Pure Sector 2: Differentiation 1

Aims

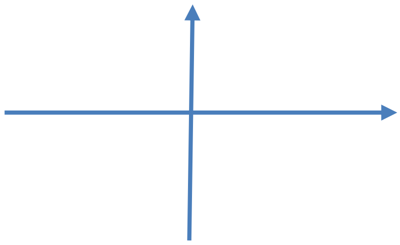
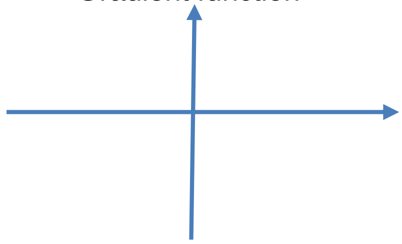
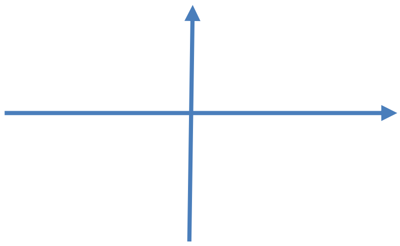
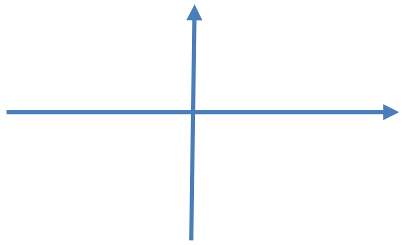
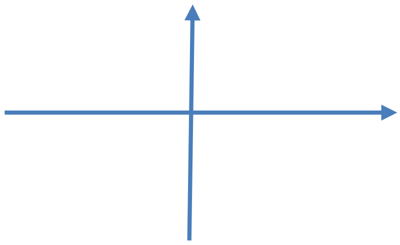
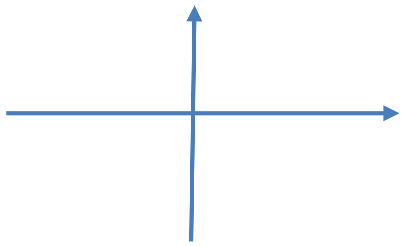
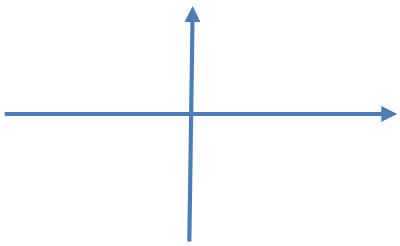
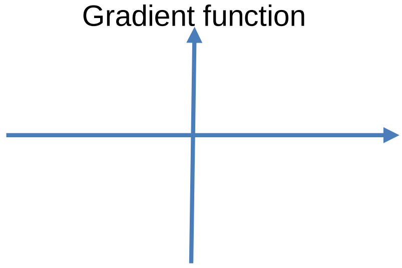
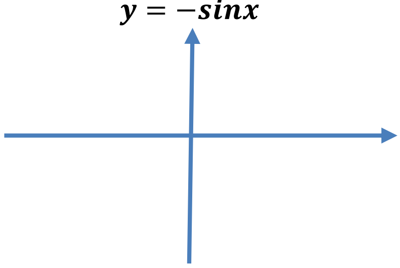
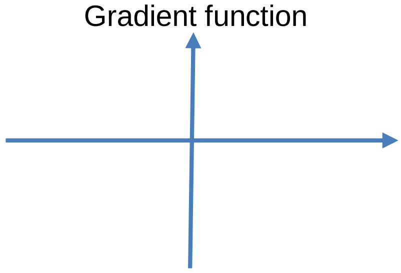
- Use and apply differentiation from first principles for small powers of n in the function x^n, $sinx$ and $cosx$.
- Differentiate $f(x) = x^n$ for rational values of n.
- Sketch the gradient of a curve and find the gradient using differentiation.
- Differentiate $\sin kx$ and $\cos kx$.

Sketching the Gradient Function

Tips for sketching gradient functions:

- When the graph is increasing the gradient is positive.
- When the graph is decreasing the gradient is negative.
- When the tangent is horizontal the gradient is zero. A point on the graph where this happens is called a **stationary point** or **turning point**.

<u>Example 1</u>

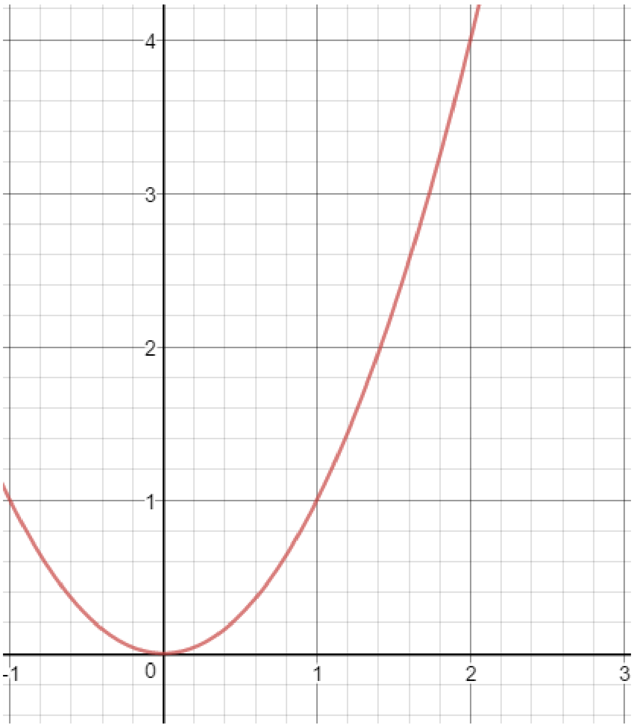
$\boldsymbol{y = 3x + 4}$ 	Gradient function 
$\boldsymbol{y = (x + 2)(x - 3)}$ 	Gradient function 
$\boldsymbol{y = (x - 2)^2(x + 5)}$ 	Gradient function 
$\boldsymbol{y = e^x + 1}$ 	Gradient function 
$\boldsymbol{y = -sinx}$ 	Gradient function 

Differentiation from first principles of x^n

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

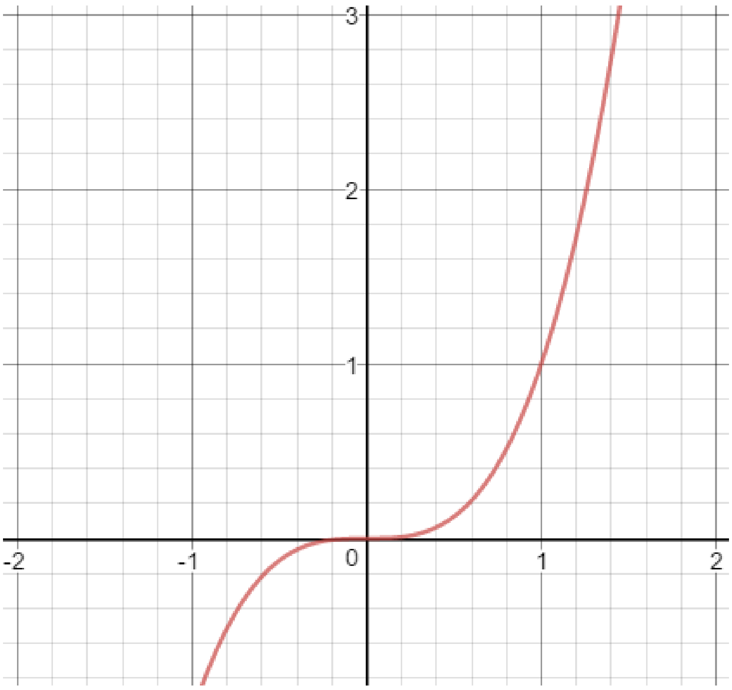
Example 2

$\boldsymbol{f(x) = x^2}$



Example 3

$\boldsymbol{f(x) = x^3}$



Example 4

a. Prove from first principles that the derivative of $y = 4$ is zero.

b. Differentiate $f(x) = x^2 - 6x$ from first principles.

c. Prove from first principles that if $f(x) = x^4$ then $f'(x) = 4x^3$

Differentiating power functions

$$y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$$

$$f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$$

Where a and n are constants

Note- the derivative is the same for an equation or a function the notation is just different

Example 5

Find the derivative with respect to x of the following functions and simplify where possible-

a. $y = x^2$

b. $y = 7x^{100}$

c. $f(x) = -2x^{10}$

d. $y = 3x^{-2}$

e. $y = 4x^5$

f. $f(x) = 2x^{\frac{1}{3}}$

g. $y = -5x$

h. $f(x) = 2$

i. $y = -6x^{-1}$

j. $y = 8x^{\frac{1}{2}}$

Exercise A

Find the derivative with respect to x of the following functions and simplify where possible-

a. $y = x^5$

b. $f(x) = 9x^{-3}$

c. $y = x^9$

d. $y = 7x^2$

e. $f(x) = 6x^{\frac{1}{3}}$

f. $f(x) = x^7$

g. $y = x^{12}$

h. $y = 2x^9$

i. $f(x) = x^{20}$

j. $y = \frac{3}{5}x^{-1}$

k. $y = \frac{7}{8}x^3$

l. $f(x) = \frac{2}{5}x^2$

m. $y = 2x^{14}$

n. $y = x^{-\frac{3}{2}}$

o. $f(x) = 4x^{11}$

p. $y = \frac{3x^3}{4}$

q. $y = 7x^3$

r. $f(x) = -4x^{-7}$

s. $f(x) = -2x^{17}$

t. $f(x) = \frac{3}{2}x^7$

u. $y = -x^{\frac{9}{7}}$

v. $y = \frac{2}{3}x^{-3}$

w. $y = 12x$

x. $y = -\frac{5x^3}{2}$

y. $f(x) = 4$

z. $y = x^{100}$

Example 6

Find the gradient of the curve $y = 3x^5$ at the point $(2, 96)$

Example 7

Find the coordinates of the point on the curve $f(x) = 8x^2$ where the gradient is 64.

Example 8

The radius, r cm, of a circular ink spot is given by $r = \frac{t^2}{8} + 1$, where t is the number of seconds after it first appears. Find the rate of change of the radius after 4s.

Exercise B

Find the gradient of these functions at the points indicated -

a. $y = x^2 \ at \ (3,9)$

b. $y = x^3 \ at \ (-1,-1)$

c. $y = x^5 \ at \ (2,32)$

d. $y = 4x^2 \ at \ (5,100)$

e. $y = 3x^3 \ at \ (-4,-192)$

f. $y = -x^5 \ at \ (1,-1)$

g. $y = 10x^2 \ at \ (6,36)$

h. $y = -7x^4 \ at \ (-2,-112)$

i. $y = 12x \ at \ (13,156)$

j. $y = 3x^2 \ at \ (-7,147)$

k. $y = -10x^7 \ at \ (1,-10)$

l. $y = 5x^5 \ at \ (2,160)$

Exercise C

Find the gradient of these functions at the points indicated –

a. $y = x^{-2} \ at \ \left(2,\frac{1}{4}\right)$

b. $y = 3x^{\frac{1}{2}} \ at \ (9,9)$

c. $f(x) = -2x^{-3} \ at \ (1,-2)$

d. $y = 4x^{-\frac{1}{4}} \ at \ (16,2)$

Example 9

a. Find the coordinates of the point on the curve $f(x) = \frac{1}{2}x^2$ where the gradient is 7

b. Find the coordinates of the point on the curve $f(x) = 3x^4$ where the gradient is -96

c. Find the coordinates of the points on the curve $y = x^3$ where the gradient is 12

Differentiating term by term

Example 10

Differentiate $f(x) = 3x^2 + 5x - 3$ with respect to x

Example 11

Find the gradient of the curve $y = 3x - 8x^2$ at the point $P\ (3, -63)$

Example 12

Find the coordinates of the point on the curve $y = 2x^2 - 3x + 4$ where it is parallel to the line $y = 5x + 3$.

Example 13

Differentiate $y = x^{-2} + 3x^{\frac{1}{2}} - 4x^{-2}$ with respect to x

Example 14

Differentiate $y = 2x^{-5} - 4x^{-\frac{1}{4}} + 6$ with respect to x

Differentiating with fractional and negative indices

The Laws of Indices

- $x^m \times x^n = x^{m+n}$
- $x^m \div x^n = x^{m-n}$
- $(x^m)^n = x^{mn}$

- $x^0 = 1$
- $\frac{1}{x^m} = x^{-m}$
- $\sqrt[n]{x} = x^{\frac{1}{n}}$

Always convert to exponent form before differentiating

Example 15

Find the derivative with respect to x of the following functions and simplify where possible-

a. $y = \frac{3}{x^2}$

b. $y = \sqrt[5]{x}$

c. $y = 7\sqrt{x}$

d. $f(x) = \frac{1}{4x^3}$

e. $f(x) = \sqrt[3]{x^2} + 2x + 3$

f. $y = \frac{5}{4\sqrt{x}} - 2x^{-1}$

g. $f(x) = \frac{5}{x^3} - \frac{2}{3x^2} + \frac{1}{4x}$

h. $y = \frac{7}{x^3} - \frac{4}{3x} + 8x^2$

Differentiating Products and Fractions

Always expand/simplify before differentiating

Example 16

Find the derivative with respect to x of the following functions and simplify where possible-

a. $y = (x-6)(x+2)$

b. $f(x) = (3x+2)(2x-5)$

c. $y = x(x+2)(2x+3)$

d. $f(x) = 2x^{\frac{1}{2}}(3x^2 - x)$

e. $y = \frac{x^6 - 8x^3 + 7}{x}$

Differentiation of sin x and cos x from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 17

$f(x) = sinx$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - sinx}{h}$$

Use the compound angle formula from your formula book

From your formula book:

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Trigonometry: small angles

For small angle θ,

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta$

As $h \to 0$, use small angle approximations for $\cos h$ and $\sin h$

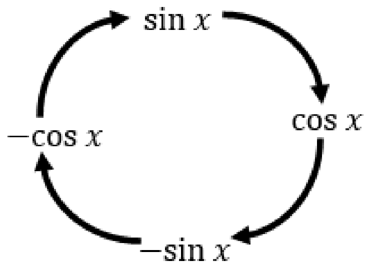
Example 18

Show that the derivative of $\cos x$ is $-\sin x$ by using differentiation from first principles.

If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

MUST be in radians



Chain Rule

If $u = g(x)$ and $y = f(u)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

or

If $u = g(x)$ and $y = fg(x)$, then $\frac{dy}{dx} = f'(u) \times g'(x)$

Example 19

Differentiate the following expressions:

a. $y = (2x - 3)^5$

b. $y = (5x - 2)^8$

c. $y = (7 - x)^2$

d. $y = (4x - 1)^{-3}$

e. $y = 3(2 - 3x)^{\frac{7}{2}}$

f. $y = (4x^3 - 2)^7 + x^2$

Chain Rule - Exponentials and Logarithms

If $y = e^x$ then $\frac{dy}{dx} = e^x$

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \ln x$ then $y = \frac{1}{x}$

If $y = \ln(f(x))$ then $y = \frac{f'(x)}{f(x)}$

<u>Example 20</u>

Differentiate the following expressions:

a. $y = e^{2x}$

b. $y = 3x^{-3} - e^{3x-1}$

c. $y = 4e^{3-2x}$

d. $y = e^{(x^2-5x+9)}$

e. $y = \ln 3x$

f. $y = \ln(4x - 6) + 5\sqrt{x}$

g. $y = 3\ln(x^2 + 4x - 1)$

h. $y = 2\ln(8 - 5x) - 10e^{4x}$

Chain Rule – Trigonometry

If $y = \sin\left(f(x)\right)$ then $\frac{dy}{dx} = f'(x)\cos\left(f(x)\right)$

If $y = \cos\left(f(x)\right)$ then $\frac{dy}{dx} = -f'(x)\sin\left(f(x)\right)$

<u>Example 21</u>

Differentiate the following:

a. $y = \sin 2x$

b. $f(x) = 2\cos 4x$

c. $f(x) = -\sin 5x$

d. $y = 3\sin x - 7\cos 2x$

e. $y = \cos 6x - 2\sin 8x$

f. $y = 2x^2 - 5sinx$

g. $f(x) = cosx + \frac{3}{x^2}$

h. $f(x) = \frac{2x^4 - 7sinx}{3}$