Pure Sector 2: Trigonometry 3

Aims:

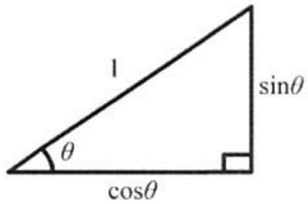
- To understand and use trigonometric identities
- To use trigonometric identities to solve more complex equations
- To construct proofs involving trigonometric functions and identities

Trigonometric Identities

You need to learn the following trig identities:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1$$



Example 1

Solve the equation $3\sin x = 4\cos x$ for $0 \le x \le 360°$ giving your answer to three significant figures.

$\frac{\sin x}{\cos x} = \frac{4}{3}$ $(\cos x \neq 0)$

$\tan x = \frac{4}{3}$

$x = 53.13\ldots, 180 + 53.13\ldots$

$= 53.13\ldots, 233.13\ldots$

$= \underline{53.1, 233}$

Example 2

Solve the equation $2\sin^2\theta = 3\cos\theta$ for $0 \le \theta \le 2\pi$ giving your answer to two decimal places.

$2(1-\cos^2\theta) = 3\cos\theta$

$2 - 2\cos^2\theta = 3\cos\theta$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -2$ (no roots)

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{5\pi}{3}$

$= \underline{1.05, 5.24}$

Proving Trigonometric Identities

Example 3

Show that $\frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \tan\theta$

$$LHS = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \tan\theta$$

$$= RHS$$

Example 4

Prove that $(\cos x - \tan x)^2 + (\sin x + 1)^2 = 2 + \tan^2 x$

$$LHS = (\cos x - \tan x)^2 + (\sin x + 1)^2$$

$$= \cos^2 x - 2\cos x \tan x + \tan^2 x + \sin^2 x + 2\sin x + 1$$

$$= (\cos^2 x + \sin^2 x) - 2\cancel{\cos x}\left(\frac{\sin x}{\cancel{\cos x}}\right) + 2\sin x + 1 + \tan^2 x$$

$$= 1 + 1 + \tan^2 x$$

$$= 2 + \tan^2 x$$

$$= RHS$$

Exam Question

(a) Solve the equation $\tan x = -3$ in the interval $0° \leqslant x \leqslant 360°$, giving your answers to the nearest degree. *(3 marks)*

(b) (i) Given that

$$7\sin^2\theta + \sin\theta\cos\theta = 6$$

show that

$$\tan^2\theta + \tan\theta - 6 = 0$$ *(3 marks)*

(ii) Hence solve the equation $7\sin^2\theta + \sin\theta\cos\theta = 6$ in the interval $0° \leqslant \theta \leqslant 360°$, giving your answers to the nearest degree. *(4 marks)*

a) $x = \tan^{-1}(-3)$

$= (-71.565...), 180 - 71.565..., 360 - 71.565...$

$= 108.4..., 288.4...$

$= \underline{108, 288}$

b)(i) $\dfrac{7\sin^2\theta}{\cos^2\theta} + \dfrac{\sin\theta\cos\theta}{\cos^2\theta} = \dfrac{6}{\cos^2\theta}$ $(\cos\theta \neq 0)$

$7\tan^2\theta + \tan\theta = \dfrac{6}{\cos^2\theta}$

$7\tan^2\theta + \tan\theta = \dfrac{6\sin^2\theta + 6\cos^2\theta}{\cos^2\theta}$

$7\tan^2\theta + \tan\theta = 6\tan^2\theta + 6$

$\tan^2\theta + \tan\theta - 6 = 0$

(ii) $(\tan\theta + 3)(\tan\theta - 2) = 0$

$\tan\theta = -3$ or $\tan\theta = 2$

$\theta = 108, 288$ $\theta = 63.43..., 243.43...$

$\underline{\theta = 63, 108, 243, 288}$

Further Trigonometric Identities

We can use the identities you have just met to form new identities.

$$\boxed{\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}}$$

If you divide each term in the identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ you get:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Which simplifies to:

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Similarly if you divide each term in the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ you get:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Which simplifies to:

$$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

These identities are used to solve trigonometric equations and to prove other identities. They are NOT in the formula booklet you must memorise them!

Example 5

Solve the equation $\sec^2 x = 4 + 2\tan x$, giving all solutions for $0° \leq x \leq 360°$.

$(1+\tan^2 x) = 4 + 2\tan x$

$\tan^2 x - 2\tan x - 3 = 0$

$(\tan x - 3)(\tan x + 1) = 0$

$\tan x = 3$ or $\tan x = -1$

$\underline{x = 72, 135, 252, 315}$

Example 6

Solve the equation $\operatorname{cosec}^2 x = 5 + 3\cot x$, giving all solutions for $0 \leq x \leq 2\pi$ to 3sf.

$(1+\cot^2 x) = 5 + 3\cot x$

$\cot^2 x - 3\cot x - 4 = 0$

$(\cot x - 4)(\cot x + 1) = 0$

$\cot x = 4$ or $\cot x = -1$

$\tan x = \frac{1}{4}$ $\quad \tan x = -1$

$\underline{x = 0.245, 2.36, 3.39, 5.50}$

Proving Trigonometric Identities

Example 7

Prove the identity $\sec^2 A - \operatorname{cosec}^2 A = (\tan A + \cot A)(\tan A - \cot A)$

$$\begin{aligned}
RHS &= (\tan A + \cot A)(\tan A - \cot A) \\
&= \tan^2 A - \cot^2 A \\
&= (\sec^2 A - 1) - (\operatorname{cosec}^2 A - 1) \\
&= \sec^2 A - 1 - \operatorname{cosec}^2 A + 1 \\
&= \sec^2 A - \operatorname{cosec}^2 A \\
&= LHS
\end{aligned}$$

Example 8

Prove the identity $\tan^2 A - \cot^2 A = (\sec A - \operatorname{cosec} A)(\sec A + \operatorname{cosec} A)$

$$\begin{aligned}
RHS &= (\sec A - \operatorname{cosec} A)(\sec A + \operatorname{cosec} A) \\
&= \sec^2 A - \operatorname{cosec}^2 A \\
&= (\tan^2 A + 1) - (\cot^2 A + 1) \\
&= \tan^2 A - \cot^2 A \\
&= LHS
\end{aligned}$$

Exam Question

5. Solve, for $0 \leqslant \theta \leqslant 180°$,

$$2\cot^2 3\theta = 7 \operatorname{cosec} 3\theta - 5$$

Give your answers in degrees to 1 decimal place.

(10)

5. $2\cot^2 3\theta = 7\csc 3\theta - 5$ $\boxed{0 \leq 3\theta \leq 540}$

$2(\csc^2 3\theta - 1) = 7\csc 3\theta - 5$

$2\csc^2 3\theta - 2 = 7\csc 3\theta - 5$

$2\csc^2 3\theta - 7\csc 3\theta + 3 = 0$

$(2\csc 3\theta - 1)(\csc 3\theta - 3) = 0$

$\csc 3\theta = \frac{1}{2}$ or $\csc 3\theta = 3$

(NO ROOTS)

$\sin 3\theta = \frac{1}{3}$

$3\theta = 19.471\ldots, 160.528\ldots, 379.471\ldots, 520.528\ldots$

$\theta = 6.490\ldots, 53.509\ldots, 126.490\ldots, 173.509$

$= \underline{6.5, 53.5, 126.5, 173.5}$