9 (a) Find $\int t \cos\left(\frac{\pi}{4}t\right) \mathrm{d}t$.

[3 marks]

(b) (i) Solve the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x^4 t \cos\left(\frac{\pi}{4}t\right)$$

given that $x = 1$ when $t = 0$, giving your answer in terms of π in the form $\mathrm{f}(x) = \mathrm{g}(t)$.

[4 marks]

(ii) Find the value of x when $t = 15$, giving your answer to three significant figures.

[2 marks]

Q 9	Solution	Mark	Total	Comment
(a)	$u = t \quad v' = \cos\left(\frac{\pi}{4}t\right)$ $u' = 1 \quad v = \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right)$ $\int t\cos\left(\frac{\pi}{4}t\right)\,dt$			
	$= t\,.\,a\sin\left(\frac{\pi}{4}t\right) \pm \int 1.\ a\sin\left(\frac{\pi}{4}t\right)$ (dt)	**M1**		Basically correct attempt at parts leading to terms of this form.
	$= \frac{4}{\pi}t\sin\left(\frac{\pi}{4}t\right) - \frac{4}{\pi}\int \sin\left(\frac{\pi}{4}\right)\,(\text{dt})$	**A1**		Correct intermediate result.
	$= \frac{4}{\pi}t\sin\left(\frac{\pi}{4}t\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}t\right)$	**A1ft**	3	**ft** on a - i.e. $a\,t\sin\left(\frac{\pi}{4}t\right) +$ $a^2\cos\left(\frac{\pi}{4}t\right)$
(b)(i)	$\frac{\text{dx}}{\text{dt}} = 3x^4\,t\cos\left(\frac{\pi}{4}t\right)$ $\int \frac{1}{3x^4}\,dx = \int t\cos\left(\frac{\pi}{4}t\right)\,dt$	**B1**		Any correct separation of variables including integral signs and 'dx' & 'dt'.
	$-\frac{1}{9}x^{-3}$	**B1**		
	$= \frac{4}{\pi}t\sin\left(\frac{\pi}{4}t\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}t\right)$ (+ C) Using $x = 1$ when $t = 0$ **correctly** to find C $C = -\frac{1}{9} - \frac{16}{\pi^2}$	**M1**		Substitution into required form seen $px^{-3} = q\,t\sin\left(\frac{\pi}{4}t\right) + r\cos\left(\frac{\pi}{4}t\right)$ or values of individual terms or implied by correct final line. **M0** if candidate thinks terms in t both give 0.
	$-\frac{1}{9x^3} = \frac{4}{\pi}t\sin\left(\frac{\pi}{4}t\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}t\right) - \frac{1}{9} - \frac{16}{\pi^2}$	**A1**	4	ACF but exact and in the required form of $f(x) = g(t)$–apply ISW if necessary.
(ii)	Using $t = 15$ to **find** a value for x	**M1**		Substitution should be seen and into the candidate's final answer for (b)(i) or implied by a **correct** final answer.
	$x = 0.199$	**A1**	2	
			9	

7 (a) Using a suitable substitution, or otherwise, find

$$\int \frac{x}{(7+2x^2)^2}\,\mathrm{d}x$$

[3 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x\,\mathrm{e}^{4y}}{(7+2x^2)^2}$$

given that $y = 0$ when $x = 2$.

Give your answer in the form $y = \mathrm{f}(x)$.

[6 marks]

(a)	$u=7+2x^2$ gives $\frac{\mathrm{du}}{\mathrm{dx}}=4x$	**M1**		OE – e.g. $\mathrm{du}=4x\,\mathrm{dx}$ etc.
	$\int\frac{x}{(7+2x^2)^2}\,\mathrm{d}x=\int\frac{1}{4u^2}\,\mathrm{d}u\ =-\frac{1}{4u}$	**dM1**		Integral all in u of the form $\int\frac{k}{u^2}\,\mathrm{d}u$ leading to $\pm\frac{k}{u}$.
	$=-\frac{1}{4(7+2x^2)}\quad(+c)$	**A1**	**3**	OE – e.g. $-\frac{1}{4}(7+2x^2)^{-1}$
(b)	$\int e^{-4y}\mathrm{d}y=\int\frac{3x}{(7+2x^2)^2}\,\mathrm{d}x$	**B1**		Correct separation seen and notation including integral signs and dy & dx.
	LHS $=-\frac{1}{4}e^{-4y}$	**B1**		
	RHS $=-\frac{3}{4(7+2x^2)}$	**B1ft**		ft on 3 x (a) from a correct integrand
	$-\frac{1}{4}e^{-4y}=-\frac{3}{4(7+2x^2)}+C$			
	$x=2$ and $y=0$ to find C $\quad\left(=-\frac{1}{5}\right)$	**M1**		Used correctly in an expression of the form $pe^{-4y}=\frac{q}{7+2x^2}+C$
	$-\frac{1}{4}e^{-4y}=-\frac{3}{4(7+2x^2)}-\frac{1}{5}$ OE	**A1**		
	$y=-\frac{1}{4}\ln\left(\frac{3}{7+2x^2}+\frac{4}{5}\right)$	**A1**	**6**	ACF - e.g. $-\frac{1}{4}\ln\left(\frac{43+8x^2}{5(7+2x^2)}\right)$ or $\frac{1}{4}\ln\left(\frac{5(7+2x^2)}{43+8x^2}\right)$ etc.

8 (a) It is given that $\frac{1}{x(k-x)}$ can be expressed as $A\left(\frac{1}{x}+\frac{1}{k-x}\right)$ where A and k are positive constants. Find A in terms of k.

[2 marks]

(b) A rumour is spreading through a school of 1200 pupils. The rate at which the rumour is spreading can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{x(1200-x)}{3600}$$

where x is the number of pupils who have heard the rumour t hours after 11.00 am.

By 11.00 am, 300 pupils have heard the rumour. Taking $t=0$ as 11.00 am, use integration to solve this differential equation to show that

$$t=3\ln\left(\frac{3x}{1200-x}\right)$$

[5 marks]

(c) Use this model to:

(i) find the time of day by which half of the pupils will have heard the rumour, giving your answer to the nearest 5 minutes

[2 marks]

(ii) find x in terms of t and hence find the number of pupils who will have heard the rumour by 3.00 pm.

[3 marks]

Q8	Solution	Mark	Total	Comment
(a)	$A\left(\frac{1}{x}+\frac{1}{k-x}\right)=A\left(\frac{k-x+x}{x(k-x)}\right)$ $\left(=A\left(\frac{k}{x(k-x)}\right)\right)$	M1		To compare with $\frac{1}{x(k-x)}$
	Comparing gives $Ak=1$ so $A=\frac{1}{k}$	A1	2	**NMS** $A=\frac{1}{k}$ scores 2/2
	Alternative: $\frac{1}{x(k-x)}=\frac{A}{x}+\frac{A}{k-x}$ $\rightarrow 1=A(k-x)+Ax$ **(M1)** $\rightarrow$ $1=Ak$ $\rightarrow$ $A=\frac{1}{k}$ **(A1)**			
(b)	$\frac{1}{x(1200-x)}=\lambda\left(\frac{1}{x}+\frac{1}{1200-x}\right)$	M1		Separation of x terms into two fractions. Common multiple needn't be correct.
	$\lambda\int\frac{1}{x}+\frac{1}{1200-x}\,\mathrm{d}x$ $\left(=\frac{1}{3600}\int \mathrm{d}t\right)$			
	$\lambda\ (\ln x-\ln(1200-x))$	A1 A1		Correct log integrations (LHS)
	$=\frac{t}{3600}\quad(+C)$			
	$\frac{1}{1200}(\ln 300-\ln 900)=C\ \rightarrow\ C=\frac{1}{1200}\ln\left(\frac{1}{3}\right)$	dM1		Using $x=300$ and $t=0$ to find a value for C
	$t=3\ln\left(\frac{3x}{1200-x}\right)$	A1	5	AG be convinced
	AG!!! - **must** see some evidence of **correct** log manipulation before awarding the final **A1** e.g. $\ln x-\ln(1200-x)=\frac{t}{3}+\ln\left(\frac{1}{3}\right)\rightarrow \ln x-\ln(1200-x)=\frac{t}{3}+\ln\left(\frac{1}{3}\right)\rightarrow\ln\left(\frac{3x}{1200-x}\right)=\frac{t}{3}$ to answer **Alternative Method** $\frac{dx}{dt}=\frac{x(1200-x)}{3600}$ $\rightarrow$ $\frac{dt}{dx}=\frac{3600}{x(1200-x)}$ $\rightarrow$ $\frac{dt}{dx}=3\left(\frac{1}{x}+\frac{1}{1200-x}\right)$ **M1** (using (a)) $\rightarrow t=3\ln x$ **A1** $-\ 3\ln(1200-x)$ **A1** (+C) then as above for **dM1 A1**			
(c)(i)	Using $x=600$ $\quad t=3\ln\left(\frac{1800}{1200-600}\right)$ OE	M1		e.g. $t=3\ln 3$ or 3.29…
	14.20 or 2.20(p.m.)	A1	2	**NMS**: correct time scores **2/2**
(c)(ii)	$t=3\ln\left(\frac{3x}{1200-x}\right)$ $\rightarrow$ $\frac{3x}{1200-x}=e^{\frac{t}{3}}$	M1		OE for RHS - e.g. $\sqrt[3]{e^t}$
	$x=\frac{1200e^{\frac{t}{3}}}{3+e^{\frac{t}{3}}}$	A1		
	(when $t=4$) $\quad x=670$	B1	3	CAO ; **not** 670.094…
	OE could also include $e^t=\left(\frac{3x}{1200-x}\right)^3$ etc. It is possible to score **M1 A0 B1.**			
	Total	**12**		

8 It is given that $\theta = \tan^{-1}\left(\frac{3x}{2}\right)$.

(a) By writing $\theta = \tan^{-1}\left(\frac{3x}{2}\right)$ as $2\tan\theta = 3x$, use implicit differentiation to show that $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{k}{4+9x^2}$, where k is an integer.

[3 marks]

(b) Hence solve the differential equation

$$9y(4+9x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosec} 3y$$

given that $x = 0$ when $y = \frac{\pi}{3}$. Give your answer in the form $\mathrm{g}(y) = \mathrm{h}(x)$.

[7 marks]

Q8	Solution	Mark	Total	Comment
(a)	$\theta = \tan^{-1}\left(\frac{3x}{2}\right) \Rightarrow 2\tan\theta = 3x$			
	$2\sec^2\theta.\frac{\mathrm{d}\theta}{\mathrm{d}x} = 3$	B1		or $2\sec^2\theta = 3\,\frac{dx}{d\theta}$
	$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{3x}{2}\right)^2$	M1		Use of correct identity to get $\sec^2\theta$ in terms of x ; condone missing bracket.
	$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{3}{2\left(1+\frac{9x^2}{4}\right)} = \frac{6}{4+9x^2}$	A1	3	**Correct** algebra with $k = 6$. $\frac{k}{4+9x^2}$ **is given** so be convinced
(b)	$9y(4+9x^2)\frac{dy}{\mathrm{d}x} = \mathrm{cosec}\,3y$			
	$\int 9y\sin 3y\ \mathrm{d}y = \int \frac{1}{4+9x^2}\,\mathrm{d}x$	B1		**Correct** separation including dy and dx at 'ends' of integrals (do not penalise unless the dy or dx is directly under the fraction bar), the integral signs at the 'front' and the integrands. The 9 can be on RHS as $\frac{1}{9}$.
	LHS : parts with $u = (9)y$ and $\mathrm{d}v = \sin 3y$	M1		**M1** for $\mathrm{p\,y\cos 3y + q\int\cos 3y\,dy}$
	$(9)\left(-\frac{1}{3}\,y\cos 3y + \int\frac{1}{3}\cos 3y\ \mathrm{d}y\right)$	A1		
	$-3y\cos 3y + \sin 3y$	A1		If 9 used on the LHS
	RHS: $\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$	B1ft		ft on k from (a) i.e. $\frac{1}{k}\tan^{-1}\left(\frac{3x}{2}\right)$.
	$-3y\cos 3y + \sin 3y = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$			
	Using $x = 0$ and $y = \frac{\pi}{3}$ to find c	M1		Must have an expression of form $\mathrm{p\sin 3y + qy\cos 3y = r\tan^{-1}\left(\frac{3x}{2}\right) + c}$ and use x=0 and $y = \frac{\pi}{3}$ to find c. PI by a correct **ft** value for c.
	$\sin\pi - 3.\frac{\pi}{3}.\cos\pi = 0 + c$ gives $c = \pi$			
	$-3y\cos 3y + \sin 3y = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + \pi$	A1	7	OE but must be a complete, correct expression

8 (a) A pond is initially empty and is then filled gradually with water. After t minutes, the depth of the water, x metres, satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sqrt{4 + 5x}}{5(1 + t)^2}$$

Solve this differential equation to find x in terms of t.

[7 marks]

(b) Another pond is gradually filling with water. After t minutes, the surface of the water forms a circle of radius r metres. The rate of change of the radius is inversely proportional to the area of the surface of the water.

(i) Write down a differential equation, in the variables r and t and a constant of proportionality, which represents how the radius of the surface of the water is changing with time.

(You are not required to solve your differential equation.)

[3 marks]

(ii) When the radius of the pond is 1 metre, the radius is increasing at a rate of 4.5 metres per second. Find the radius of the pond when the radius is increasing at a rate of 0.5 metres per second.

[2 marks]

Q8	Solution	Mark	Total	Comment
(a)(i)	$\int \frac{1}{\sqrt{4+5x}}\,dx = \int \frac{1}{5(1+t)^2}\,dt$	B1		Correct separation and notation seen on a single line somewhere in their solution
	$a(4+5x)^{\frac{1}{2}}$ or $b(1+t)^{-1}$	M1		OE $a\sqrt{4+5x}$ or $b\left(\frac{1}{1+t}\right)$
	$\frac{2}{5}(4+5x)^{\frac{1}{2}}$	A1		OE $\frac{2}{5}\sqrt{4+5x}$
	$-\frac{1}{5}(1+t)^{-1} \quad (+C)$	A1		OE $-\frac{1}{5(1+t)}$
	$x=0,\ t=0 \Rightarrow C=1$	m1		Use $(0,0)$ to find a constant
	$\frac{2}{5}(4+5x)^{\frac{1}{2}} = 1-\frac{1}{5}(1+t)^{-1}$	A1		**OE**
	$x=\frac{5}{4}\left(1-\frac{(1+t)^{-1}}{5}\right)^2-\frac{4}{5}$	A1	7	ACF eg $x=\frac{1}{20}\left(\frac{4+5t}{1+t}\right)^2-\frac{4}{5}$
(b)(i)	$\frac{dr}{dt}$	B1		Seen; allow R for r
	$\frac{1}{r^2}$	M1		$\frac{1}{r^2}$ seen **;** allow R for r
	$\frac{dr}{dt}=\frac{k}{r^2}$	A1	3	Any constant k including $\frac{c}{\pi}$ but **not** including variable t Must use R or r consistently
(ii)	$\left(\frac{dr}{dt}\right)=4.5=\frac{k}{1^2}$ or $4.5=\frac{c}{\pi\times 1^2}$	M1		**Use** $\frac{dr}{dt}=4.5$ with $r=1$ to find a value for the constant
	$0.5=\frac{4.5}{r^2}\Rightarrow r=3$ (metres)	A1	2	
	Total		**12**	

5.

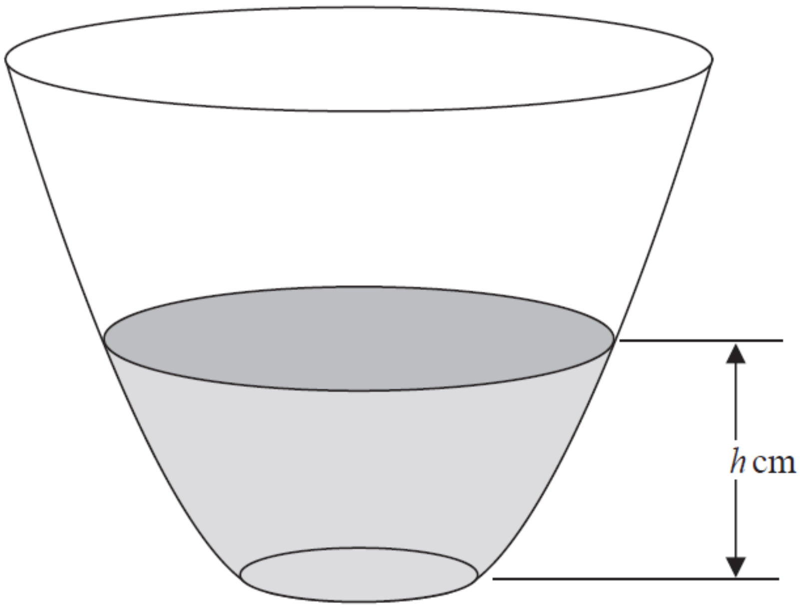

Figure 3

A container with a circular cross-section is shown in Figure 3.

Initially the container is empty. At time t seconds after water begins to flow into the container, the height of water in the container is h cm.

The height of water in the container satisfies the differential equation

$$\frac{dh}{dt} = \frac{k}{h+4} \qquad 0 \leqslant h \leqslant 35$$

where k is a constant.

When $h = 16$, the height of water in the container is increasing at a rate of $0.6\,\text{cm s}^{-1}$

(a) Find the value of k.

(1)

(b) Find the time taken to fill the container with water from empty to a height of 30 cm.

(4)

Given that the water flows into the container at a constant rate of $96\pi\,\text{cm}^3\,\text{s}^{-1}$

(c) find the volume of water in the container when $h = 30$

Give your answer in cm^3 to 3 significant figures.

(2)

(a)	$\left\{h=16, \frac{dh}{dt}=0.6 \Rightarrow 0.6=\frac{k}{16+4} \Rightarrow\right\}$ $k=12$	$k=12$	B1
			(1)
(b) **Way 1**	$\left\{\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{12}{h+4} \Rightarrow \int (h+4)\mathrm{d}h = \int 12\ \mathrm{d}t\right\}$		
	$\frac{h^2}{2}+4h=12t\ \{+c\}$	$h \pm 4 \to \alpha h^2+\beta h;\ \alpha, \beta \neq 0$	M1
		Correct integration with a follow through on their k found in (a). Ignore limits or $+c$	A1ft
	$\left\{\left[\frac{h^2}{2}+4h\right]_0^{30}=\left[12t\right]_0^T\right\}$		
	$\frac{900}{2}+120=12T \Rightarrow T=...$	**dependent on the previous M mark** Applies $h=30$ to an integrated equation of the form $\alpha h^2+\beta h=\gamma t+c;\ \alpha, \beta, \gamma \neq 0$ (c can be 0) and solves to find a value for the time taken	dM1
	$\left\{T=\frac{570}{12} \Rightarrow\right\}$ $T=47.5$ (seconds)	47.5	A1
			(4)
(c) **Way 1**	$\left\{\frac{\mathrm{d}V}{\mathrm{d}t}=96\pi \Rightarrow\right\}$ Volume $=96\pi(47.5)$	96π(their "47.5"), where their "47.5" is positive	M1
	$\{=4560\pi=14325.6625...\}$ $=14300$ (cm^3) (3 sf)	14300	A1 **cao**
			(2)

4.

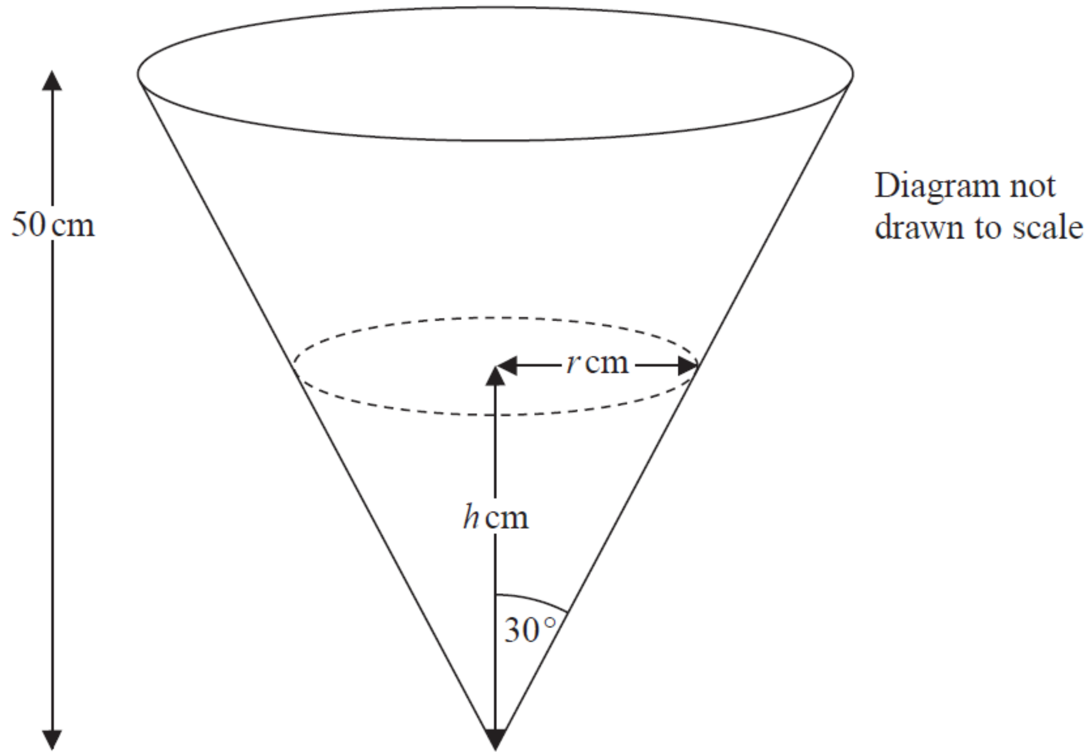

Figure 1

A water container is made in the shape of a hollow inverted right circular cone with semi-vertical angle of 30°, as shown in Figure 1. The height of the container is 50 cm.

When the depth of the water in the container is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(a) Show that $V = \frac{1}{9}\pi h^3$

[*You may assume the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(2)

Given that the volume of water in the container increases at a constant rate of 200 cm^3 s^{-1},

(b) find the rate of change of the depth of the water, in cm s^{-1}, when $h = 15$
Give your answer in its simplest form in terms of π.

(4)

4. (a)	$\frac{r}{h} = \tan 30 \Rightarrow r = h\tan 30 \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $\frac{h}{r} = \tan 60 \Rightarrow r = \frac{h}{\tan 60} \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $\frac{r}{\sin 30} = \frac{h}{\sin 60} \Rightarrow r = \frac{h\sin 30}{\sin 60} \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $h^2 + r^2 = (2r)^2 \Rightarrow r^2 = \frac{1}{3}h^2$	Correct use of trigonometry to find r in terms of h **or** correct use of Pythagoras to find r^2 in terms of h^2	M1
	$\left\{V = \frac{1}{3}\pi r^2 h \Rightarrow\right\} V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h \Rightarrow V = \frac{1}{9}\pi h^3$ *	Correct proof of $V = \frac{1}{9}\pi h^3$ or $V = \frac{1}{9}h^3\pi$ Or shows $\frac{1}{9}\pi h^3$ or $\frac{1}{9}h^3\pi$ with some reference to $V =$ in their solution	A1 *
			[2]
(b) **Way 1**	$\frac{dV}{dt} = 200$		
	$\frac{dV}{dh} = \frac{1}{3}\pi h^2$	$\frac{1}{3}\pi h^2$ o.e.	B1
	Either • $\left\{\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt} \Rightarrow\right\} \left(\frac{1}{3}\pi h^2\right)\frac{dh}{dt} = 200$ • $\left\{\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} \Rightarrow\right\} \frac{dh}{dt} = 200 \times \frac{1}{\frac{1}{3}\pi h^2}$	**either** $\left(\text{their } \frac{dV}{dh}\right) \times \frac{dh}{dt} = 200$ **or** $200 \div \left(\text{their } \frac{dV}{dh}\right)$	M1
	When $h = 15,\ \frac{dh}{dt} = 200 \times \frac{1}{\frac{1}{3}\pi(15)^2} \left\{= \frac{200}{75\pi} = \frac{600}{225\pi}\right\}$	**dependent on the previous M mark**	dM1
	$\frac{dh}{dt} = \frac{8}{3\pi}$ (cm s^{-1})	$\frac{8}{3\pi}$	A1 **cao**
			[4]

6. Given that $y = 2$ when $x = -\frac{\pi}{8}$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{3\cos^2 2x} \qquad -\frac{1}{2} < x < \frac{1}{2}$$

giving your answer in the form $y = \mathrm{f}(x)$.

(6)

6.	$\frac{dy}{dx} = \frac{y^2}{3\cos^2 2x}$; $-\frac{1}{2} < x < \frac{1}{2}$; $y = 2$ at $x = -\frac{\pi}{8}$		
	$\int \frac{1}{y^2}\,dy = \int \frac{1}{3\cos^2 2x}\,dx$	Separates variables as shown Can be implied by a correct attempt at integration Ignore the integral signs	B1
	$\int \frac{1}{y^2}\,dy = \int \frac{1}{3}\sec^2 2x\,dx$		
	$-\frac{1}{y} = \frac{1}{3}\left(\frac{\tan 2x}{2}\right)\ \{+c\}$	$\pm\frac{A}{y^2} \to \pm\frac{B}{y}$; $A, B \neq 0$	M1
		$\pm\lambda\tan 2x$	M1
		$-\frac{1}{y} = \frac{1}{3}\left(\frac{\tan 2x}{2}\right)$	A1
	$-\frac{1}{2} = \frac{1}{6}\tan\left(2\left(-\frac{\pi}{8}\right)\right) + c$	Use of $x = -\frac{\pi}{8}$ and $y = 2$ in an integrated equation ***containing a constant of integration***, e.g. c	M1
	$-\frac{1}{2} = -\frac{1}{6} + c \Rightarrow c = -\frac{1}{3}$		
	$-\frac{1}{y} = \frac{1}{6}\tan 2x - \frac{1}{3} = \frac{\tan(2x) - 2}{6}$		
	$y = \frac{-1}{\frac{1}{6}\tan 2x - \frac{1}{3}}$ or $y = \frac{6}{2 - \tan 2x}$ or $y = \frac{6\cot 2x}{-1 + 2\cot 2x}$ $\left\{-\frac{1}{2} < x < \frac{1}{2}\right\}$		A1 o.e.
			[6]

7.

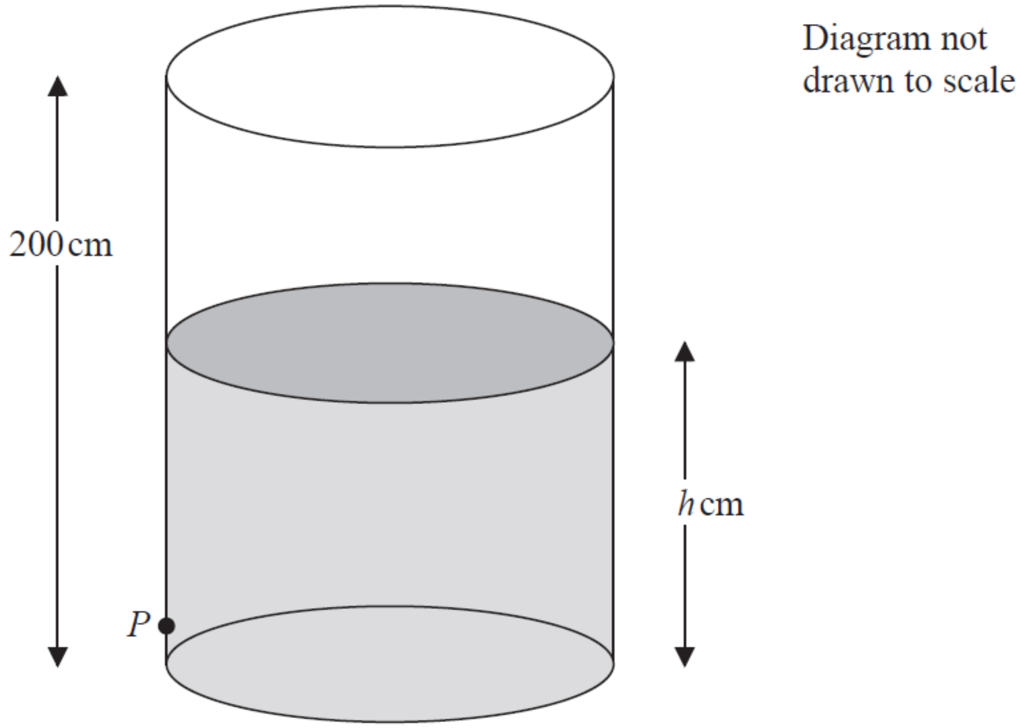

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

(a)	$-1.1 = k\sqrt{(130-9)}$ □ $k = ...$	Substitutes $h = 130$ and either $\frac{\mathrm{d}h}{\mathrm{d}t} = -1.1$ or $\frac{\mathrm{d}h}{\mathrm{d}t} = 1.1$ into the printed equation and rearranges to give $k = ...$	M1
	so, $k = -\frac{1}{10}$ or -0.1	$k = -\frac{1}{10}$ or -0.1	A1
			[2]
(b) **Way 1**	$\int \frac{\mathrm{d}h}{\sqrt{(h-9)}} = \int k\,\mathrm{d}t$	Separates the variables correctly. $\mathrm{d}h$ and $\mathrm{d}t$ should not be in the wrong positions, although this mark can be implied by later working. Ignore the integral signs.	B1
	$\int (h-9)^{-\frac{1}{2}}\,\mathrm{d}h = \int k\,\mathrm{d}t$		
	$\frac{(h-9)^{\frac{1}{2}}}{(\frac{1}{2})} = kt\,(+c)$	Integrates $\frac{\pm\lambda}{\sqrt{(h-9)}}$ to give $\pm\mu\sqrt{(h-9)}$; λ, μ □ 0	M1
		$\frac{(h-9)^{\frac{1}{2}}}{(\frac{1}{2})} = kt$ or $\frac{(h-9)^{\frac{1}{2}}}{(\frac{1}{2})} = (\text{their } k)t$, with/without $+ c$, or equivalent, which can be un-simplified or simplified.	A1
	$\{t = 0, h = 200$ □ $\}$ $2\sqrt{(200-9)} = k(0) + c$	Some evidence of applying both $t = 0$ and $h = 200$ to changed equation containing a constant of integration, e.g. c or A	M1
	□ $c = 2\sqrt{191}$ □ $2(h-9)^{\frac{1}{2}} = -0.1t + 2\sqrt{191}$ $\{h = 50 \Rightarrow\}$ $2\sqrt{(50-9)} = -0.1t + 2\sqrt{191}$ $t = ...$	**dependent on the previous M mark** Applies $h = 50$ and their value of c to their changed equation and rearranges to find the value of $t = ...$	dM1
	$t = 20\sqrt{191} - 20\sqrt{41}$ or $t = 148.3430145... = 148$ (minutes) (nearest minute)	$t = 20\sqrt{191} - 20\sqrt{41}$ isw or awrt 148	A1 cso

4. The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{dx}{dt} = -\frac{5}{2}x, \qquad t \geqslant 0$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(a) solve the differential equation, giving x in terms of t. You should show all steps in your working and give your answer in its simplest form.

(4)

(b) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams. Give your answer to the nearest minute.

(3)

(a) **Way 1**	$\int \frac{1}{x}\,dx = \int -\frac{5}{2}\,dt$	Separates variables as shown. dx and dt should not be in the wrong positions, though this mark can be implied by later working. Ignore the integral signs.	B1
	$\ln x = -\frac{5}{2}t + c$	Integrates both sides to give **either** $\pm\frac{\alpha}{x} \to \pm\alpha\ln x$ **or** $\pm k \to \pm kt$ (with respect to t); $k, \alpha \neq 0$	M1
		$\ln x = -\frac{5}{2}t + c$, including "$+c$"	A1
	$\{t = 0, x = 60 \Rightarrow\}$ $\ln 60 = c$ $\ln x = -\frac{5}{2}t + \ln 60 \Rightarrow$ $\underline{x = 60e^{-\frac{5}{2}t}}$ or $\underline{x = \frac{60}{e^{\frac{5}{2}t}}}$	Finds their c and uses correct algebra to achieve $x = 60e^{-\frac{5}{2}t}$ or $x = \frac{60}{e^{\frac{5}{2}t}}$ with **no incorrect working seen**	A1 cso
			[4]
(a) **Way 2**	$\frac{dt}{dx} = -\frac{2}{5x}$ or $t = \int -\frac{2}{5x}\,dx$	**Either** $\frac{dt}{dx} = -\frac{2}{5x}$ **or** $t = \int -\frac{2}{5x}\,dx$	B1
	$t = -\frac{2}{5}\ln x + c$	Integrates both sides to give **either** $t = ...$ **or** $\pm\alpha\ln px;\ \alpha \neq 0,\ p > 0$	M1
		$t = -\frac{2}{5}\ln x + c$, including "$+c$"	A1
	$\{t = 0, x = 60 \Rightarrow\}$ $c = \frac{2}{5}\ln 60 \Rightarrow t = -\frac{2}{5}\ln x + \frac{2}{5}\ln 60$ $\Rightarrow -\frac{5}{2}t = \ln x - \ln 60 \Rightarrow$ $\underline{x = 60e^{-\frac{5}{2}t}}$ or $\underline{x = \frac{60}{e^{\frac{5}{2}t}}}$	Finds their c and uses correct algebra to achieve $x = 60e^{-\frac{5}{2}t}$ or $x = \frac{60}{e^{\frac{5}{2}t}}$ with **no incorrect working seen**	A1 cso
			[4]
(b)	$20 = 60e^{-\frac{5}{2}t}$ or $\ln 20 = -\frac{5}{2}t + \ln 60$	Substitutes $x = 20$ into an equation in the form of **either** $x = \pm\lambda e^{\pm\mu t} \pm \beta$ **or** $x = \pm\lambda e^{\pm\mu t \pm \alpha\ln\delta x}$ **or** $\pm\alpha\ln\delta x = \pm\mu t \pm \beta$ **or** $t = \pm\lambda\ln\delta x \pm \beta$; $\alpha, \lambda, \mu, \delta \neq 0$ and β can be 0	M1
	$t = -\frac{2}{5}\ln\left(\frac{20}{60}\right)$ $\{= 0.4394449...\text{ (days)}\}$ **Note: t must be greater than 0**	**dependent on the previous M mark** Uses correct algebra to achieve an equation of the form of either $t = A\ln\left(\frac{60}{20}\right)$ or $A\ln\left(\frac{20}{60}\right)$ or $A\ln 3$ or $A\ln\left(\frac{1}{3}\right)$ o.e. or $t = A(\ln 20 - \ln 60)$ or $A(\ln 60 - \ln 20)$ o.e. ($A \in$ □, $t > 0$)	dM1
	$\Rightarrow t = 632.8006... = 633$ (to the nearest minute)	awrt 633 **or** 10 hours and awrt 33 minutes	A1 cso
	Note: dM1 can be implied by $t =$ awrt 0.44 from no incorrect working.		

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t\,, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - \mathrm{e}^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

7. (a)	$\frac{2}{P(P-2)} = \frac{A}{P} + \frac{B}{(P-2)}$	
	$2 \equiv A(P-2) + BP$ — Can be implied.	M1
	$A = -1,\ B = 1$ — Either one.	A1
	giving $\frac{1}{(P-2)} - \frac{1}{P}$ — See notes. **cao, aef**	A1
		[3]
(b)	$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t$	
	$\int \frac{2}{P(P-2)}\,\mathrm{d}P = \int \cos 2t\,\mathrm{d}t$ — can be implied by later working	B1 oe
	$\ln(P-2) - \ln P = \frac{1}{2}\sin 2t \quad (+c)$ — $\pm\lambda\ln(P-2) \pm \mu\ln P$, $\lambda \neq 0,\ \mu \neq 0$	M1
	$\ln(P-2) - \ln P = \frac{1}{2}\sin 2t$	A1
	$\{t = 0, P = 3 \Rightarrow\}\quad \ln 1 - \ln 3 = 0 + c \quad \{\Rightarrow\ c = -\ln 3 \text{ or } \ln(\tfrac{1}{3})\}$ — See notes	M1
	$\ln(P-2) - \ln P = \frac{1}{2}\sin 2t - \ln 3$ $\ln\left(\frac{3(P-2)}{P}\right) = \frac{1}{2}\sin 2t$	
	$\frac{3(P-2)}{P} = \mathrm{e}^{\frac{1}{2}\sin 2t}$ — Starting from an equation of the form $\pm\lambda\ln(P-\beta) \pm \mu\ln P = \pm K\sin\delta t + c$, $\lambda,\mu,\beta,K,\delta \neq 0$, applies a fully correct method to eliminate their logarithms. **Must have a constant of integration that need not be evaluated (see note)**	M1
	$3(P-2) = P\mathrm{e}^{\frac{1}{2}\sin 2t} \Rightarrow 3P - 6 = P\mathrm{e}^{\frac{1}{2}\sin 2t}$ gives $3P - P\mathrm{e}^{\frac{1}{2}\sin 2t} = 6 \Rightarrow P(3 - \mathrm{e}^{\frac{1}{2}\sin 2t}) = 6$ — A complete method of rearranging to make P the subject. **Must have a constant of integration that need not be evaluated (see note)**	dM1
	$P = \frac{6}{(3 - \mathrm{e}^{\frac{1}{2}\sin 2t})}$ * — Correct proof.	A1 * cso
(c)	$\{\text{population} = 4000 \Rightarrow\}\ P = 4$ — States $P = 4$ or applies $P = 4$	M1
	$\frac{1}{2}\sin 2t = \ln\left(\frac{3(4-2)}{4}\right) \left\{= \ln\left(\frac{3}{2}\right)\right\}$ — Obtains $\pm\lambda\sin 2t = \ln k$ or $\pm\lambda\sin t = \ln k$, $\lambda \neq 0,\ k > 0$ where λ and k are numerical values and λ can be 1	M1
	$t = 0.4728700467...$ — anything that rounds to 0.473. Do not apply isw here	A1
		[3]