Pure Sector 4: Numerical Methods

Aims

- Locate roots of $f(x) = 0$ by considering changes of sign and understand when this method can fail.
- Solve equations approximately using simple iterative methods; draw associated cobweb and staircase diagrams.
- Solve equations using the Newton-Raphson method and other recurrence relations of the form *xn* + 1= g(*xn*)
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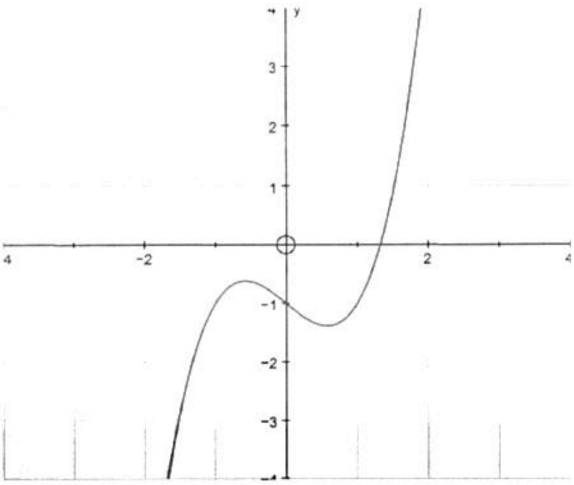
Bounds for Roots



If we are unable to solve an equation exactly, it is sometime useful to find approximate solutions. For example to solve $x^3 = x + 1$ we would rewrite it as $x^3 - x - 1 = 0$ and then try to factorise it. If we can't factorise then we could find approximate values for x where the graph crosses the x-axis. To do this we can sketch the graph to see how many roots (solutions) there are and roughly what the values of each root will be.

Locating a root allows us to say that a root lies between two values, these are called bounds. We can sketch a graph to do this or find bounds by change of sign. The change of sign method only works when the function is continuous. A continuous function does not change sign in an interval which contains an even number of roots.

> If $f(x)$ is **continuous** and changes sign between $x = b$ and $x = c$, then the equation $f(x) = 0$ has a root α, where $b < \alpha < c$.

Example 1

Show that the equation $x^3 + x^2 + 3 = 0$ has a root between -1.8 and -1.9.

$f(-1.8) = (-1.8)^3 + (-1.8)^2 + 3 = 0.408 > 0$

$f(-1.9) = (-1.9)^3 + (-1.9)^2 + 3 = -0.249 < 0$

$f(x)$ is continous and there is a change of sign $\therefore -1.9 < \alpha < -1.8$

Example 2

Find the integer bounds for any roots of the equation $x^3 - 1 = x$.

> You first need to rearrange so $f(x) = 0$. If you have two equation $f(x) - g(x) = 0$.

$f(x) = x^3 - 1 - x$

$f(1) = 1^3 - 1 - 1 = -1 < 0$

$f(2) = 2^3 - 1 - 2 = 5 > 0$

$f(x)$ is continous and there is a change of sign $\therefore 1 < \alpha < 2$

Example 3

The curve $y = 3^x$ intersects the curve $y = 10 - x^3$ at the point where $x = \alpha$. Show that α lies between 1 and 2.

$f(x) = 10 - x^3 - 3^x$

$f(1) = 10 - 1^3 - 3^1 = 6 > 0$

$f(2) = 10 - 2^3 - 3^2 = -7 < 0$

$f(x)$ is continous and there is a change of sign $\therefore 1 < \alpha < 2$

Example 4

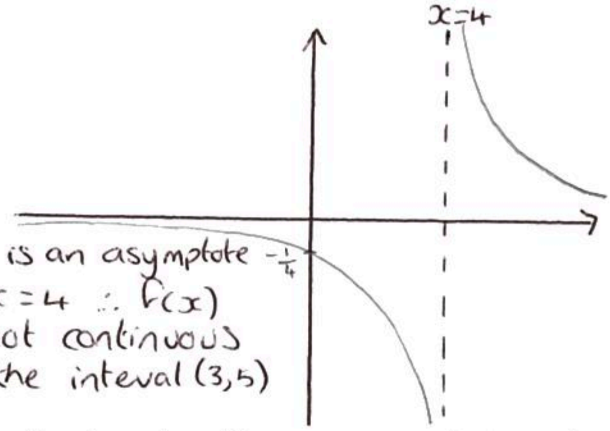
$$f(x) = \frac{1}{x-4}, x \neq 4$$

a) Show that there is a sign change across the interval (3,5)
b) Using a suitable sketch, explain why this method is not appropriate.

a) $f(3) = \dfrac{1}{3-4} = -1 < 0$

$f(5) = \dfrac{1}{5-4} = 1 > 0$

b)



There is an asymptote at $x = 4$ $\therefore$ $f(x)$ is not continuous in the inteval (3,5)

Iteration

If an equation has a solution α, then you can use an interactive formula written as $x_{n+1} = g(x_n)$ to solve the equation numerically. If the starting point x_1 is close to α, then then iterative formula produces a sequence $x_2, x_3, \ldots..$ which can converge to α.

Example 5

A sequence is defined by $x_{n+1} = \frac{x_n - 3}{1+2x_n}, x_1 = 2$. Find the values of x_2, x_3 and x_4.

$x_1 = 2$

$x_2 = \dfrac{2-3}{1+2(2)} = -0.2$

$x_3 = -\dfrac{16}{3}$

$x_4 = \dfrac{25}{29}$

Example 6

For each of the sequences, find the values of x_2, x_3 and x_4 and state whether the sequence is convergent or divergent. If the sequence is convergent state the limit, L.

a) $x_{n+1} = 1 + 2x_n$, $x_1 = 5$

b) $x_{n+1} = \frac{5x_n+1}{1+2x_n}, x_1 = 2$

a) $x_1 = 5$

$x_2 = 11$

$x_3 = 23$

$x_4 = 47$

$\therefore$ divergent

b) $x_1 = 2$

$x_2 = 2.2$

$x_3 = 2.222$

$x_4 = 2.224$

Convergent $L = 2.22$ (3sf)

Example 7

a) Show that $x^3 - 3x - 5 = 0$ can be rearranged into the form $x = \sqrt[3]{3x+5}$

$x^3 = 3x + 5$

$x = \sqrt[3]{3x+5}$

b) Using the iterative formula $x_{n+1} = \sqrt[3]{3x_n + 5}$, find a solution of $x^3 - 3x - 5 = 0$, correct to 3dp

$x_1 = 1$

$x_2 = 2$

$x_3 = 2.22398$

$x_4 = 2.26837$

$x_5 = 2.276967$

$x_6 = 2.27862$

$x_7 = 2.27894$ $\quad \therefore \ x = 2.279$ (3dp)

Example 8

a) Show that $f(x) = x^3 - 6x + 2$ has a root, α, between 0 and 1.

$f(0) = 0^3 - 6(0) + 2 = 2 > 0$

$f(1) = 1^3 - 6(1) + 2 = -3 < 0$

$f(x)$ is continuous and there is a change of sign $\therefore 0 < \alpha < 1$

b) Show that the equation $f(x) = 0$ can be rearranged into the form $x = \sqrt{\frac{6x-2}{x}}$ and comment on the suitability of the iterative formula $x_{n+1} = \sqrt{\frac{6x_n - 2}{x_n}}$ with $x_1 = 0.5$ for estimating α.

$x^3 - 6x + 2 = 0$

$x^3 = 6x - 2$

$x^2 = \frac{6x-2}{x}$

$x = \sqrt{\frac{6x-2}{x}}$

$x_1 = 0.5$

$x_2 = 1.414$

$x_3 = 2.141$

$x_4 = 2.251$

$\vdots$

$x_8 = 2.262$

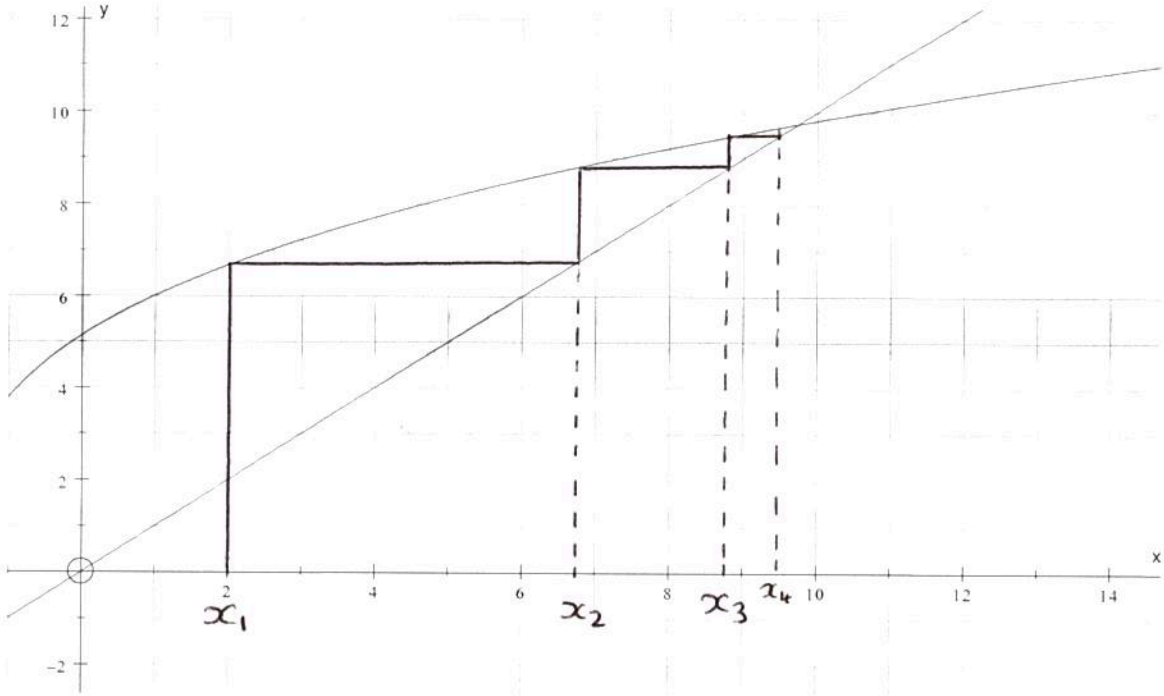
$\therefore$ the formula is not suitable as it converges on a different root not α.

Cobweb and Staircase Diagrams

Always start by going up/down to the CURVE then across to the LINE then CURVE etc...

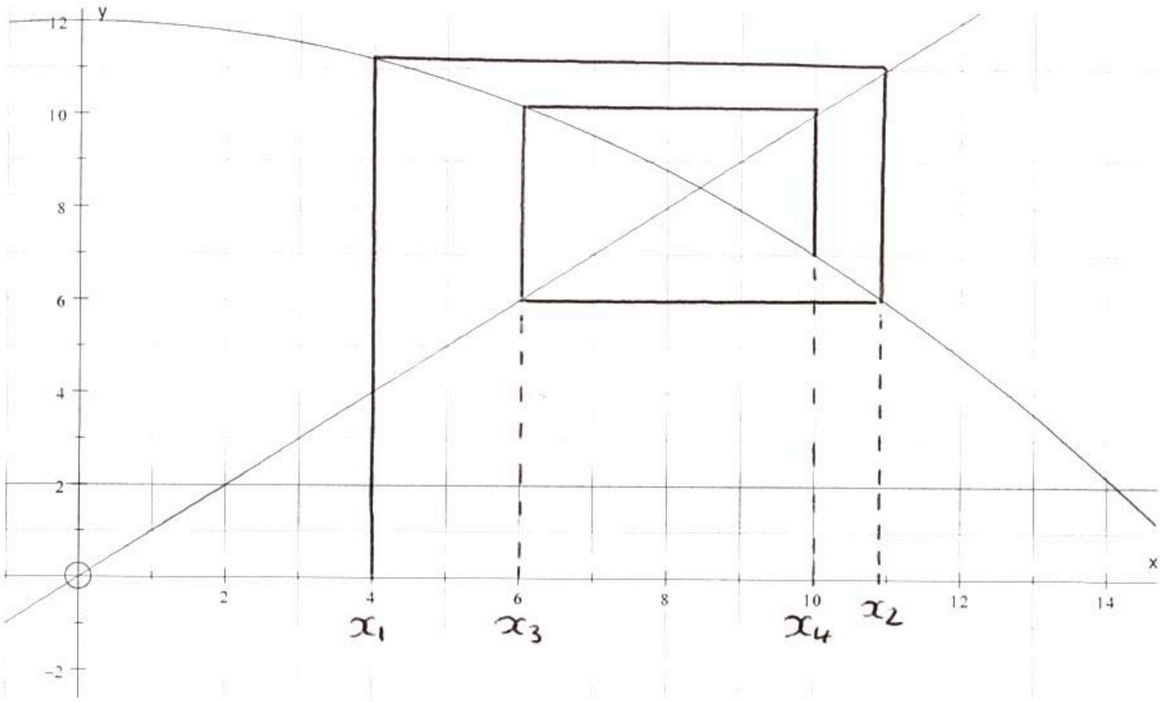
Example 8

Solve $x^3 - 81x - 135 = 0$, using the iterative formula $x_{n+1} = \sqrt[3]{81x_n + 135}$, $x_1 = 2$



Example 9

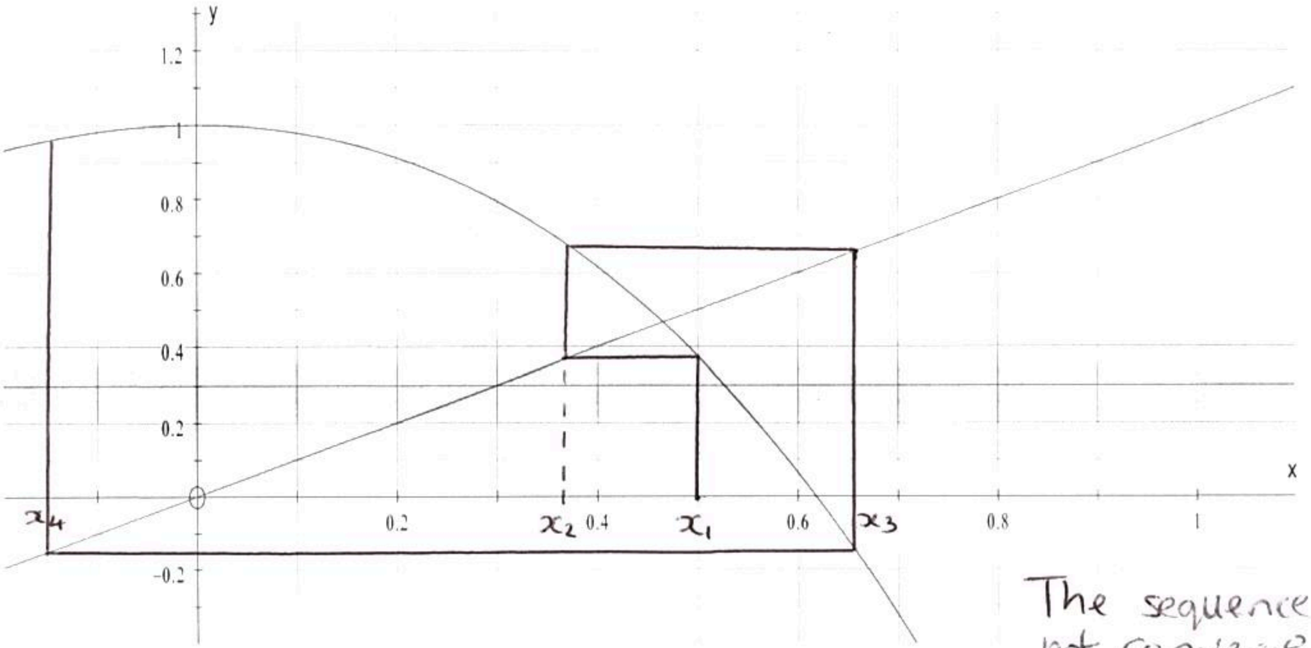
Find out whether this sequence will converge from a starting point of $x_1 = 4$



The sequence will converge from $x_1 = 4$

<u>Example 10</u>

Find out whether this sequence will converge from a starting point of $x_1 = 0.5$



The sequence does not converge.

Exam Question

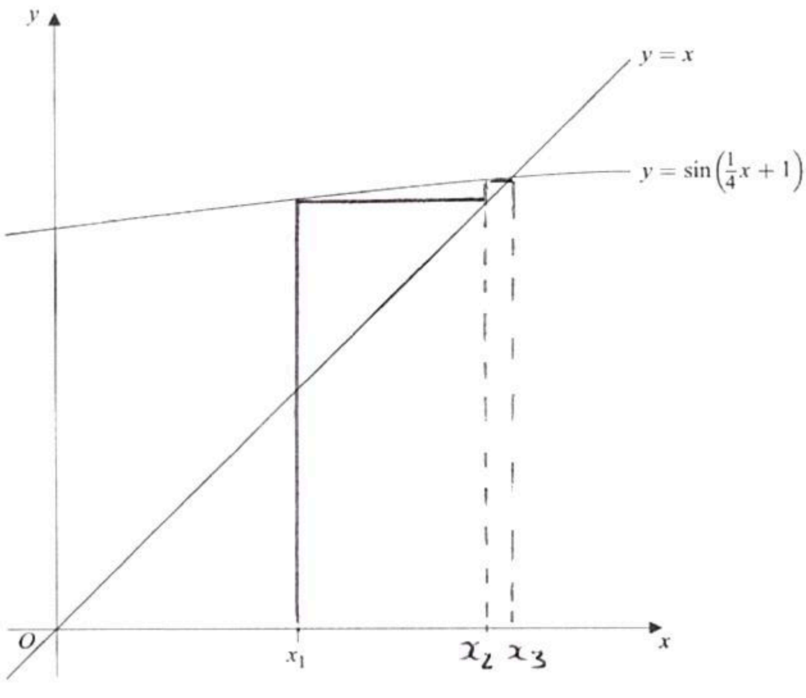
The equation $\sin^{-1} x = \frac{1}{4}x + 1$ can be rewritten as $x = \sin\left(\frac{1}{4}x + 1\right)$.

(i) Use the iteration $x_{n+1} = \sin\left(\frac{1}{4}x_n + 1\right)$ with $x_1 = 0.5$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

$x_2 = 0.844$
$x_3 = 0.936$

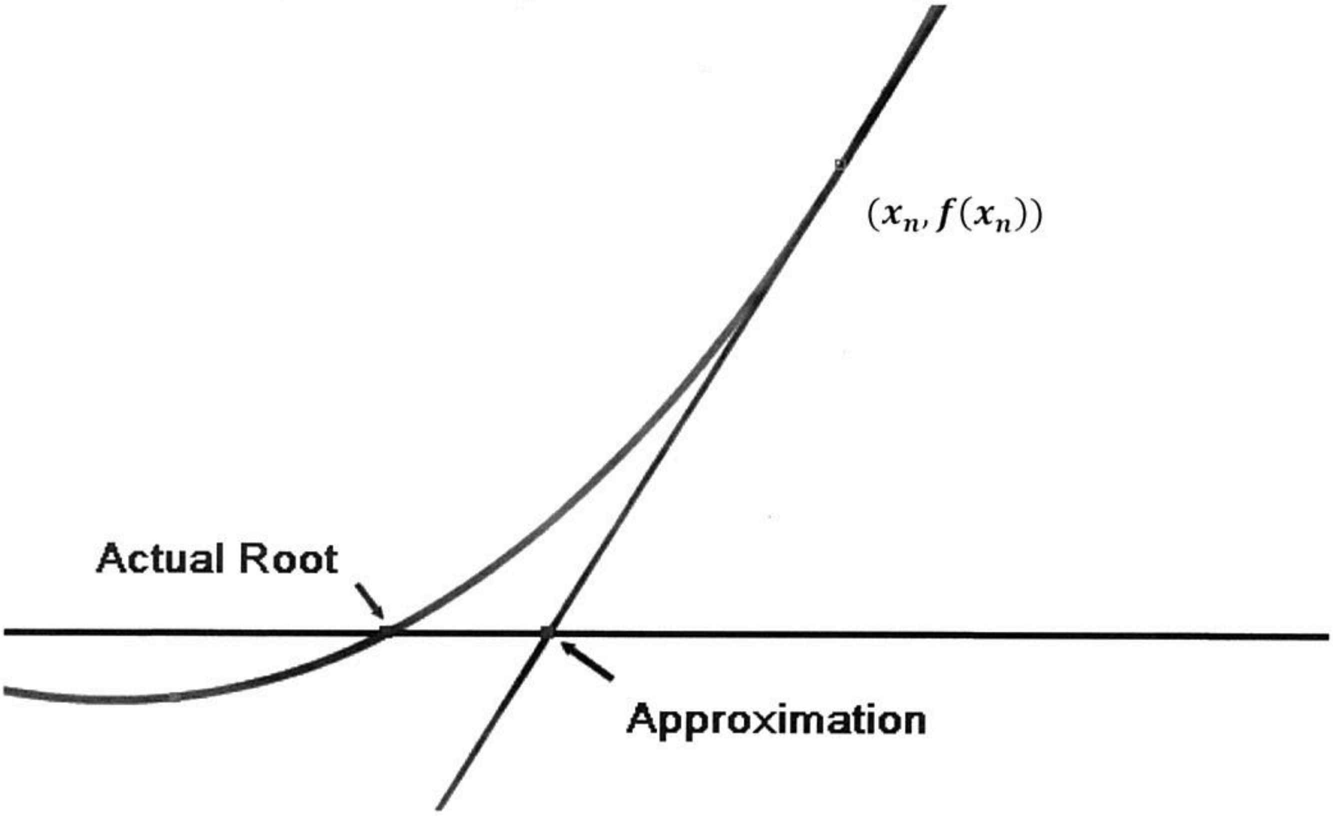
(ii) The sketch on **Figure 1** shows parts of the graphs of $y = \sin\left(\frac{1}{4}x + 1\right)$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*



Newton-Raphson

Newton-Raphson is another method to estimate the root of an equation. It uses the gradient of a curve at the point to find a tangent and then tells you where the root of that tangent is.



The **Newton-Raphson** iterative formula for solving $f(x) = 0$ is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method may fail when:

- The function cannot be differentiated.
- The first approximation is close to or on the x coordinate of a stationary point so the iteration diverge or converge to a different root.

Example 11

For $f(x) = x^3 - 7$, take $x_1 = 2$ as a first approximation to $f(x) = 0$, and use the Newton-Raphson method to find a second approximation, x_2.

$f(x) = x^3 - 7$

$f'(x) = 3x^2$

$x_1 = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{2^3 - 7}{3(2)^2}$$

$$= \frac{23}{12}$$

Example 12

a) For $f(x) = \sin x$, take $x_1 = 4$ as a first approximation to $f(x) = 0$, and use the Newton-Raphson method to find an approximation to π, correct to 6 decimal places.

$f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 4$

$x_2 = 4 - \dfrac{\sin 4}{\cos 4} = 2.8421\ldots$

$x_3 = 2.842 - \dfrac{\sin 2.842}{\cos 2.842} = 3.1508\ldots$

$x_4 = 3.1415925\ldots$

$x_5 = 3.1415926\ldots$

$x_6 = 3.1415926\ldots$

$\therefore \pi = 3.141593$ (6dp)

b) Explain why this would not work with $x_1 = 1$.

When starting from $x_1 = 1$ the sequence converges to the root 0 not π.

c) Explain why this would not work with $x_1 = \frac{\pi}{2}$

$\cos\left(\frac{\pi}{2}\right) = 0 \quad \therefore$

$x_2 = \dfrac{\pi}{2} - \dfrac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}}$ is undefined.

(cannot divide by 0)

Exam Questions

The equation

$$x^3 - x^2 + 4x - 900 = 0$$

has exactly one real root, α.

Taking $x_1 = 10$ as a first approximation to α, use the Newton–Raphson method to find a second approximation, x_2, to α. Give your answer to four significant figures.

(3 marks)

$f'(x) = 3x^2 - 2x + 4$

$x_1 = 10$

$$x_2 = 10 - \frac{10^3 - 10^2 + 4(10) - 900}{3(10)^2 - 2(10) + 4}$$

$$= \frac{700}{71}$$

$$= 9.859$$

A curve has equation $y = x^3 - 3x + 3$.

(a) Show that the curve intersects the x-axis at the point $(\alpha, 0)$ where $-3 < \alpha < -2$.

(b) A student attempts to find α using the Newton-Raphson method with $x_1 = -1$.

Explain why the student's method fails.

a) $f(-3) = (-3)^3 - 3(-3) + 3 = -15 < 0$

$f(-2) = (-2)^3 - 3(-2) + 3 = 1 > 0$

Change of sign and $f(x)$ is continuous $\therefore -3 < \alpha < -2$

b) $f'(x) = 3x^2 - 3$

When $f'(x) = 3(-1)^2 - 3 = 0 \quad \therefore \; x_2 = -1 - \dfrac{(-1)^3 - 3(-1) + 3}{3(-1)^2 - 3}$

is undefined.