7 Alex is investigating changes in the selling price of houses. One particular house was sold on 1 January 1950 for £3000. This house was resold on 1 January 2010 for £$240\ 000$.

Alex proposes a model

$$P = Ak^t$$

for the selling price, £P, of this house, where t is the time in years after 1 January 1950 and A and k are constants.

(a) (i) Write down the value of A.

[1 mark]

(ii) Show that, correct to six decimal places, $k = 1.075767$

[2 marks]

(iii) Use logarithms and this model to estimate the year during which the selling price of this house first reached £$450\ 000$.

[3 marks]

(b) For another house that was sold on 1 January 1960 for £5100, Alex proposes the model

$$Q = 5100 \times 1.0785^t$$

for the selling price, £Q, of this house, t years after 1 January 1960.

(i) Use this model to estimate the selling price of this house 10 years previously on 1 January 1950, giving your answer to the nearest £100.

[2 marks]

(ii) Use logarithms to find the year during which the two models predict that the selling price of these two houses will be the same.

[4 marks]

Q 7	Solution	Mark	Total	Comment
(a)(i)	$A = 3000$	B1	1	From $P = 3000$ when $t = 0$.
(ii)	$240\,000 = 3\,000\,k^{60}$ $k^{60} = 80$ PI $k = \sqrt[60]{80}$ $= 1.075767$	M1 A1	 2	OE: $\ln 240000 = \ln 3000 + 60 \ln k$ $\ln k = \frac{\ln 240000 - \ln 3000}{60}$ or better See below
	To earn the **A1** we must see either a correct exact answer for $k$or $\ln k$ (as shown) or any correct decimal answer to greater accuracy $(1.075766873 \ldots)$ followed by the printed 6 decimal place value.			
(iii)	$450\,000 = 3\,000\,k^{t}$ $k^{t} = 150$ $t \ln k = \ln 150$ OE $t = 68.6 \ldots$ 2018	M1 dM1 A1	 3	Setting up an equation for k^t Correct method for a linear equation in t or $t = log_k 150$ - PI by $t = 68.6 \ldots/69$. Accept 2019 but from correct working.
	As implied, the first two marks can be earned without a numerical value for k but are for correctly reaching a linear equation in t - e.g. as above or such as $\ln 450\,000 = \ln 3\,000 + t \ln k$ etc.			
(b)(i)	$Q = 5\,100 \times 1.0785^{-10}$ $= 2\,400$	M1 A1	 2	Using $t = -10$ in given expression. Must be this value but allow 2400.00
(ii)	**<u>Alternative 1 - using models as given</u>** $5100 \times 1.0785^{t-10} = 3000 \times k^{t}$ $\ln 5100 + (t - 10) \ln 1.0785$ $= \ln 3000 + t \ln k$ $t = \frac{\ln 3000 - \ln 5100 + 10 \ln 1.0785}{\ln 1.0785 - \ln k}$ or $88.7 \ldots$ (Year) 2038	M1 dM1 A1 A1	 4	k needn't be numerical or could be wrong to earn first two marks. OE: $\ln 1.7 + (t - 10) \ln 1.0875 = t \ln k$ Using rules of logs **correctly** $t = \frac{10 \ln 1.0785 - \ln 1.7}{\ln 1.0785 - \ln k}$ or $88.7 \ldots$ Accept 2039 from correct working.

5 In a conservation area, a disease is spreading amongst two species of wild animal, P and Q, which is reducing their numbers.

Previous experience has shown that the number of each of the species P and Q can be modelled by

$$\mathrm{p}(t) = 4500\mathrm{e}^{-\frac{1}{20}t} \quad \text{and} \quad \mathrm{q}(t) = 3000\mathrm{e}^{-\frac{1}{40}t} \text{ respectively}$$

where t is the time in weeks after the disease is first detected.

This outbreak of the disease was first detected on 1 May.

(a) Use the two models to find:

(i) the number of species P on 1 May;

[1 mark]

(ii) the number of species Q after 36 weeks from 1 May, giving your answer to the nearest 10;

[1 mark]

(iii) after how many weeks the number of species P will first fall below 1500.

[2 marks]

(b) Use logarithms and the two models to calculate the value of t when the number of species Q will be four times that of species P. Give your answer to the nearest whole number.

[3 marks]

(c) When $t = T$ the number of species Q first exceeds that of species P by 300.

(i) Use this information and the two models to derive a quadratic equation in x where $x = \mathrm{e}^{-\frac{1}{40}T}$.

[2 marks]

(ii) Hence find the number of days after 1 May when this difference of 300 animals will first occur. Give your answer to the nearest day.

[3 marks]

Q5	Solution	Mark	Total	Comment
(a)(i)	4500	B1	1	
(a)(ii)	1220	B1	1	
(a)(iii)	$4500e^{-\frac{1}{20}t} < 1500$			
	$\frac{1}{20}t > \ln 3$ or $-\frac{1}{20}t < \ln\frac{1}{3}$ or better	M1		Correctly converting from exponential to logarithmic form
	22	A1	2	Allow 21.97… **NMS** scores **B2** for 22 or 21.97…
(b)	$(Q = 4P \Rightarrow)$ $3000e^{-\frac{1}{40}t} = 4(4500)e^{-\frac{1}{20}t}$ OE	M1		Setting up a correct equation but **M0** if logs **not** used later.
	$\frac{t}{40} = \ln 6$ or $-\frac{t}{40} = \ln\frac{1}{6}$ OE	A1		e.g. $\ln 3000 - \frac{t}{40} = \ln 18000 - \frac{t}{20}$
	72	A1	3	CAO
(c)(i)	$3000e^{-\frac{1}{40}T} - 4500e^{-\frac{1}{20}T} = 300$	M1		Setting up a correct equation – could include both *x* and *T* (or *t*).
	$\left(x = e^{-\frac{1}{40}T}\right) \Rightarrow 3000x - 4500x^2 = 300$ $15x^2 - 10x + 1 = 0$	A1	2	Correct quadratic in x (ACF) - apply ISW for wrong cancelling or rearranging.
(c)(ii)	$(x) = \frac{10 \pm \sqrt{40}}{30}$ (0.12… or 0.54…)	dM1		
	$T = 24.3(41\ldots)$	A1		Allow 24
	$= 170$ (days)	A1	3	Accept October 18th if 170 not seen.
			12	

4 The mass of radioactive atoms in a substance can be modelled by the equation

$$m = m_0 k^t$$

where m_0 grams is the initial mass, m grams is the mass after t days and k is a constant. The value of k differs from one substance to another.

(a) (i) A sample of radioactive iodine reduced in mass from 24 grams to 12 grams in 8 days.

Show that the value of the constant k for this substance is 0.917004, correct to six decimal places.

[1 mark]

(ii) A similar sample of radioactive iodine reduced in mass to 1 gram after 60 days.

Calculate the initial mass of this sample, giving your answer to the nearest gram.

[2 marks]

(b) The half-life of a radioactive substance is the time it takes for a mass of m_0 to reduce to a mass of $\frac{1}{2}m_0$.

A sample of radioactive vanadium reduced in mass from exactly 10 grams to 8.106 grams in 100 days.

Find the half-life of radioactive vanadium, giving your answer to the nearest day.

[4 marks]

Q4	Solution	Mark	Total	Comment
(a)(i)	$m = m_0 k^t$ Using $m = 12, m_0 = 24$ and $t = 8$ $k^8 = \frac{1}{2}$ or $k = \left(\sqrt[8]{0.5}\right)$ $= 0.917004$	B1	1	$12 = 24k^8$ OE e.g. $k = \left(\frac{1}{2}\right)^{\frac{1}{8}}$ Must see a correct exact expression for k or k^8 or k=0.91700404(32....) to at least 8 d. p. AG be convinced
	Note that **AG** so to earn the mark they must show us a correct exact expression for k or k^8. Accept such as $k = e^{\left(\frac{ln0.5}{8}\right)}$ or $e^{-0.086643...}$ or $\left(\frac{1}{2}\right)^{\frac{1}{8}}$ or 0.91700404(32 ...) as sufficient evidence but withhold the mark if a clear error has been made – e.g. $k = \sqrt[\frac{1}{8}]{0.5}$. Candidates who work with logs must reach an expression such as $\log k = \frac{\log 12 - \log 24}{8}$ first.			
(a)(ii)	$1 = m_0(0.917004)^{60}$ $m_0 = 181$	M1 A1	 2	or $m_0 = (0.917004)^{-60}$ PI by **A1** later Must be 181 no **ISW**
	NMS scores **SC2** for 181 **only** but sight of greater accuracy (181.0198...) implies **M1** if 181 not seen.			
(b)	$m = m_0 k^t$ $8.106 = 10 \times k^{100}$ $k = \sqrt[100]{0.8106}$ OE $\frac{1}{2}m_0 = m_0 k^t$ $k^t = \frac{1}{2}$ $t \log k = \log\left(\frac{1}{2}\right)$ $t = \frac{\log\left(\frac{1}{2}\right)}{\log k}$ $= 330$	 M1 A1 M1 A1	 4	 OE: e.g. $k = e^{\ln(0.8106)/100}$ A linear equation in t from $k^t = \frac{1}{2}$ e.g. $t = log_k(0.5)$ Must be 330 No **ISW**
	For guidance, for first **A1**, $k = 0.9979$ ... **PI** by later correct work. The first **M1** is for a correct interpretation of the information given so could equally be awarded for an expression involving logs of k such as ln 8.106 = ln10 + 100 ln k then **A1** for a correct expression for lnk such as $\ln k = \frac{\ln 8.106 - \ln 10}{100}$ or, using base 10, $\log k = \frac{\log 8.106 - 1}{100}$. Those who use the value of k from (a) could only score **M0 A0 M1 A0**. **NMS** scores **SC4** for 330 **only** but sight of greater accuracy (330.1006...) implies **M1 A1 M1** if 330 not seen.			

8. A study is being carried out on two colonies of ants.

The number of ants N_A in colony A, t years after the start of the study, is modelled by the equation

$$N_A = 3000 + 600e^{0.12t} \qquad t \in \mathbb{R}, t \geqslant 0$$

Using the model,

(a) find the time taken, from the start of the study, for the number of ants in colony A to double. Give your answer, in years, to 2 decimal places.

(5)

(b) Show that $\dfrac{dN_A}{dt} = pN_A + q$, where p and q are constants to be determined.

(3)

The number of ants N_B in colony B, t years after the start of the study, is modelled by the equation

$$N_B = 2900 + Ce^{kt} \qquad t \in \mathbb{R}, t \geqslant 0$$

where C and k are positive constants.

According to this model, there will be 3100 ants in colony B one year after the start of the study and 3400 ants in colony B two years after the start of the study.

(c) (i) Show that $k = \ln\left(\dfrac{5}{2}\right)$

(ii) Find the value of C.

(4)

Question	Scheme	Marks
8 (a)	Substitute $N_A = 7200$ in $N_A = 3000 + 600e^{0.12t}$	B1
	$\Rightarrow e^{0.12t} = 7$	M1 A1
	$\Rightarrow t = \dfrac{\ln 7}{0.12} = 16.22$ years	M1, A1
		(5)
(b)	Differentiates to achieve $\dfrac{dN_A}{dt} = \beta e^{0.12t}$ $\left[\dfrac{dN_A}{dt} = 600 \times 0.12e^{0.12t}\right]$	M1
	Substitutes $e^{0.12t} = \dfrac{N_A - 3000}{600}$ into $\dfrac{dN_A}{dt} = \beta e^{0.12t}$ OR Substitutes $600e^{0.12t} = N_A - 3000$ into $\dfrac{dN_A}{dt} = \alpha \times 600e^{0.12t}$	dM1
	$\Rightarrow \dfrac{dN_A}{dt} = 0.12(N_A - 3000) = 0.12N_A - 360$ or $\dfrac{3}{25}N_A - 360$	A1
		(3)
(c)(i)	$200 = Ce^k$ and $500 = Ce^{2k}$	M1
	$e^k = \dfrac{500}{200} \Rightarrow k = \ldots$ or $e^{-k} = \dfrac{200}{500} \Rightarrow k = \ldots$	dM1
	$k = \ln\left(\dfrac{5}{2}\right)$ * cso	A1*
(ii)	80	B1
		(4)
		(12 marks)

3. The value of a car is modelled by the formula

$$V = 16000e^{-kt} + A, \qquad t \geqslant 0,\ t \in \mathbb{R}$$

where V is the value of the car in pounds, t is the age of the car in years, and k and A are positive constants.

Given that the value of the car is £17 500 when new and £13 500 two years later,

(a) find the value of A,

(1)

(b) show that $k = \ln\left(\frac{2}{\sqrt{3}}\right)$

(4)

(c) Find the age of the car, in years, when the value of the car is £6000

Give your answer to 2 decimal places.

(4)

3(a)	$A = 1500$	B1
		(1)
(b)	Sub $t = 2, V = 13500 \Rightarrow 16000\mathrm{e}^{-2k} = 12000$	M1
	$\Rightarrow e^{-2k} = \frac{3}{4} \quad 0.75 \quad \text{oe}$	A1
	$\Rightarrow k = -\frac{1}{2}\ln\frac{3}{4}, = \ln\sqrt{\frac{4}{3}} = \ln\left(\frac{2}{\sqrt{3}}\right)$	dM1, A1*
		(4)
(c)	Sub $6000 = 16000\mathrm{e}^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} + '1500' \Rightarrow \mathrm{e}^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} = C$	M1
	$\Rightarrow \mathrm{e}^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} = \frac{45}{160} = 0.28125$	A1
	$\Rightarrow T = -\frac{\ln\left(\frac{45}{160}\right)}{\ln\left(\frac{2}{\sqrt{3}}\right)} = 8.82$	M1 A1
		(4)

8.

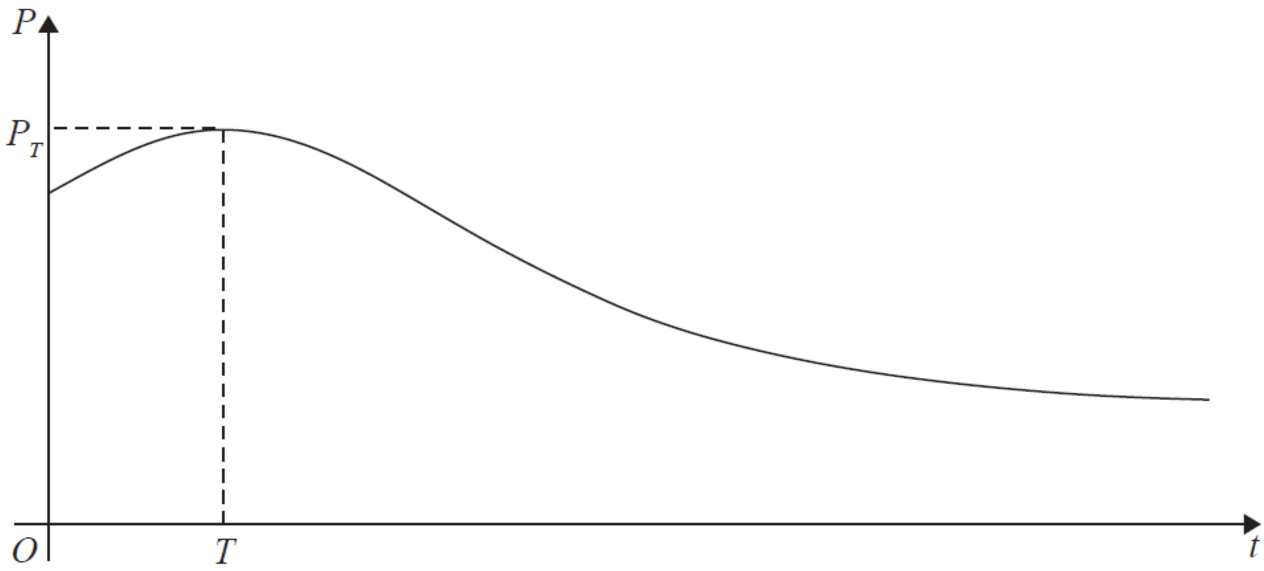

Figure 3

The number of rabbits on an island is modelled by the equation

$$P = \frac{100e^{-0.1t}}{1 + 3e^{-0.9t}} + 40, \qquad t \in \mathbb{R}, t \geqslant 0$$

where P is the number of rabbits, t years after they were introduced onto the island.

A sketch of the graph of P against t is shown in Figure 3.

(b) Find $\frac{dP}{dt}$

(3)

The number of rabbits initially increases, reaching a maximum value P_T when $t = T$

(c) Using your answer from part (b), calculate

(i) the value of T to 2 decimal places,

(ii) the value of P_T to the nearest integer.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

For $t > T$, the number of rabbits decreases, as shown in Figure 3, but never falls below k, where k is a positive constant.

(d) Use the model to state the maximum value of k.

(1)

8 (a)	$P_0 = \frac{100}{1+3} + 40 = 65$	B1	
			(1)
	$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{kt} = C\mathrm{e}^{kt}$	M1	
(b)	$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\left(1+3\mathrm{e}^{-0.9t}\right)\times -10\mathrm{e}^{-0.1t} - 100\mathrm{e}^{-0.1t}\times -2.7\mathrm{e}^{-0.9t}}{\left(1+3\mathrm{e}^{-0.9t}\right)^2}$	M1 A1	
			(3)
(c)(i)	At maximum $-10\mathrm{e}^{-0.1t} - 30\mathrm{e}^{-0.1t}\times\mathrm{e}^{-0.9t} + 270\mathrm{e}^{-0.1t}\times\mathrm{e}^{-0.9t} = 0$		
	$\mathrm{e}^{-0.1t}\left(-10+240\mathrm{e}^{-0.9t}\right) = 0$		
	$\mathrm{e}^{-0.9t} = \frac{10}{240}$ oe $\mathrm{e}^{0.9t} = 24$	M1	
	$-0.9t = \ln\left(\frac{1}{24}\right) \Rightarrow t = \frac{10}{9}\ln(24) = 3.53$	M1, A1	
(c) (ii)	Sub $t = 3.53 \Rightarrow P_T = 102$	A1	
			(4)
(d)	40	B1	

9. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams and t is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given.

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places.

(2)

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

(b) show that the **total** amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places.

(2)

No more doses of the antibiotic are given. At time T hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

(c) Show that $T = a\ln\left(b + \dfrac{b}{e}\right)$, where a and b are integers to be determined.

(4)

4. Water is being heated in an electric kettle. The temperature, θ °C, of the water t seconds after the kettle is switched on, is modelled by the equation

$$\theta = 120 - 100e^{-\lambda t}, \qquad 0 \leqslant t \leqslant T$$

(a) State the value of θ when $t = 0$

(1)

Given that the temperature of the water in the kettle is 70 °C when $t = 40$,

(b) find the exact value of λ, giving your answer in the form $\dfrac{\ln a}{b}$, where a and b are integers.

(4)

When $t = T$, the temperature of the water reaches 100 °C and the kettle switches off.

(c) Calculate the value of T to the nearest whole number.

(2)

9(a)	Subs $D = 15$ and $t = 4$ $\quad x = 15e^{-0.2\times4} = 6.740\ (mg)$	M1A1
		(2)
(b)	$15e^{-0.2\times7} + 15e^{-0.2\times2} = 13.754(mg)$	M1A1*
		(2)
(c)	$15e^{-0.2\times T} + 15e^{-0.2\times(T+5)} = 7.5$ $15e^{-0.2\times T} + 15e^{-0.2\times T}e^{-1} = 7.5$	M1
	$15e^{-0.2\times T}(1+e^{-1}) = 7.5 \Rightarrow e^{-0.2\times T} = \dfrac{7.5}{15(1+e^{-1})}$	dM1
	$T = -5\ln\left(\dfrac{7.5}{15\left(1+e^{-1}\right)}\right) = 5\ln\left(2+\dfrac{2}{e}\right)$	A1, A1
		(4)
		(8 marks)

4(a)	$(\theta =)20$	B1
		(1)
(b)	Sub $t = 40, \theta = 70 \Rightarrow 70 = 120 - 100e^{-40\lambda}$	
	$\Rightarrow e^{-40\lambda} = 0.5$	M1A1
	$\Rightarrow \lambda = \dfrac{\ln 2}{40}$	M1A1
		(4)
(c)	$\theta = 100 \Rightarrow T = \dfrac{\ln 0.2}{-\text{their}'\lambda'}$	M1
	$T = \text{awrt } 93$	A1
		(2)