Pure Sector 3: Functions

Aims:

- Decide whether a mapping is function and whether it is one-one or many-one.
- Find composite functions given two or more functions.
- Find the inverse of a function including the range and domain.
- Sketch graphs of inverse functions.
- Understand the conditions for a function to have an inverse.

Definitions

A mapping is a rule for x (the input). E.g. $x \to 4x+1, x \to x^2-4, x \to \pm\sqrt{x}$

A function is a mapping in which there is only one possible output/answer for each input. A function may be one to one or many to one. So in the examples above:

NB Many is defined as two or more.

__________________ is **one – many**, so is NOT a function.

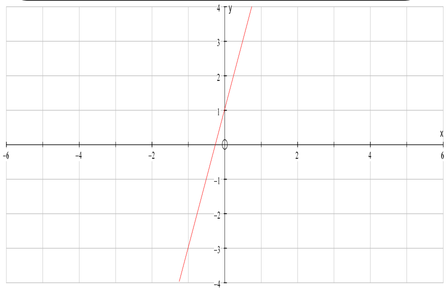
__________________ is a **one – one** function and can be written as $f(x)=$__________ or $f: x \to$__________

__________________ is a **many – one** function and can be written as $g(x)=$_________ or $g: x \to$__________

Graphs of Functions

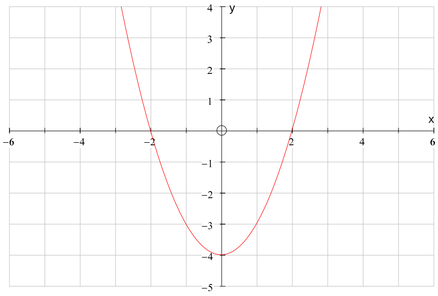
$x \to 4x+1$

Is a ___________ function as any vertical or horizontal line will cut the graph once so the output is unique.



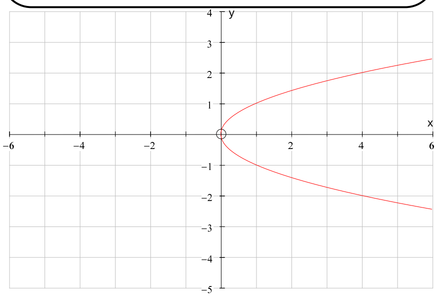
$x \to x^2-4$

Is a ___________ function as any vertical line will cut the curve only once but a horizontal line cuts the curve



$x \to +\sqrt{x}$

Is not a function as you can draw a vertical line that cuts the curve more than once so the output is not unique.



Example 1

The functions f and g are defined for all real values of x and are such that $f(x)=4x+1$ and $g(x)=x^2-4$

(a) Find $f(-4)$ and $g(5)$.

(b) Find the two values of x for which $f(x)=g(x)$.

Reminders:

Range and Domain

The set of all possible inputs of a mapping or function is called the **domain** (x values).

The set of outputs for a particular set of inputs for a mapping or function is called the **range** (y values).

Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \dots\}$
$\{x_1, x_2, \dots\}$	the set with the elements $x_1, x_2, \dots$
$\{x: \dots\}$	the set of all x such that...
$\mathbb{Z}$	the set of integers $\{0, \pm 1, \pm 2, \pm 3, \dots\}$
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, \dots\}$
$\mathbb{Z}_0^+$	the set of non-negative integers $\{0, 1, 2, 3, \dots\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers $\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}^+\right\}$

<u>Example 2</u>
The function f is defined by $f(x) = e^x + 5$. State the greatest possible domain and range of f.

Alternative Set Notation

$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leq x \leq b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leq x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leq b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$

<u>Example 3</u>
The function f is defined by $f(x) = \sin x$, $x \in \mathbb{R}$. State the range of f using interval notation.

<u>Example 4</u>
The function f is defined by $f(x) = x^2 + x - 3$.

a) State the range of f.
b) The domain of f is restricted to $[-1, 2)$. State the new range of f.

Composite Functions

The term composition is used when one operation is performed after another operation. Sometimes you have two given functions such as f and g and need to perform one function after another.

Suppose $f(x) = x^2$ and $g(x) = 2 + 3x$. What is $f[g(x)]$?

$$f[g(x)] = f(2 + 3x) = (2 + 3x)^2$$

This is usually written without the extra brackets as $fg(x)$ and fg or $f \circ g$ is said to be the composite function.

Although we write $fg(x)$ the function g operates first because it is closest to x.

Example 5
The function f and g are defined by $f(x) = x^3 + 5$ and $g(x) = 3 - 2x$. Find:

a) $fg(x)$

b) $gf(x)$

c) $fg(2)$

Generally $fg(x) \neq gf(x)$.

Example 6
The function f and g are defined by $f(x) = x^3$ and $g(x) = \frac{1}{x+2}$.

a) Find $fg(x)$

b) Solve $fg(x) = 1$

Example 7
The function f and g are defined by $f: x \to 4 - x^2$ and $g: x \to \ln x$

a) State the range of f as an interval in set notation.

b) State the greatest possible domain of g in set notation.

c) Find the composite function $g \circ f$ and state its greatest possible domain in set notation.

Inverse Functions

Notation: The inverse of $f(x)$ is written as $f^{-1}(x)$. This does not mean $\frac{1}{f(x)}$.

An inverse function reverses the effect of a function.
For example, if $f(4) = 7$ then $f^{-1}(7) = 4$ and if $f: x \to x + 3$ then $f^{-1}: x \to x - 3$.

Since an inverse function reverses the effect of a function the **domain** of f is the **range** of f^{-1} and the **range** of f is the **domain** of f^{-1}.

Only one-one functions have an inverse function
(A many-one function does not have an inverse function)

Example 8

Find the inverse of the function $f(x) = \frac{1-6x}{2}, x \in \mathbb{R}$

Step1	Write $y = f(x)$.	
Step 2	Rearrange the equation to make x the new subject.	
Step 3	Swap x and y over.	
Step 4	The new expression for y is equal to $f^{-1}(x)$	

Example 9

Find the inverse of the function $f(x) = \frac{1}{x-3}, x \in \mathbb{R}, x \neq 3$

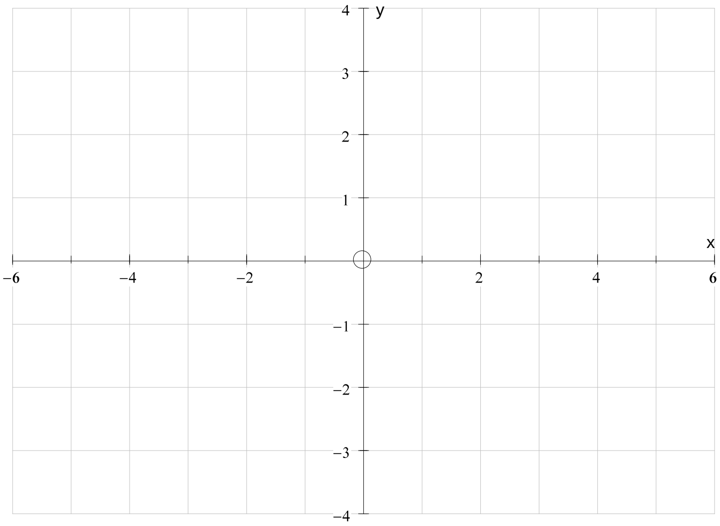
Step1	Write $y = f(x)$.	
Step 2	Rearrange the equation to make x the new subject.	
Step 3	Swap x and y over.	
Step 4	The new expression for y is equal to $f^{-1}(x)$	

Graphing Functions and their Inverses

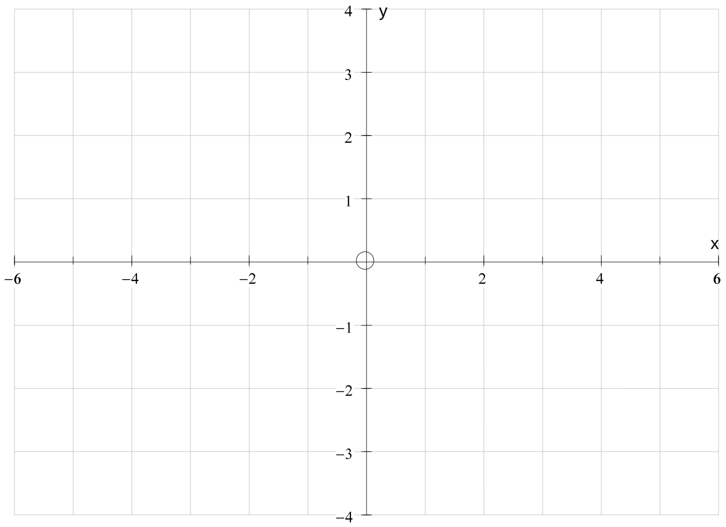
Example 10

Find the following inverses and sketch both $f(x)$ and $f^{-1}(x)$ on the axes given:

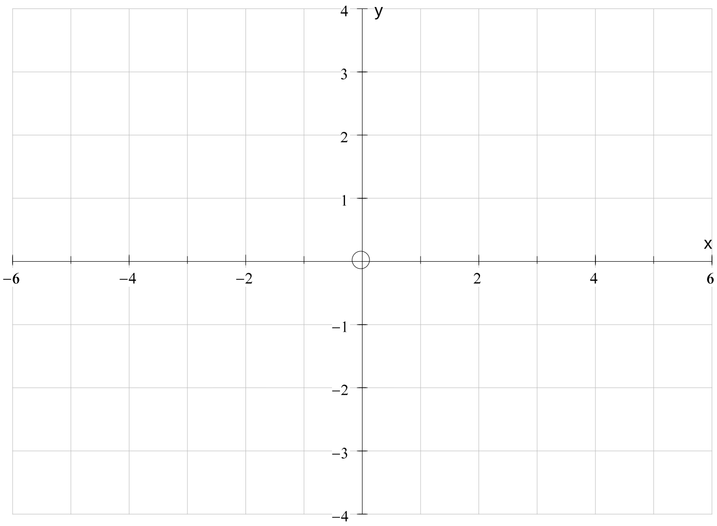
1. $f(x) = 4x - 3, x \in \mathbb{R}$



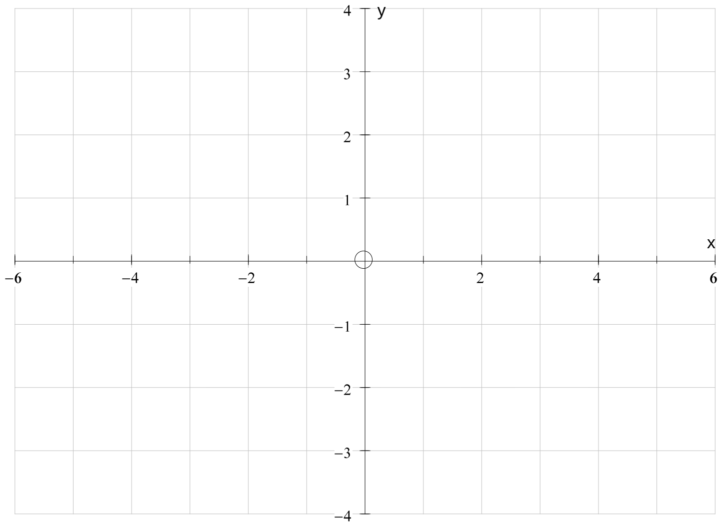
2. $f(x) = 1 - \frac{x}{2}, x \in \mathbb{R}$



3. $f(x) = 2(x^3 - 1), x \in \mathbb{R}$



4. $f(x) = 3x + 1, x \geq 0$



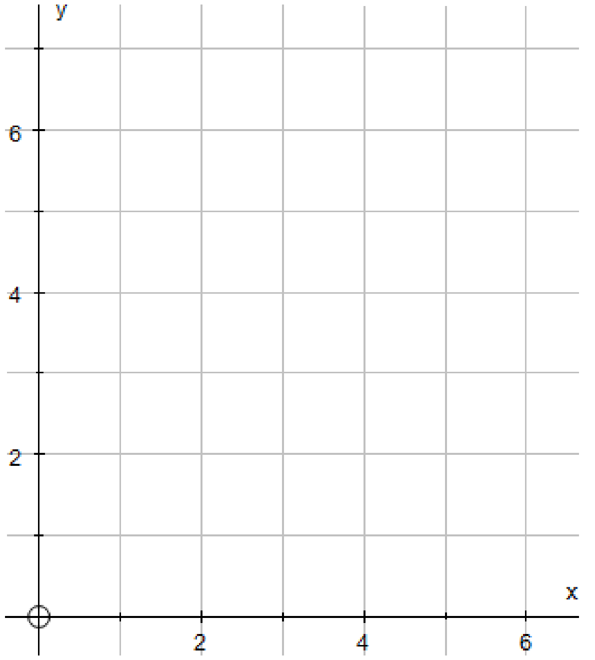
What do you notice?

Example 11

For the function $f(x) = 3 + 2x, x < 0$

a) Sketch $f(x)$ and $f^{-1}(x)$ on the same axes

b) Use your sketches to write down: (i) the range of $f(x)$ (ii) the domain & range of $f^{-1}(x)$



The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Exam Question

The functions f and g are defined with their respective domains by

$$f(x) = 2 - x^4 \qquad \text{for all real values of } x$$

$$g(x) = \frac{1}{x-4} \qquad \text{for real values of } x, \; x \neq 4$$

(a) State the range of f. *(2 marks)*

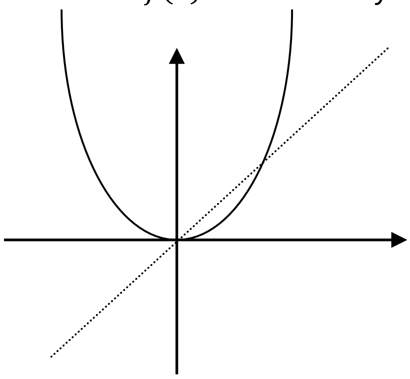
(b) Explain why the function f does not have an inverse. *(1 mark)*

(c) (i) Write down an expression for $fg(x)$. *(1 mark)*

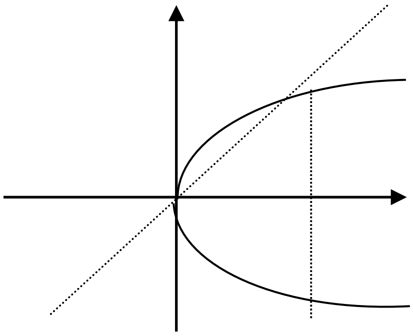
(ii) Solve the equation $fg(x) = -14$. *(3 marks)*

Restricting the Domain

The function $f(x) = x^2$ is many-one



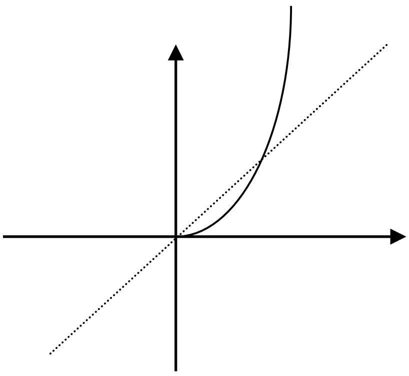
If we reflect in $y = x$ we get



$f(x) = x^2, x \in \mathbb{R}$ has no inverse

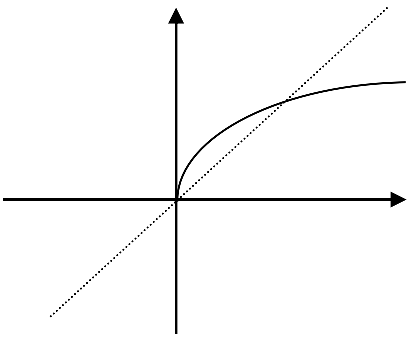
One-many so it is not a function

But if we can restrict the domain so it becomes one-one then there will be an inverse:



$f(x) = x^2, x \geq 0$

(Now one-one)



$f^{-1}(x) = \sqrt{x}, x \in \mathbb{R}^+$

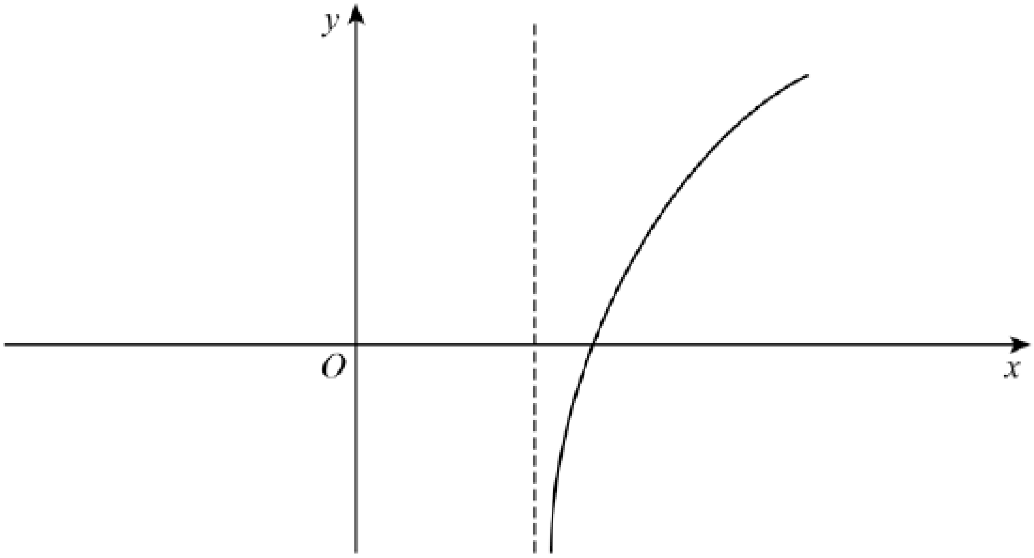
(Now also one-one)

If you have a function $f(x)$ and $f^{-1}(x)$ are the same then $f(x)$ is called a self-inverse.

Exam Style Question

The curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \ln(2x-3)$, is sketched below.

The domain of f is $\{x \in \mathbb{R} : x > \frac{3}{2}\}$



(a) The inverse of f is f^{-1}

(i) Find $\mathrm{f}^{-1}(x)$

(ii) State the range of f^{-1}

(iii) Sketch the curve with equation $y = \mathrm{f}^{-1}(x)$, indicating the value of the y-coordinate of the point where the curve intersects the y-axis.

(b) The function g is defined by $\mathrm{g}(x) = \mathrm{e}^{2x} - 4$, for all real values of x

(i) Find $\mathrm{gf}(x)$, giving your answer in the form $(ax-b)^2 - c$, where a, b and c are integers

(ii) Write down an expression for $\mathrm{fg}(x)$, and hence find the exact solution of the equation $\mathrm{fg}(x) = \ln 5$

Inverse Trigonometric Functions

For a function to have an inverse it must be one-one. The sine function defined over the domain $x \in \mathbb{R}$ is a many-one function and so does not have an inverse. In order for trigonometric functions to have an inverse we much restrict the domain so that it becomes a one-one function but still takes all real values in the range (i.e. $-1 \leq \sin x \leq 1$).

$f(x) = \sin x$

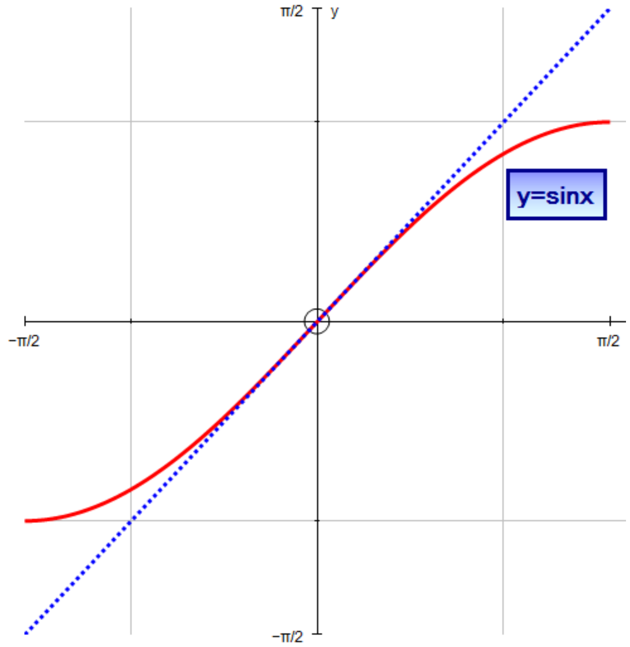
Restricted Domain:

Range:

$f^{-1}(x) = \sin^{-1} x \; = \arcsin x$

Domain:

Range:



NB: $\sin^{-1} x \neq \frac{1}{\sin x}$

In an exam you many be asked to give the coordinates of the end points of the graph.

End points of $y = \sin^{-1} x$:

$f(x) = \cos x$

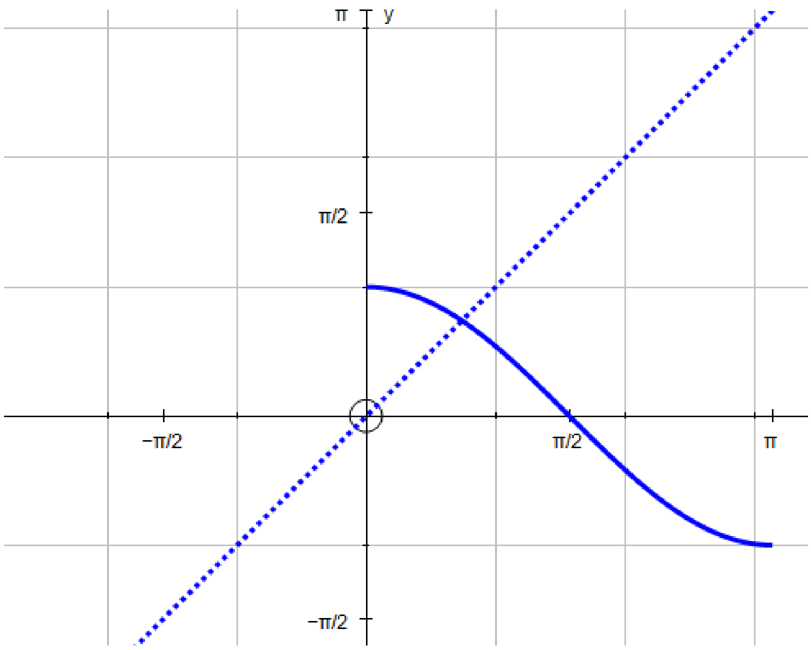
Restricted Domain:

Range:

$f^{-1}(x) = \cos^{-1} x = \arccos x$

Domain:

Range:



End points of $y = \cos^{-1} x$:

NB: $\cos^{-1} x \neq \frac{1}{\cos x}$

$\boxed{f(x) = \tan x}$

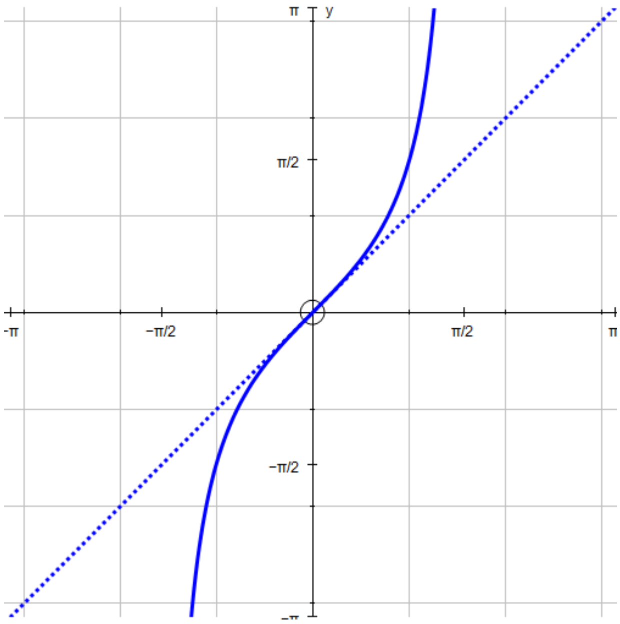
Restricted Domain:

Range:

$\boxed{f^{-1}(x) = \tan^{-1} x = \arctan x}$

Domain:

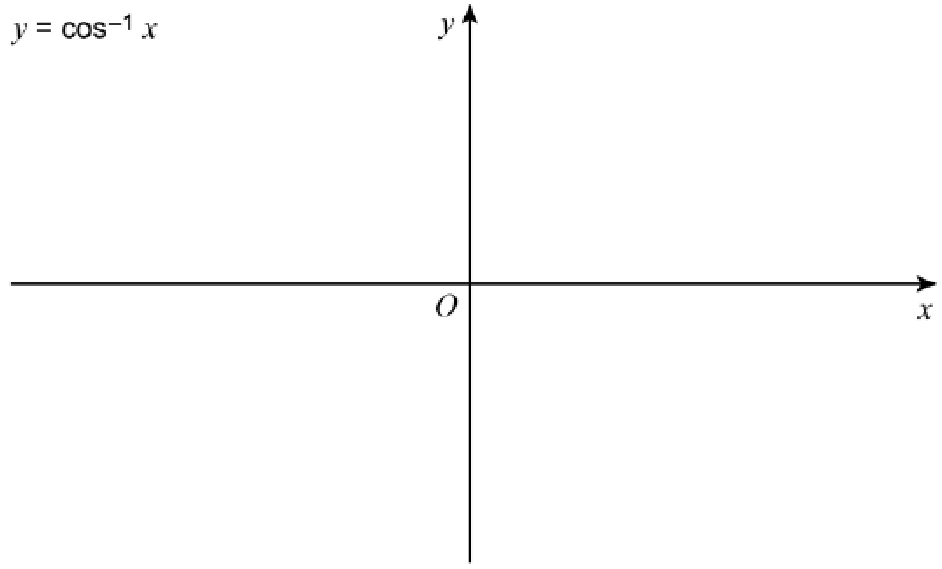
Range:

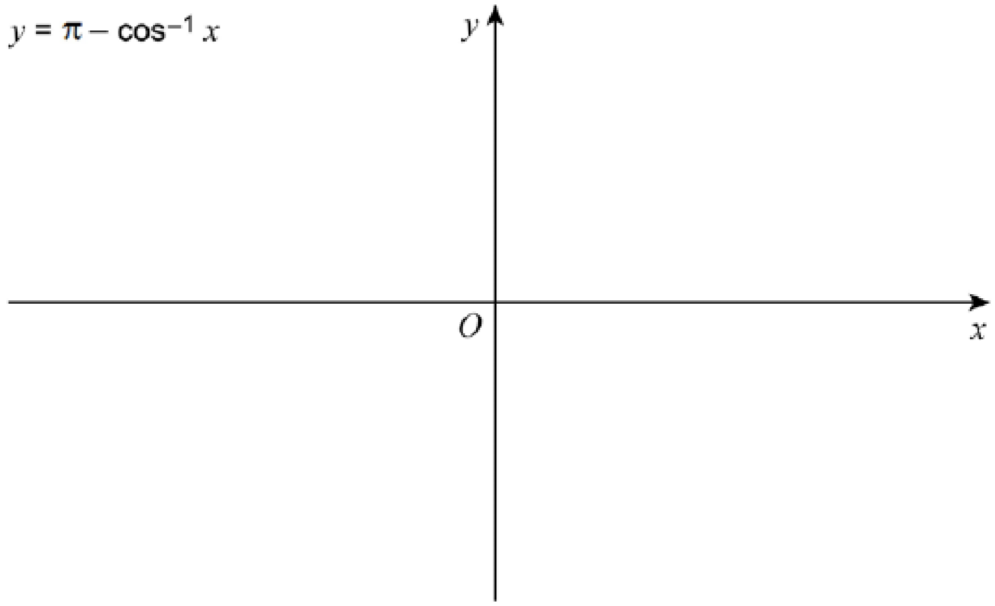


Exam Style Questions

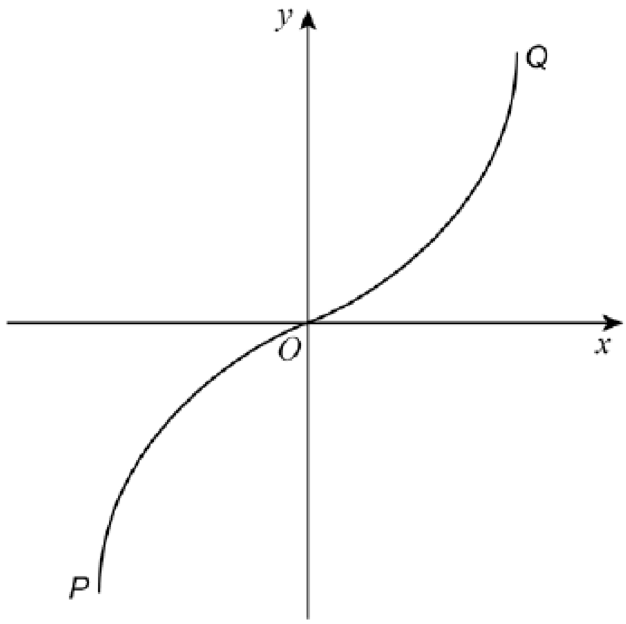
6 (a) Sketch the graph of $y = \cos^{-1}x$, where y is in radians on the first set of axes below.
State the coordinates of the end points of the graph.

(b) Sketch the graph of $y = \pi - \cos^{-1}x$, where y is in radians on the second axes below.
State the coordinates of the end points of the graph.





3 (a) The sketch shows the graph of $y = \sin^{-1} x$



Write down the coordinates of the points P and Q, the end points of the graph.

(b) Sketch the graph of $y = \sin^{-1}(x - 1)$