Statistics Sector 1: Hypothesis Tests for a Proportion

Aims

- Carry out a hypothesis test for a proportion using a binomial distribution, using critical values or p-values.
- Interpret the findings of a hypothesis test in context.

Definitions

Null Hypothesis (H_0):
an assertion that a parameter in a statistical model takes a particular value, and is assumed true until experimental evidence suggests otherwise.

Alternative Hypothesis (H_1):
expresses the way in which the value of a parameter may deviate from that specified in the null hypothesis, and is assumed true when the experimental evidence suggests that the null hypothesis is false.

Test statistic:
a function of a sample of observations which provides a basis for testing the validity of the null hypothesis.

Critical region:
the null hypothesis is rejected when a calculated value of the test statistic lies within this region.

Acceptance region:
the null hypothesis is accepted when a calculated value of the test statistic lies within this region.

Critical value:
the value which determines the boundary of the critical region.

Significance level:
the size of the critical region. It is the probability of incorrectly rejecting the null hypothesis.

One-tailed test:
the critical region is located wholly at one end of the sampling distribution of the test statistic.
H_1 involves $<$ or $>$ but not both.

Two-tailed test:
the critical region comprises areas at both ends of the sampling distribution of the test statistic.
H_1 involves $\neq$.

Exact Test for the Proportion, p, of a Binomial Distribution

1) State the hypothesis.
   $H_0: p = value$
   $H_1: p < value,\ p > value\ or\ p \neq value$

   Where p is the population proportion of………
   Alternatively write the hypothesis in words

2) Write down the observed data X out of n and the distribution of X under H_0

3) Declare the hypothesis test.
   A one/two tailed test of the population proportion at the % significance level, using $X \sim B(n,p)$ where X is…..

4) The test statistic is X

5) Use either critical values or probability to decide whether to accept or reject the H_0

   Critical Values
   If X falls in critical region then reject H_0

   Probability
   Calculate $P(a\ value\ as\ or\ more\ extreme\ than\ X)$ using $X \sim B(n,p)$

   If $H_1: p <$ then $P(X \leq\ obs\ data)$
   If $H_1: p >$ then $P(X \geq\ obs\ data)$

   If $H_1: p \neq$: then look to see if X is less or greater than than the mean np
   If X less then the mean then $P(X \leq\ obs\ data)$
   If X greater than the mean then $P(X \geq\ obs\ data)$

6) Compare this probability with the significance level or significance level divided by two if two tailed.
   If probability is less than the significance level then reject H_0.

7) Make a conclusion in the context of the original question.

Example 1

Until recently, an average of 60 out of every 100 patients has survived a particularly severe infection. When a new drug is administered to a random sample of 15 patients with the infection, 12 survive.

Does this provide evidence, at the 10% level of significance, that the new drug is more effective than existing treatments?

Hypotheses

$H_0: p=0.6$
$H_1: p>0.6$ (one tailed)

Observed Data

12 out of 15 survive use $X \sim B(15, 0.6)$

Test Statistic

$X=12$

P (a value as or more extreme than the observed data)

$$P(X \geq 12) = 1 - P(X \leq 11)$$
$$= 1 - 0.9095$$
$$= 0.0905 < 0.1 \therefore \text{Reject } H_0$$

Compare this with the significance level

Conclusion

There is significant evidence to suggest that the new drug is more effective than existing treatments, at the 10% significance level.

Example 2

Tulip bulbs are sold in packets of 50 mixed colours: red, yellow and white. Random samples of bulbs are obtained and put into packets. A packet is selected at random and the number of white tulips resulting is found to be 15.

Investigate, at the 5% level of significance, the claim that 20% of the bulbs sold in such packets result in white tulips.

Hypotheses

$H_0: p = 0.2$

$H_1: p \neq 0.2$ (two tailed)

Observed Data

15 out of 50 turned white, use $X \sim B(50, 0.2)$

Test Statistic

$X = 15$

P (a value as or more extreme than the observed data)

Two tailed – mean $= np = 50 \times 0.2 = 10$ T.S. Mean $15 > 10$ $\therefore$ use $P(x \geq 15)$

$P(x \geq 15) = 1 - P(x \leq 14)$

$= 0.0607$

Compare this with the significance level – two tailed so $\frac{5\%}{2} = 2.5\%$

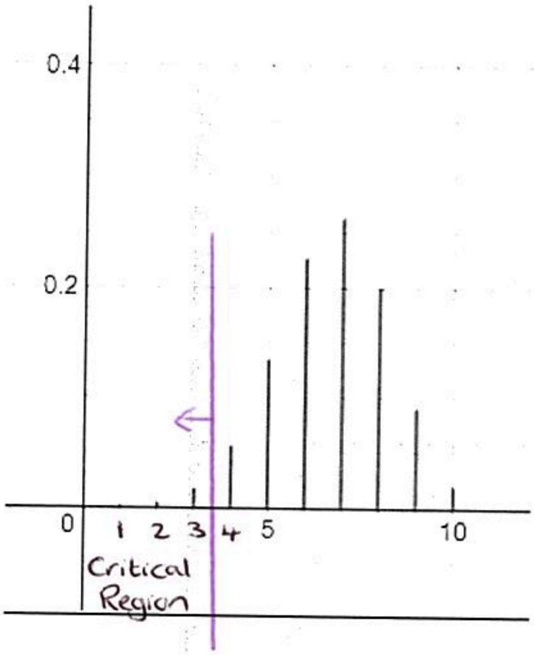
$0.0607 > 0.025$ $\therefore$ Accept H_0

Conclusion

There is not significant evidence to suggest that the proportion of bulbs resulting in white tulips is not 20%, at the 5% sig level.

Example 3

Determine the critical region for $H_0: X \sim \mathrm{B}(10, 0.67)$ for a lower one-tailed hypothesis test at the 5% significance level.



Dist - BINM - InvB

Area = 0.05

n = 10

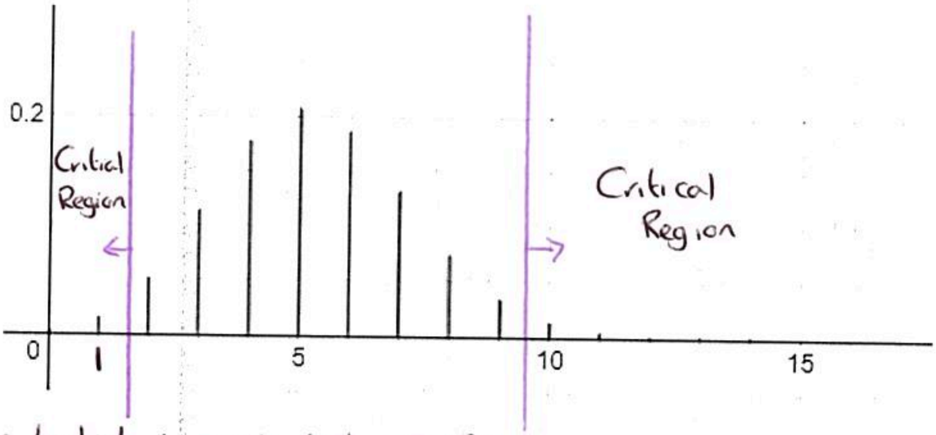
p = 0.67 gives xINV = 4

$P(x \le 4) = 0.0732 \quad > 5\%$

$P(x \le 3) = 0.0185 \quad < 5\%$

Example 4

Determine the critical region for $H_0: X \sim \mathrm{B}(17, 0.31)$ for a two-tailed hypothesis test at the 5% significance level.



Two tailed ∴ each tail = 2.5%

Lower Tail

$P(x \le 2) = 0.0657 \quad > 2.5\%$

$P(x \le 1) = 0.0157 \quad < 2.5\%$

InvB Area = 0.025

n = 17

p = 0.31

Upper Tail

$P(x \ge 9) = 1 - P(x \le 8) = 0.0492 \quad > 2.5\%$

$P(x \ge 10) = 1 - P(x \le 9) = 0.0162 \quad < 2.5\%$

InvB Area = 0.975

n = 17

p = 0.31

Example 5

During busy periods at a call centre, callers either get through to an operator immediately or are put on hold. A large survey revealed that 20% of callers were put on hold.

The call centre increases the number of operators with the intention of reducing the proportion of callers who are put on hold. A hypothesis test is carried out at the 5% level, to examine whether the centre has been successful in increasing the proportion of callers to get through immediately.

After the change, a random sample of 10 callers is taken and the number who get through immediately is recorded. By considering the critical region, what conclusions can you draw and how would you suggest improving the hypothesis test?

Hypotheses ← callers who get through immediately.

$H_0: p = 0.8$

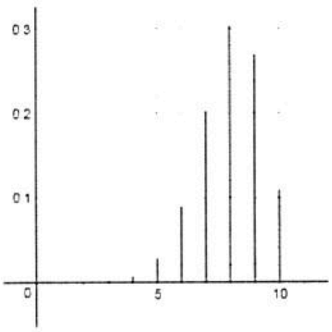
$H_1: p > 0.8$ (one tailed)

Observed Data

Not given. Use $X \sim B(10, 0.8)$

~~Test Statistic~~

Critical Region



InvN

area = 0.95

$n = 10$

$p = 0.8$ $\quad$ x Inv = 10

$P(x \geq 10) = P(x = 10) = 0.1074 > 0.05$,

so there is no critical region.

Conclusion

There is no critical region so the test would always result in accepting H_0.

This indicates the sample is too small and the test should be repeated with a larger sample.

Exam Questions

A newspaper article claimed that fewer than 30 per cent of cyclists stop at zebra crossings when a pedestrian is waiting to use the crossing.

Sabrina collects data at several zebra crossings. She records 20 occasions at random when a cyclist approaches a crossing whilst a pedestrian is waiting to use it. On 4 of these occasions, the cyclist stops.

(a) Carry out a hypothesis test to investigate the claim made in the newspaper article. Use an exact binomial distribution and the 10% significance level. *(6 marks)*

(b) Give one reason why a binomial distribution may not be an appropriate model in this situation. *(1 mark)*

a) $H_0: p = 0.3$

$H_1: p < 0.3$ (one tailed)

4 out of 20 cyclists stop, use $X \sim B(20, 0.3)$

$P(X \leq 4) = 0.2375 > 0.1 \therefore$ Accept H_0

There is no significant evidence to suggest the proportion of cyclists stopping is fewer than 30%, at the 10% sig level.

b) More than one cyclist may get to the crossing at the same time.

Probability of stopping may be affect by things such as weathe etc.

A bed manufacturer claims that 15 per cent of adults sleep on average for at least 8 hours each night. Rowan is asked by his teacher to investigate her belief that a higher percentage of teenagers sleep on average for at least 8 hours each night. Rowan finds that, of the 14 teenagers in his A-level Statistics class, 4 of them report an average of at least 8 hours sleep each night.

(i) Use Rowan's data and an exact distribution to test, at the 5% significance level, whether the corresponding population percentage for teenagers is higher than 15 per cent. *(6 marks)*

$H_0: p = 0.15$

$H_1: p > 0.15$ (one tailed)

4 out of 14 teenagers, use $X \sim B(14, 0.15)$

$P(X \geq 4) = 1 - P(X \leq 3)$

$= 1 - 0.8535$

$= 0.1465 > 0.1 \therefore$ Accept H_0

There is no significant evidence to suggest that the population percentage for teenagers sleeping on average for at least 8 hour each night is highe than 15%, at the 5% sig level.

James is a guitarist in a rock band which is about to start a 14-night tour. James usually uses Britepick guitar strings, which he changes before each performance. The thinnest string on a guitar, the top-E string, is the one most likely to break and, for James, the probability that this happens during a 1-hour performance is 0.02.

James is thinking of using Pluckwell strings rather than Britepick strings in the future and has bought some Pluckwell top-E strings to use each night of the 14-night tour. He finds that he breaks a top-E string during the band's 1-hour performance on 2 of these 14 nights.

(i) Use a binomial distribution to investigate, at the 5% level of significance, whether Pluckwell top-E strings are more likely to break than Britepick top-E strings. *(5 marks)*

(ii) Name one other factor besides reliability that James should consider when deciding whether to change his brand of strings. *(1 mark)*

(iii) Irrespective of the data collected during the tour, explain why it would not have been possible to investigate, at the 5% level of significance, whether Pluckwell top-E strings are less likely to break than Britepick top-E strings. You should support your explanation with an appropriate binomial probability. *(2 marks)*

i) $H_0: p = 0.02$

$H_1: p > 0.02$

Breaks 2 out of 14 nights, use $X \sim B(14, 0.02)$

$P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - 0.96896\ldots$

$= 0.0310 < 0.05$ ∴ Reject H_0

There is significant evidence to suggest Pluckwell strings are more likely to break than Britepick strings, at the 5% sig level.

ii) Price, avaliability, etc.

iii) $P(X=0) = 0.7536 > 0.05$ ∴ you would always Accept H_0 so there is no critical region.