Pure Sector 1: Logarithms

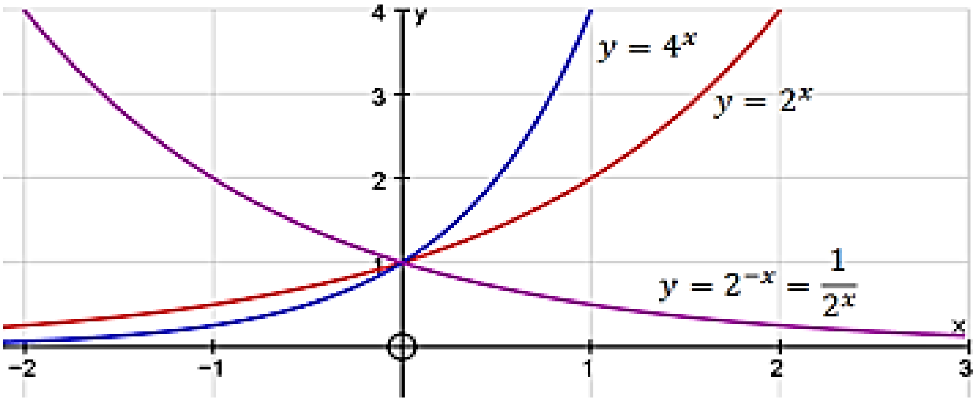
Aims

- To be able to convert between exponentials and logarithms
- To know and be able to use the laws of logarithms
- Solve equations in index form

Exponential functions

$y = 2^x$, $y = 4^x$, $y = 2^{-x}$

are exponential functions



Logarithms

- The logarithm of N to a base a is written as $log_a N$
- The log of a number N is the power that the base number is raised by to give N
- $N = a^x \quad \Leftrightarrow \quad x = \log_a N$
  Exponential/index form Logarithmic form

Where the logarithm has no base number the default is base 10. $\log a$ means $\log_{10} a$

<u>Example 1</u>

Write the following exponentials in logarithmic form:

a. $10^2 = 100$

b. $a^0 = 1$

c. $2^3 = 8$

d. $a^1 = a$

e. $9^{\frac{1}{2}} = 3$

f. $2^x = 27$

<u>Example 2</u>

Write the following logarithms in exponential form:

a. $\log_3 81 = 4$

b. $\log_7 7 = 1$

c. $\log_8 2 = \frac{1}{3}$

d. $\log_5 1 = 0$

e. $\log_{10} 1000 = 3$

f. $\log_5 x = 2$

The Exponential Function and the Natural Logarithm

The exponential function is written as e^x.

This is a particular base that is defined so that the y co-ordinate of a point on the curve is the same as the gradient of the curve at that point. This is true for every point on the curve $y = e^x$ and will be very useful when we start to look at tangents and normal lines of exponential curves (see your differentiation notes later).

The natural logarithm is the logarithm with base e. However, we write $\ln x$ rather than $\log_e x$

$\log_e A$ is written as: $\ln A$

Therefore, using the definition of logarithms

$$y = e^x \quad \Leftrightarrow \quad x = \ln y$$

$$\ln e = 1$$

$$e^{\ln x} = x = \ln e^x$$

Example 3

Simplify/solve each of the following:

a) $e^x = a$

b) $\ln x = 5$

c) $\ln y = x$

d) $e^{\ln 3}$

e) $\ln e^2$

f) $\ln \sqrt{e}$

g) $\ln e^{-4}$

h) $e^{2\ln 5}$

i) $e^{-3\ln 2}$

j) $\ln 2e^{-2}$

Laws of Logarithms

$$\log_a x + \log_a y = \log_a xy$$
$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$
$$\log_a x^n = n \log_a x$$

$$\ln x + \ln y = \ln xy$$
$$\ln x - \ln y = \ln \left(\frac{x}{y}\right)$$
$$\ln x^n = n ln\, x$$

NB
To use either of the first two rules, you must make sure that there are no coefficients in front of the logarithms

Example 4

Write the following as a single logarithm:

a. $2\log_3 3 + \log_3 4$

b. $2\log_3 3 + 2\log_3 4$

c. $4\log_{10} 2 - \log_{10} 6$

d. $3\ln 2 + \ln 3$

Example 5

Express the following in terms of $log_a x$, $log_a y$ and $log_a z$:

a. $log_a(xyz) =$

b. $log_a\left(\frac{xy}{z}\right) =$

c. $log_a x\sqrt{y}\ =$

d. $log_a \sqrt{xyz} =$

Example 6

Write $2 + \log_a 3$ as a single logarithm:

Example 7

Given that $3\ln x - 4\ln y = 1$, express e in terms of x and y:

Example 8

Solve the following equations:

a. $4\log_5 x = 7.2$

b. $2\ln x = 5$

c. $6\log_2 3 - \log_2 x = 9$

d. $2\log_2(x+7) = \log_2(x+5) + 3$

e. $\ln(x+2) - \ln(x+1) = 1$

f. $\log(x+4) = \log(6x) - \log 18 + \log(x-1)$

Exam Questions

AQA C2 January 2006

7 It is given that n satisfies the equation

$$2\log_a n - \log_a(5n-24) = \log_a 4$$

(a) Show that $n^2 - 20n + 96 = 0$. *(3 marks)*

(b) Hence find the possible values of n. *(2 marks)*

Edexcel C2 June 2013

7. (i) Find the exact value of x for which

$$\log_2(2x) = \log_2(5x+4) - 3$$

(4)

(ii) Given that

$$\log_a y + 3\log_a 2 = 5$$

express y in terms of a.
Give your answer in its simplest form.

(3)

6. Given that

$$2\log_2(x+15) - \log_2 x = 6$$

(a) Show that

$$x^2 - 34x + 225 = 0$$

(5)

(b) Hence, or otherwise, solve the equation

$$2\log_2(x+15) - \log_2 x = 6$$

(2)

Find, in exact form where appropriate, the solution of each of the following equations.

b) $\ln(2w+1) = 1 + \ln(w-1)$ **(5)**

Given that

$$\log_a(b^2) + 3\log_a y = 3 + 2\log_a\left(\frac{y}{a}\right)$$

express y in terms of a and b.

Give your answer in a form not involving logarithms. *(5 marks)*

By considering a sequence of three transformations, or otherwise, sketch the graph of

$$y = \ln(2-4x),\ x \in \mathbb{R},\ x \le \tfrac{1}{2}.$$

The sketch must include the coordinates of any points where the graph meets the coordinate axes and the equations of any asymptotes **(4)**

Solving Exponential Equations

Example 9

Solve the following equations for x, giving your answer in exact form where appropriate:

a. $5^x = 63$

b. $e^{x+1} = 8$

c. $3^{2-x} = 5^x$

d. $3^{x+1} = 27^{x-3}$

e. $e^{4x} > 20$

f. $3^{-2x} < 143$

Polynomials involving Exponentials and Logarithms

<u>Example 10</u>

Solve the following equations:

a. $3^{2x} - 4 \times 3^{x} + 3 = 0$ (use the substitution $y = 3^{x}$)

b. $5^{2x} - 7 \times 5^{x} + 10 = 0$

c. $e^{2x} - 2e^{x} - 15 = 0$

d. $e^{x} - 3 - 4e^{-x} = 0$

e. $(\ln x)^2 - 3\ln x + 2 = 0$

f. $2\ln x - 7 + \frac{6}{\ln x} = 0$

Exam Questions

AQA C2 January 2006

6 (a) Write each of the following in the form $\log_a k$, where k is an integer:

(i) $\log_a 4 + \log_a 10$; *(1 mark)*

(ii) $\log_a 16 - \log_a 2$; *(1 mark)*

(iii) $3\log_a 5$. *(1 mark)*

(b) Use logarithms to solve the equation $(1.5)^{3x} = 7.5$, giving your value of x to three decimal places. *(3 marks)*

(c) Given that $\log_2 p = m$ and $\log_8 q = n$, express pq in the form 2^y, where y is an expression in m and n. *(3 marks)*

(a) Sketch the graph of $y = (0.2)^x$, indicating the value of the intercept on the y-axis.
[2 marks]

(b) Use logarithms to solve the equation $(0.2)^x = 4$, giving your answer to three significant figures.

[2 marks]

(c) Describe the geometrical transformation that maps the graph of $y = (0.2)^x$ onto the graph of $y = 5^x$.

[1 mark]

Find the exact solution of the following equation

$$e^{x} - e^{-x} = \frac{3}{2}. \qquad \textbf{(5)}$$

Solve the following exponential equation, giving the answer correct to 3 s.f.

$$4^{y} - 3\left(2^{y}\right) - 10 = 0. \qquad \textbf{(6)}$$

Challenge

Find, in its simplest form, the solution of the following simultaneous equations

$$e^{2x+4} = e\,y$$

$$\ln y = 4x + 6. \qquad \textbf{(6)}$$

Find algebraically the exact solutions to the equations

(a) $\ln(4 - 2x) + \ln(9 - 3x) = 2\ln(x + 1), \qquad -1 < x < 2$

(5)

(b) $2^x e^{3x+1} = 10$

Give your answer to (b) in the form $\dfrac{a + \ln b}{c + \ln d}$ where a, b, c and d are integers.

(5)