 120x - 7680x^{-2}$

minimum $\therefore \frac{dS}{dx} = 0$ $\quad 0 = 120x - 7680x^{-2}$

$\frac{7680}{x^2} = 120x$

$x^3 = 64 \quad x = 4$

d) $\frac{d^2S}{dx^2} = 120 + 15360x^{-3}$

$x = 4$

$\frac{d^2S}{dx^2} = 120 + \frac{15360}{64}$

$= 360 > 0 \therefore$ minimum