Pure Sector 2: Differentiation 3

Aims

- Differentiate using the product rule, the quotient rule and the chain rule, including problems involving rates of change and inverse functions.
- Find points of Inflection

$$y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$$

$$y = e^{kx} \Rightarrow \frac{dy}{dx} = ke^{kx}$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$y = \ln(f(x)) \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$

$$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$$

If $u = g(x)$ and $y = f(u)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

or

If $u = g(x)$ and $y = fg(x)$, then $\frac{dy}{dx} = f'(u) \times g'(x)$

Recap Example 1

Differentiate:

a) $y = e^{4x}$

b) $y = (2x-3)^5$

c) $y = \sin(4x)$

d) $y = 3\ln(x^2 + 4x - 1)$

e) $y = 2\cos(4x-1)$

f) $y = e^{\cos x}$

Recap Example 2

A curve has equation $y = e^{2x} - 8x$ and has a single turning point, *P*.

i) Find the x co-ordinates of *P* in an exact form, and show that its y co-ordinate is $4(1 - \ln 4)$.

ii) Find the value of $\frac{d^2y}{dx^2}$ at *P*, and hence deduce the nature of the turning point.

Recap Example 3

A curve has equation $y = \sqrt{x^2 + 3x}$ $x \geq 0$,

a) Find the equation of the tangent to the curve at point (1, 2).

b) Determine whether the curve increasing or decreasing at this point.

Product Rule

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
or
If $h(x) = f(x)g(x)$ then $\frac{dy}{dx} = f(x)g'(x) + f'(x)g(x)$

Example 5

Find the derivative of:

a) $y = x^3 \sin x$

b) $y = e^{3x} \ln x$

c) $y = 2\sqrt{x}\sin x$

Example 6

Find the equation of the tangent to the curve $y = e^{3x}\cos x$ when $x = 0$.

Exam Question

6 A curve has equation $y = e^{2x}(x^2 - 4x - 2)$.

(a) Find the value of the x-coordinate of each of the stationary points of the curve. *(6 marks)*

(b) (i) Find $\dfrac{d^2y}{dx^2}$. *(2 marks)*

(ii) Determine the nature of each of the stationary points of the curve. *(2 marks)*

The Quotient Rule

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

or

If $h(x) = \frac{f(x)}{g(x)}$ then $\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

This is given in the formulae booklet in this form.

<u>Example 7</u>

Differentiate the following with respect to x.

a)

$$y = \frac{\sin x}{x^2}$$

b)

$$y = \frac{x^3 - 1}{x^3 + 1}$$

c)

$$y = \frac{1 - x^2}{\ln(x + 1)}$$

Example 8

A curve has equation $y = \frac{x^2 - 1}{2x + 4}$. Find the x coordinates of the stationary points of the curve giving your answer in the form $a + b\sqrt{c}$.

Exam Question

A curve has the equation

$$y = \frac{2x}{\sin x}, \qquad 0 < x < \pi.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *(3 marks)*

(b) The point P on the curve has coordinates $\left(\frac{\pi}{2}, \pi\right)$.

(i) Show that the equation of the tangent to the curve at P is $y = 2x$. *(3 marks)*

(ii) Find the equation of the normal to the curve at P, giving your answer in the form $y = mx + c$. *(3 marks)*

Derivative of Other Trigonometric Functions

These are given in the formulae booklet.

$$y = \tan x \text{ then } \frac{dy}{dx} = \sec^2 x$$

$$y = \cot x \text{ then } \frac{dy}{dx} = -\operatorname{cosec}^2 x$$

$$y = \sec x \text{ then } \frac{dy}{dx} = \sec x \tan x$$

$$y = \operatorname{cosec} x \text{ then } \frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

Example 9

Differentiate with respect to x:

a) $y = \tan(3x - 4)$

b) $y = \sec 3x$

c) $f(x) = 4x - \operatorname{cosec}(2x + 1)$

d) $y = 3\cot 4x$

e) $y = 2\cos(4x^3 - 5x)$

f) $\mathrm{g}(x) = \tan^3 2x$

g) $y = \sec(3x - x^2)$

h) $f(x) = \operatorname{cosec}[(2x - 1)^4]$

i) $y = \cot e^{3x}$

Exam Questions

7 It is given that $y = \tan 4x$.

(a) By writing $\tan 4x$ as $\dfrac{\sin 4x}{\cos 4x}$, use the quotient rule to show that $\dfrac{dy}{dx} = p(1 + \tan^2 4x)$, where p is a number to be determined. *(3 marks)*

(b) Show that $\dfrac{d^2y}{dx^2} = qy(1 + y^2)$, where q is a number to be determined. *(5 marks)*

6.

a) Differentiate $y = x\sec^2 x - \tan x$ to show $\dfrac{dy}{dx} = 2x\sec^2 x\tan x$

b) Hence find the gradient of the curve when $x = \dfrac{\pi}{4}$

Inverse Functions

$$\boxed{\frac{dy}{dx} = 1 \div \frac{dx}{dy}}$$

Example 10

Differentiate $y = 3^x$

$x =$

$\frac{dx}{dy} =$

$\frac{dy}{dx} =$

Example 11

(b) (i) Find $\frac{dx}{dy}$ when $x = 2y^3 + \ln y$. *(1 mark)*

(ii) Hence find an equation of the tangent to the curve $x = 2y^3 + \ln y$ at the point $(2,1)$. *(3 marks)*

Example 12

(d) (i) Write the equation $y = 2\cos^{-1}(x-1)$ in the form $x = \mathrm{f}(y)$. *(2 marks)*

(ii) Hence find the value of $\frac{dx}{dy}$ when $y = 2$. *(3 marks)*

Mixed exam-style question A

A curve C has equation

$$y = \frac{3}{(5-3x)^2}, \qquad x \neq \frac{5}{3}.$$

The point P on C has x-coordinate 2.

Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(7)

Mixed exam-style question B

(i) Given that $y = \dfrac{\ln(x^2+1)}{x}$, find $\dfrac{dy}{dx}$.

(4)

(ii) Given that $x = \tan y$, show that $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

(5)

Mixed exam-style question C

The point P is the point on the curve $x = 2\tan\left(y + \dfrac{\pi}{12}\right)$ with y-coordinate $\dfrac{\pi}{4}$.

Find an equation of the normal to the curve at P.

(7)

Mixed exam-style question D

A curve C has equation

$$y = e^{2x}\tan x, \quad x \neq (2n+1)\frac{\pi}{2}.$$

(*a*) Show that the turning points on C occur where $\tan x = -1$.

(6)

(*b*) Find an equation of the tangent to C at the point where $x = 0$.

(2)

Mixed exam-style question E

Find the equation of the tangent to the curve $x = \cos(2y + \pi)$ at $\left(0, \frac{\pi}{4}\right)$.

Give your answer in the form $y = ax + b$, where a and b are constants to be found.

(6)

Mixed exam-style question F

Given that

$$x = \sec^2 3y, \quad 0 < y < \frac{\pi}{6},$$

(*a*) find $\frac{dx}{dy}$ in terms of y.

(2)

(*b*) Hence show that

$$\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}.$$

(4)

(*c*) Find an expression for $\frac{d^2y}{dx^2}$ in terms of x. Give your answer in its simplest form.

(4)

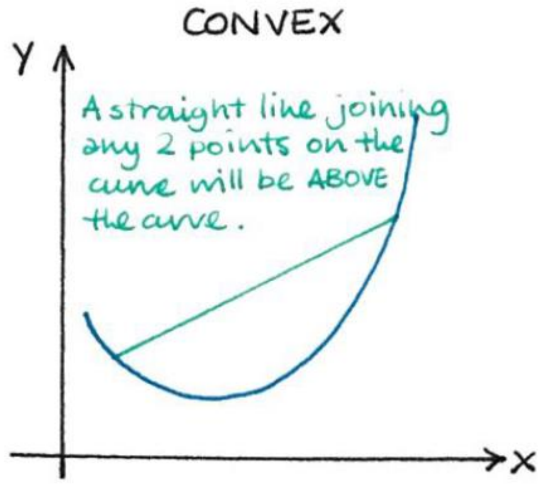
Mixed exam-style question G

A curve has equation $y = e^{3x-x^3}$. Find the exact values of the coordinates of the stationary points of the curve and determine the nature of these stationary points.

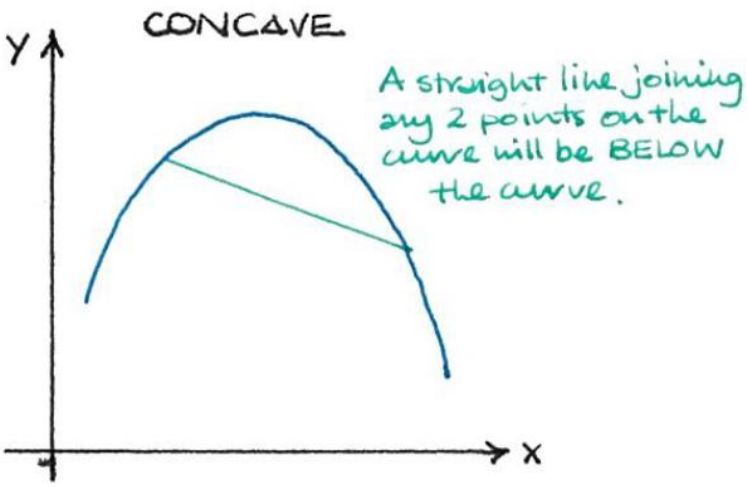
[7 marks]

Points of Inflection

Curves can be convex or concave. These descriptions refer to the shape of the curve.



Convex curves are ones that curve downwards.



Concave curves are ones that curve upwards.

A curve is convex when $\frac{d^2y}{dx^2} > 0$. (When its gradient is increasing)

A curve is concave when $\frac{d^2y}{dx^2} < 0$. (When its gradient is decreasing)

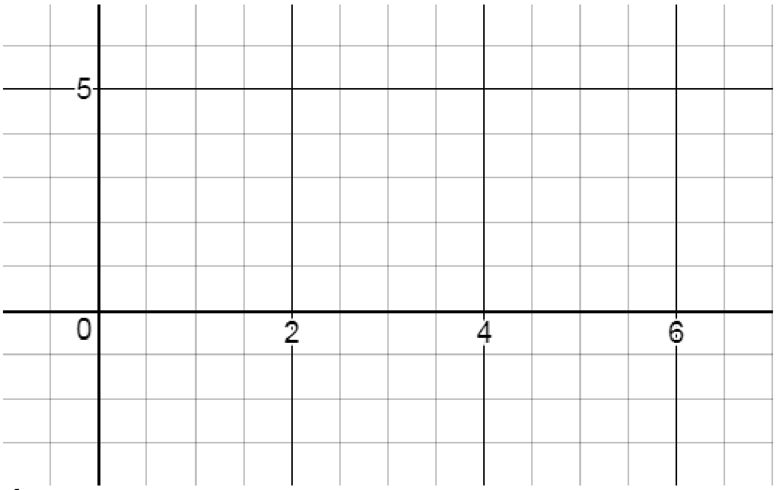
Points of inflection occur when the curve changes from convex to concave or concave to convex. (i.e. where $\frac{d^2y}{dx^2}$ changes between positive and negative or vice versa.) They may or may not be stationary.

At a point of inflection, $\frac{d^2y}{dx^2} = 0$, but not all points where $\frac{d^2y}{dx^2} = 0$ are points of inflection. Investigation either side of the point to see if the sign of $\frac{d^2y}{dx^2}$ is changing needs to be done and that section of the curve must be continuous.

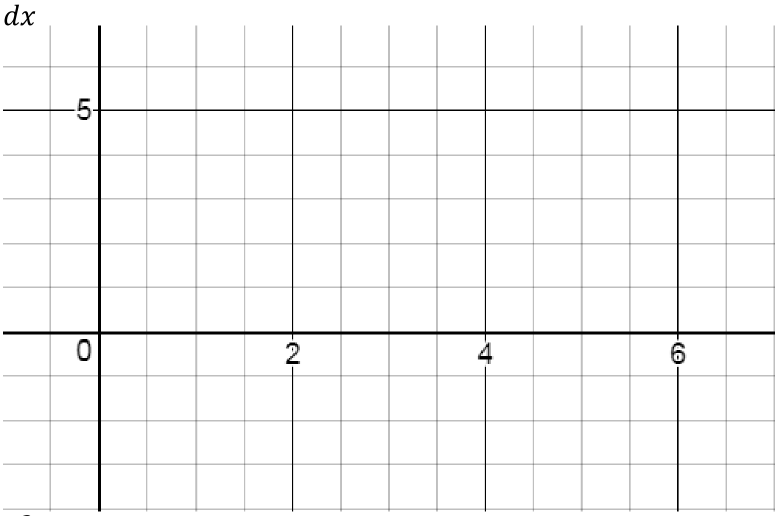
<u>Example 13</u>

By sketching the graphs of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ investigate and determine the stationary points and/or points of inflection of $y = x^3 - 6x^3 + 9x + 1$.

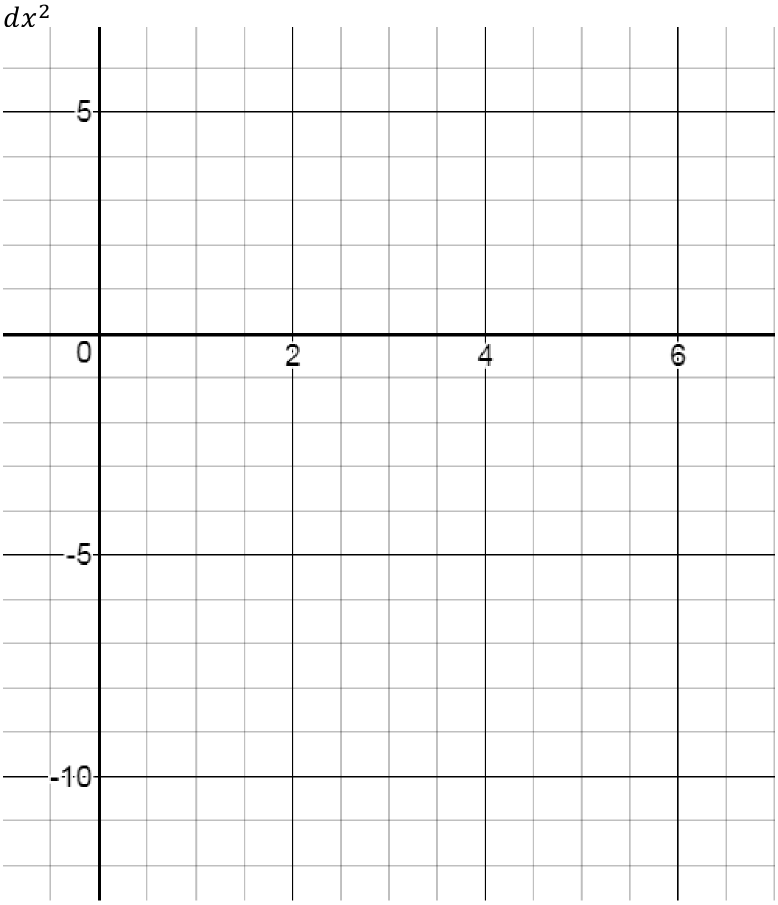
$y = x^3 - 6x^3 + 9x + 1$



$\frac{dy}{dx}$

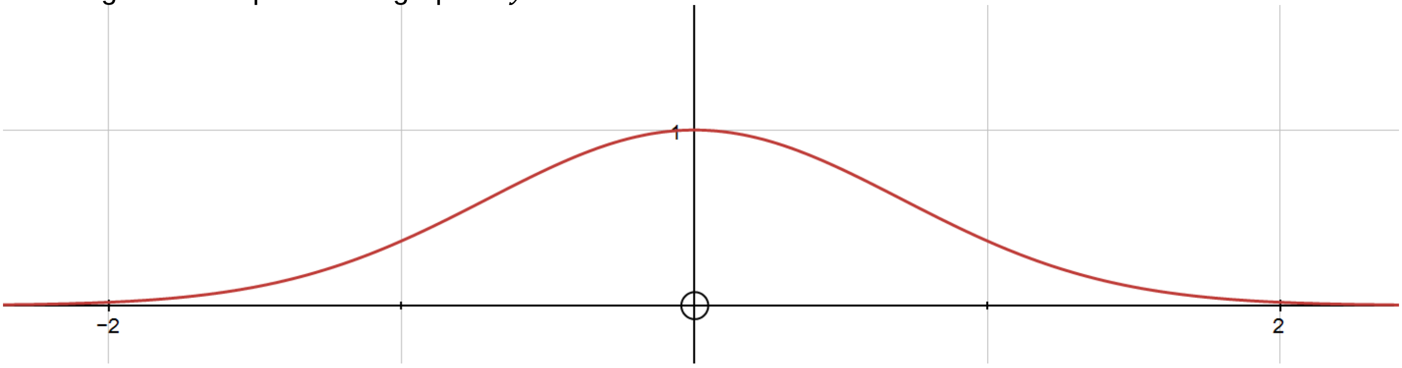


$\frac{d^2y}{dx^2}$



Example 14

The diagram show part of the graph of $y = e^{-x^2}$



a) Where is the gradient at its maximum?
b) Where is the gradient at its minimum?
c) For which values of x is the graph concave?

Example 15

A function f is defined by

$$f(x) = \frac{x}{\sqrt{2x - 2}}$$

a) Find $f'(x)$
b) Show that the graph of $y = f(x)$ has exactly one point of inflection.

Exam Questions

A curve has equation $y = xe^{\frac{x}{2}}$

Show that the curve has a single point of inflection and state the exact coordinates of this point of inflection.

[8 marks]

A curve has equation $y = 2x \cos 3x + \left(3x^2 - 4\right) \sin 3x$

(a) Find $\frac{dy}{dx}$, giving your answer in the form $\left(mx^2 + n\right) \cos 3x$, where m and n are integers.

[4 marks]

(b) Show that the x-coordinates of the points of inflection of the curve satisfy the equation

$$\cot 3x = \frac{9x^2 - 10}{6x}$$

[4 marks]