Pure Sector 4: Numerical Methods

Aims

- Locate roots of $f(x) = 0$ by considering changes of sign and understand when this method can fail.
- Solve equations approximately using simple iterative methods; draw associated cobweb and staircase diagrams.
- Solve equations using the Newton-Raphson method and other recurrence relations of the form *xn* + 1= g(*xn*)
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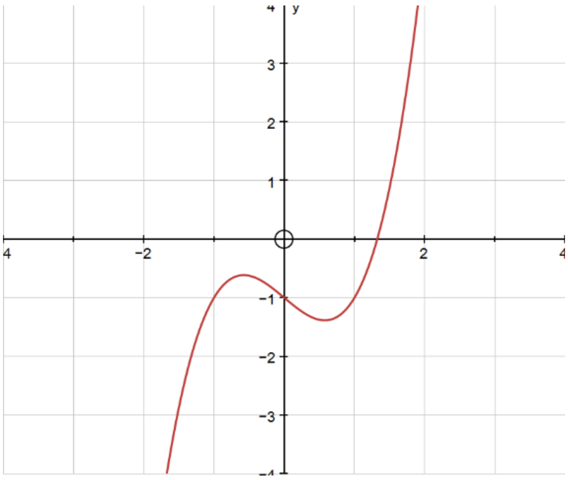
Bounds for Roots



If we are unable to solve an equation exactly, it is sometime useful to find approximate solutions. For example to solve $x^3 = x + 1$ we would rewrite it as $x^3 - x - 1 = 0$ and then try to factorise it. If we can't factorise then we could find approximate values for x where the graph crosses the x-axis. To do this we can sketch the graph to see how many roots (solutions) there are and roughly what the values of each root will be.

Locating a root allows us to say that a root lies between two values, these are called bounds. We can sketch a graph to do this or find bounds by change of sign. The change of sign method only works when the function is continuous. A continuous function does not change sign in an interval which contains an even number of roots.

> If $f(x)$ is **continuous** and changes sign between $x = b$ and $x = c$, then the equation $f(x) = 0$ has a root α, where $b < \alpha < c$.

<u>Example 1</u>

Show that the equation $x^3 + x^2 + 3 = 0$ has a root between -1.8 and -1.9.

<u>Example 2</u>

Find the integer bounds for any roots of the equation $x^3 - 1 = x$.

> You first need to rearrange so $f(x) = 0$. If you have two equation $f(x) - g(x) = 0$.

Example 3

The curve $y = 3^x$ intersects the curve $y = 10 - x^3$ at the point where $x = \alpha$. Show that α lies between 1 and 2.

Example 4

$$f(x) = \frac{1}{x - 4}, x \neq 4$$

a) Show that there is a sign change across the interval $(3,5)$
b) Using a suitable sketch, explain why this method is not appropriate.

Iteration

If an equation has a solution α, then you can use an interactive formula written as $x_{n+1} = g(x_n)$ to solve the equation numerically. If the starting point x_1 is close to α, then then iterative formula produces a sequence $x_2, x_3, \dots..$ which can converge to α.

Example 5

A sequence is defined by $x_{n+1} = \frac{x_n - 3}{1 + 2x_n}, x_1 = 2$. Find the values of x_2, x_3 and x_4.

Example 6

For each of the sequences, find the values of x_2, x_3 and x_4 and state whether the sequence is convergent or divergent. If the sequence is convergent state the limit, L.

a) $x_{n+1} = 1 + 2x_n,\ x_1 = 5$

b) $x_{n+1} = \frac{5x_n + 1}{1 + 2x_n}, x_1 = 2$

Example 7

a) Show that $x^3 - 3x - 5 = 0$ can be rearranged into the form $x = \sqrt[3]{3x+5}$

b) Using the iterative formula $x_{n+1} = \sqrt[3]{3x_n + 5}$, find a solution of $x^3 - 3x - 5 = 0$, correct to 3dp

Example 8

a) Show that $f(x) = x^3 - 6x + 2$ has a root, α, between 0 and 1.

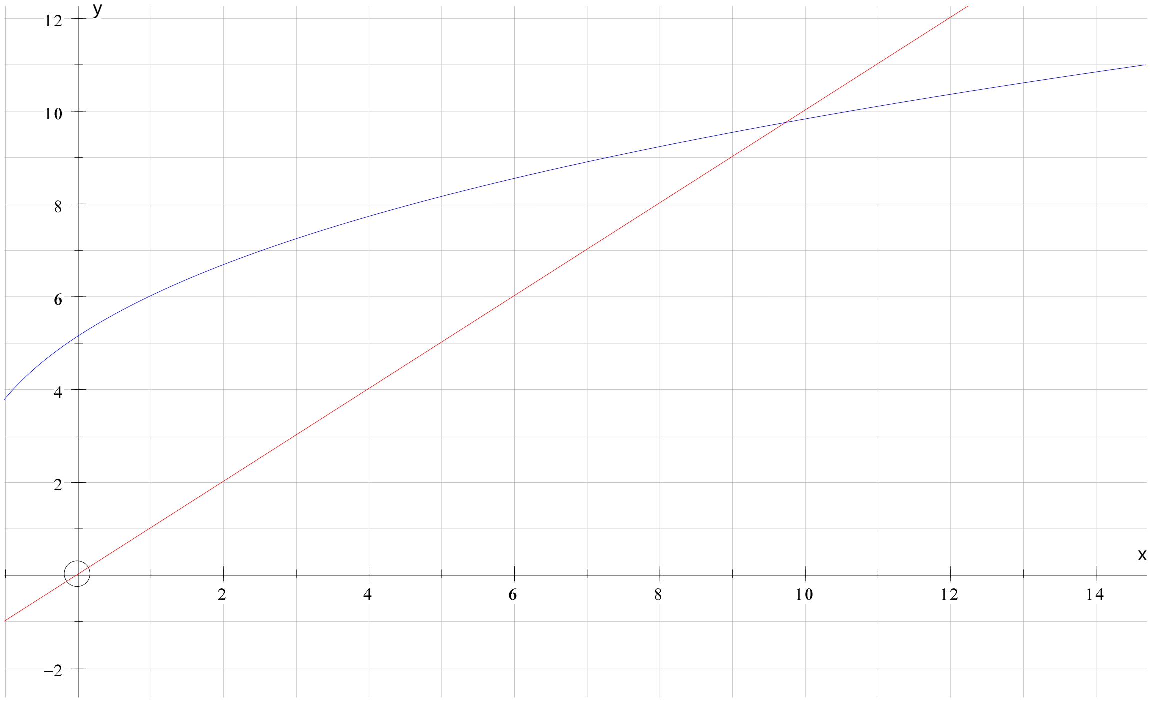
b) Show that the equation $f(x) = 0$ can be rearranged into the form $x = \sqrt{\frac{6x-2}{x}}$ and comment on the suitability of the iterative formula $x_{n+1} = \sqrt{\frac{6x_n-2}{x_n}}$ with $x_1 = 0.5$ for estimating α.

Cobweb and Staircase Diagrams

Always start by going up/down to the CURVE then across to the LINE then CURVE etc...

Example 8

Solve $x^3 - 81x - 135 = 0$, using the iterative formula $x_{n+1} = \sqrt[3]{81x_n + 135}$, $x_1 = 2$

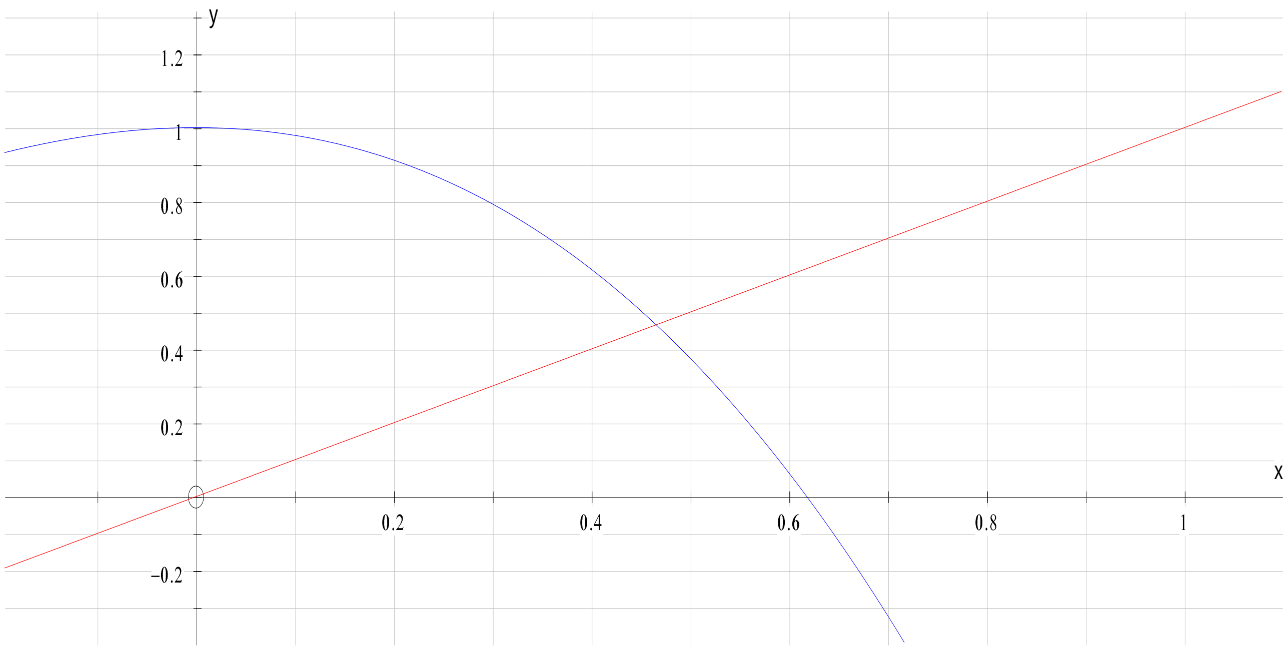


Example 9

Find out whether this sequence will converge from a starting point of $x_1 = 4$

<u>Example 10</u>

Find out whether this sequence will converge from a starting point of $x_1 = 0.5$



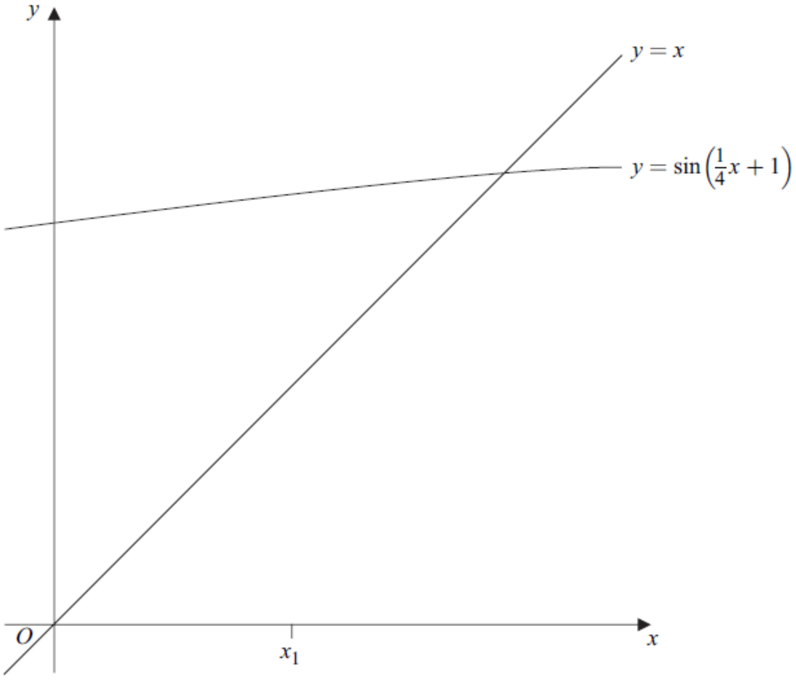
Exam Question

The equation $\sin^{-1} x = \frac{1}{4}x + 1$ can be rewritten as $x = \sin\left(\frac{1}{4}x + 1\right)$.

(i) Use the iteration $x_{n+1} = \sin\left(\frac{1}{4}x_n + 1\right)$ with $x_1 = 0.5$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

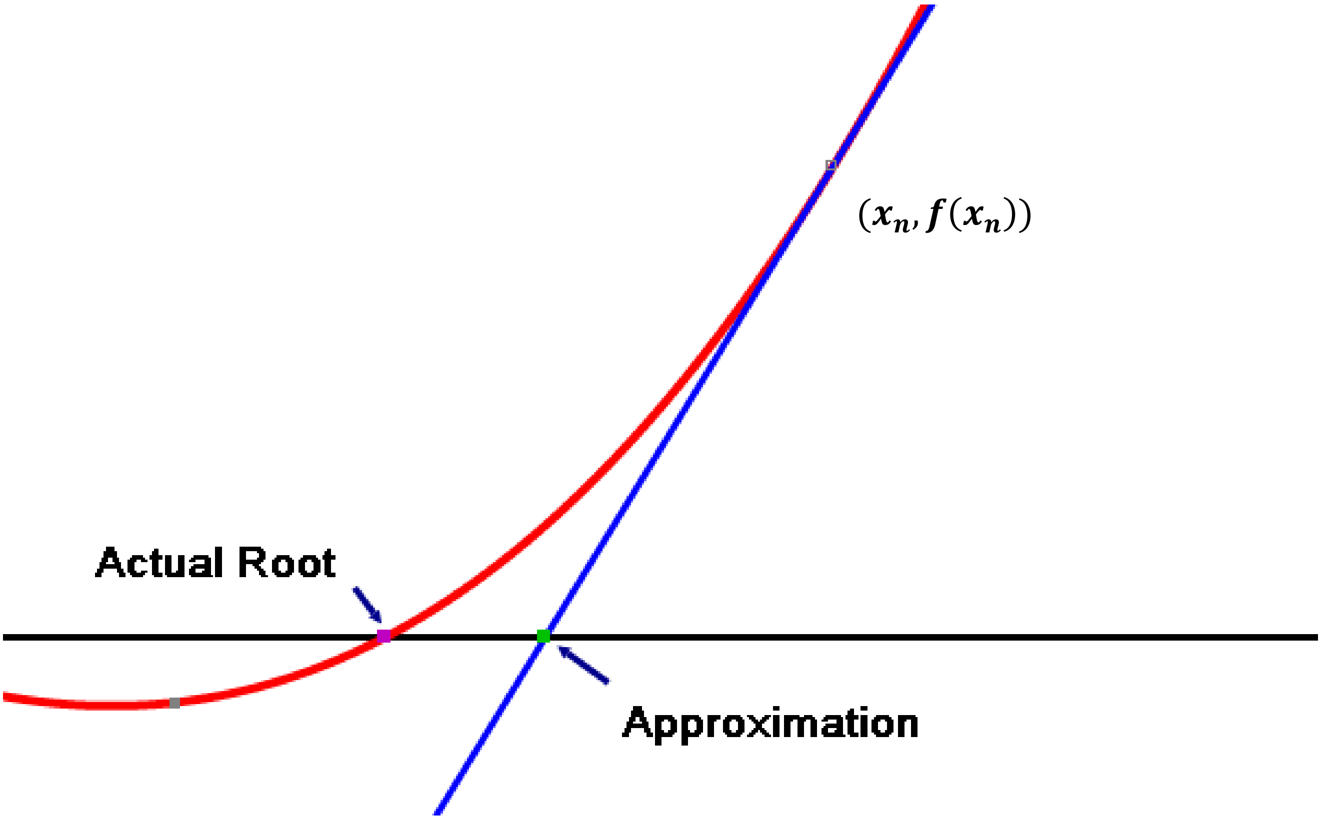
(ii) The sketch on **Figure 1** shows parts of the graphs of $y = \sin\left(\frac{1}{4}x + 1\right)$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*



Newton-Raphson

Newton-Raphson is another method to estimate the root of an equation. It uses the gradient of a curve at the point to find a tangent and then tells you where the root of that tangent is.



The **Newton-Raphson** iterative formula for solving $f(x) = 0$ is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method may fail when:

- The function cannot be differentiated.
- The first approximation is close to or on the x coordinate of a stationary point so the iteration diverge or converge to a different root.

<u>Example 11</u>

For $f(x) = x^3 - 7$, take $x_1 = 2$ as a first approximation to $f(x) = 0$, and use the Newton-Raphson method to find a second approximation, x_2.

Example 12

a) For $f(x) = \sin x$, take $x_1 = 4$ as a first approximation to $f(x) = 0$, and use the Newton-Raphson method to find an approximation to π, correct to 6 decimal places.

b) Explain why this would not work with $x_1 = 1$.

c) Explain why this would not work with $x_1 = \frac{\pi}{2}$

Exam Questions

The equation

$$x^3 - x^2 + 4x - 900 = 0$$

has exactly one real root, α.

Taking $x_1 = 10$ as a first approximation to α, use the Newton–Raphson method to find a second approximation, x_2, to α. Give your answer to four significant figures.

(3 marks)

A curve has equation $y = x^3 - 3x + 3$.

(a) Show that the curve intersects the x-axis at the point $(\alpha, 0)$ where $-3 < \alpha < -2$.

(b) A student attempts to find α using the Newton-Raphson method with $x_1 = -1$.

Explain why the student's method fails.