Pure Sector 1 : Quadratics

Aims
- Solve quadratic inequalities
- Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.
- Calculate the number of roots of a quadratic using the discriminant
- Solve simultaneous equations involving one linear and one quadratic equation

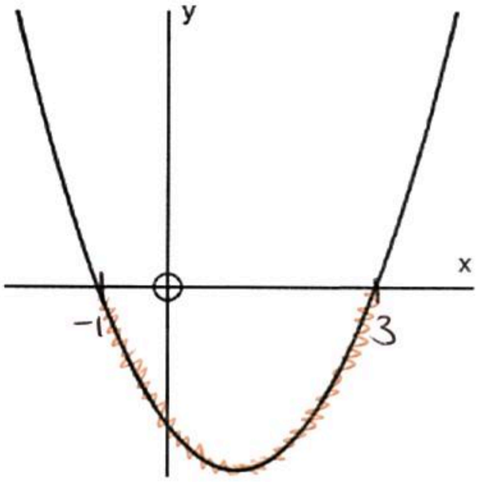
Set Notation

$\{x: \dots\}$	the set of all x such that...
$\cup$	union
$\cap$	intersection

Quadratic inequalities

Example 1

a) Consider the graph $y = x^2 - 2x - 3$ for what values of x is $y \leq 0$?



Roots: $0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x = 3, -1$

b) So for what values of x would $x^2 - 2x - 3 \leq 0$?

$-1 \leq x \leq 3$

c) What would the answer be using set notation?

$\{x : -1 \leq x \leq 3\}$

OR

$\{x : x \geq -1\} \cap \{x : x \leq 3\}$

Example 2

a) Find the range of values of x that satisfy $x^2 + x - 6 < 0$

You must draw a sketch!

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

Critical Values

$x = -3, 2$

$-3 < x < 2$

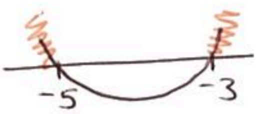
$\{x : -3 < x < 2\}$

b) Solve the inequality $x^2 + 8x + 15 \geq 0$

$x^2 + 8x + 15 = 0$

C.V.

$x = -3, -5$



$\{x : x \leq -5\} \cup \{x : x \geq -3\}$

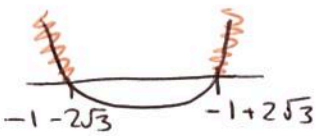
$x \leq -5$ OR $x \geq -3$

c) Find the values of x such that $x^2 + 2x - 11 > 0$

$x^2 + 2x - 11 = 0$

C.V.

$x = -1 \pm 2\sqrt{3}$



$\{x : x < -1 - 2\sqrt{3}\} \cup \{x : x > -1 + 2\sqrt{3}\}$

$x < -1 - 2\sqrt{3}$ OR $x > -1 + 2\sqrt{3}$

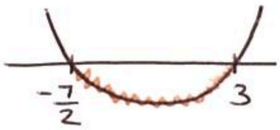
Example 3

Solve the following inequalities

a) $(x-3)(2x+7)<0$

C.V

$x=3,\ x=-\frac{7}{2}$



$-\frac{7}{2}<x<3$

b) $x^2-5x-6\leq 0$

C.V

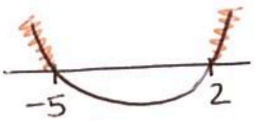
$x=6,\ x=-1$

$-1\leq x\leq 6$

c) $x^2+3x-10\geq 0$

C.V

$x=2,\ x=-5$

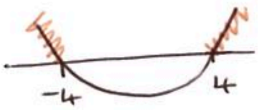


$x\leq -5$ OR $x\geq 2$

d) $x^2-16>0$

C.V

$x=4,\ x=-4$



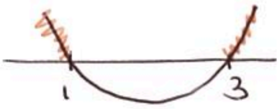
$x<-4$ OR $x>4$

e) $4x-3\leq x^2$

$x^2-4x+3\geq 0$

C.V

$x=3,\ x=1$



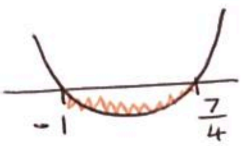
$x\leq 1$ OR $x\geq 3$

f) $3x+7>4x^2$

$4x^2-3x-7<0$

C.V

$x=\frac{7}{4},\ x=-1$

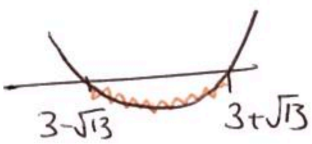


$-1<x<\frac{7}{4}$

g) $x^2-6x-4<0$

C.V.

$x=3\pm\sqrt{13}$



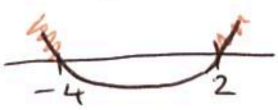
$3-\sqrt{13}<x<3+\sqrt{13}$

h) Find the values for x which satisfy both inequalities $x^2+2x>8$ and $3(2x+1)\leq 15$.

$x^2+2x-8>0$

C.V

$x=2,\ x=-4$



$x<-4$ OR $x>2$

$3(2x+1)\leq 15$

$2x+1\leq 5$

$2x\leq 4$

$x\leq 2$

BOTH

$x<-4$

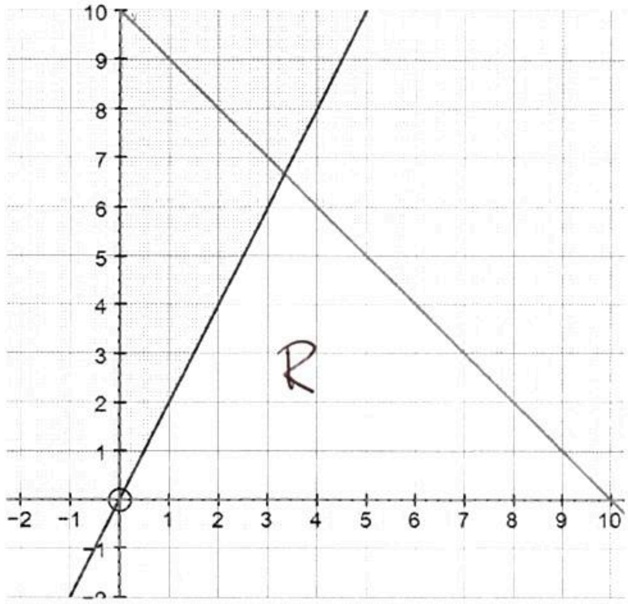
Graphing Inequalities

Dotted/dashed lines or curves are used for < or >.
Solid lines or curves are used for ≤ or ≥.

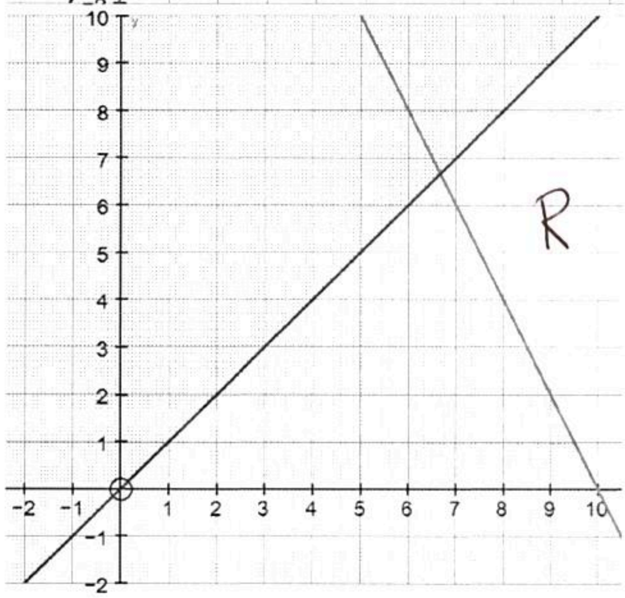
Example 4

Draw the following inequalities and label the region satisfied by all the inequalities.

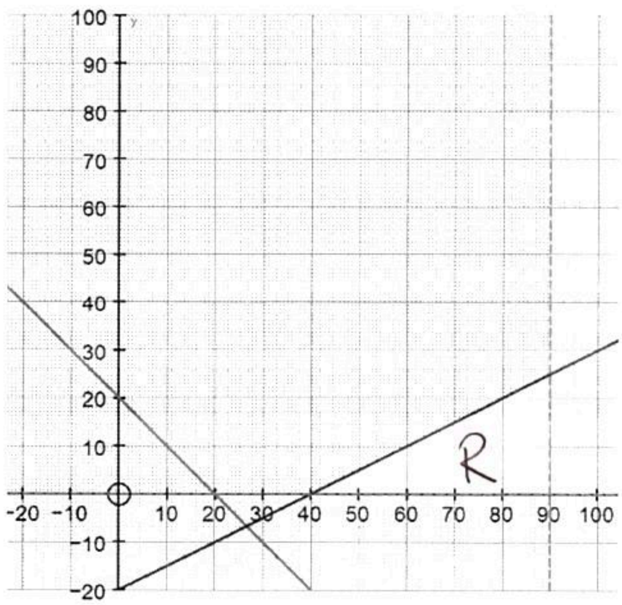
a) $y \leq 2x$,
$x + y \leq 10$,
$x > 0$,
$y > 0$



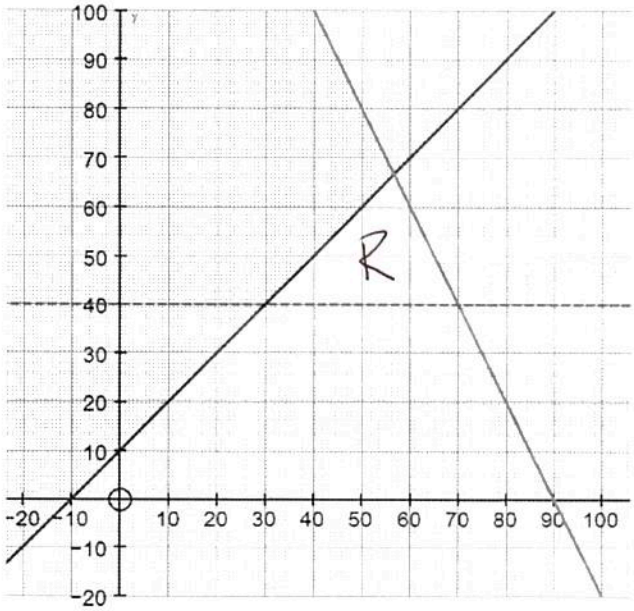
b) $2x + y \geq 20$,
$y \leq x$



c) $x + y \geq 20$,
$3x - 6y \geq 120$,
$x < 90$,
$y > 0$

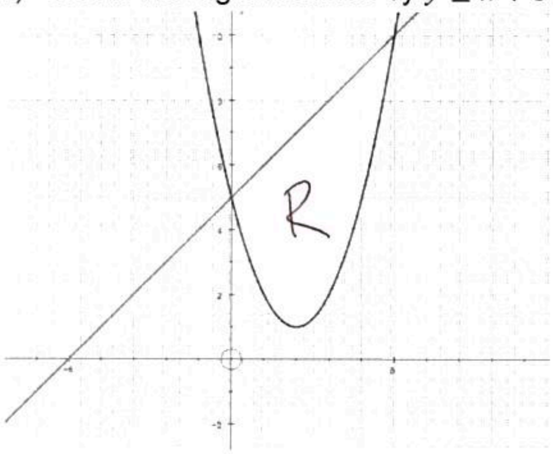


d) $y > 40$,
$y - x \leq 10$,
$2x + y \leq 180$



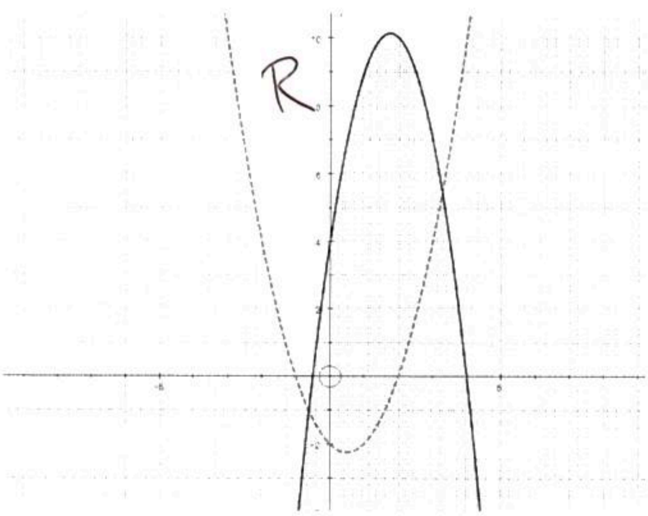
Example 5

a) Draw the graphs $y = x + 5$ and $y = x^2 - 4x + 5$ on the same diagram.
b) Shade the region defined by $y \leq x + 5$ and $y \geq x^2 - 4x + 5$



Example 6

Draw and label the region the satisfies $y > x^2 - x - 2$ and $y \geq 4 + 7x - 2x^2$.



Roots of a quadratic

The roots of a quadratic are the x intercepts of the graph (i.e. when $y = 0$).
Factorising and the quadratic formula are used to find the roots of a quadratic.

Recall, if $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac$ is called the discriminant

Example 7

For each of the following quadratic equations:

i. Find the roots
ii. Sketch the graph
iii. Put $y = 0$ and calculate $b^2 - 4ac$

a) $y = x^2 - 4x$

i) $x^2 - 4x = 0$
$x = 0, x = 4$

iii) $b^2 - 4ac$
$= (-4)^2 - 4(1)(0)$
$= 16$

b) $y = x^2 - 8x + 16$

i) $x^2 - 8x + 16 = 0$
$x = 4$

iii) $b^2 - 4ac$
$= (-8)^2 - 4(1)(16)$
$= 0$

c) $y = x^2 - 8x + 20$

i) $x^2 - 8x + 20 = 0$
NONE

iii) $b^2 - 4ac$
$= (-8)^2 - 4(1)(20)$
$= -16$

d) $y = x^2 + 6x + 8$

i) $x^2 + 6x + 8 = 0$
$x = -2, x = -4$

iii) $b^2 - 4ac$
$= (6)^2 - 4(1)(8)$
$= 4$

e) $y = 4x^2 + 4x + 1$

i) $4x^2 + 4x + 1 = 0$
$x = -\frac{1}{2}$

iii) $b^2 - 4ac$
$= (4)^2 - 4(4)(1)$
$= 0$

f) $y = 2x^2 + 3x + 7$

i) $2x^2 + 3x + 7 = 0$
NONE

iii) $b^2 - 4ac$
$= 3^2 - 4(2)(7)$
$= -47$

For $y = ax^2 + bx + c$

- If $\boldsymbol{b^2 - 4ac > 0}$ then there are **two distinct (different) real roots**. The graph will cross the x axis twice.
- If $\boldsymbol{b^2 - 4ac < 0}$ then there are **no real roots**. The graph will not cross the x axis.
- If $\boldsymbol{b^2 - 4ac = 0}$ then there are **equal roots** or **one repeated real root**. The graph will touch the x axis i.e. the x axis is a tangent to the curve.
- If $\boldsymbol{b^2 - 4ac \geq 0}$ then there are **real roots** (1 or 2). This is a common exam question!

A tangent is a line that intersects the circle/curve once i.e. it touches the circle/curve.

Example 8

How many roots do the following equations have?

a) $y = x^2 - 7x + 3$ $\quad b^2 - 4ac = (-7)^2 - 4(1)(3) = 37 > 0 \therefore$ two real roots

b) $y = 3x^2 - 7x + 5$ $\quad b^2 - 4ac = (-7)^2 - 4(3)(5) = -11 < 0 \therefore$ no real roots

c) $y = 5x^2 - x - 5$ $\quad b^2 - 4ac = (-1)^2 - 4(5)(-5) = 101 > 0 \therefore$ two real roots

d) $y = 9x^2 - 6x + 1$ $\quad b^2 - 4ac = (-6)^2 - 4(9)(1) = 0 \therefore$ one repeated root

Example 9

For the following equations determine the values of k for which the equation has equal roots

a) $x^2 - 3x + k = 0$

$a = 1$
$b = -3$
$c = k$

$$b^2 - 4ac = 0$$
$$(-3)^2 - 4(1)(k) = 0$$
$$9 - 4k = 0$$
$$k = \frac{9}{4}$$

b) $kx^2 - 6x + (8 + k) = 0$

$a = k$
$b = -6$
$c = 8 + k$

$$b^2 - 4ac = 0$$
$$(-6)^2 - 4(k)(8 + k) = 0$$
$$36 - 32k - 4k^2 = 0$$
$$k^2 + 8k - 9 = 0$$
$$k = 1 \text{ OR } k = -9$$

Example 10

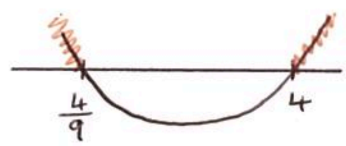
The equation $x^2 + 3(k - 2)x + (k + 5) = 0$ has two distinct real roots. Find the set of possible values for k.

$a = 1$
$b = 3(k - 2)$
$c = k + 5$

$$b^2 - 4ac > 0$$
$$[3(k-2)]^2 - 4(1)(k+5) > 0$$
$$9(k^2 - 4k + 4) - 4k - 20 > 0$$
$$9k^2 - 36k + 36 - 4k - 20 > 0$$
$$9k^2 - 40k + 16 > 0$$

C.V

$k = 4,\ k = \frac{4}{9}$



$$k < \frac{4}{9} \text{ OR } k > 4$$

Set Notation

$$\{k : k < \tfrac{4}{9}\} \cup \{k : k > 4\}$$

Example 11

$(k-1)x^2+2kx+7k-4=0$

$\downarrow$

Given that the quadratic equation $(k-1)x^2+2kx+7k=4$ has no real roots.

a) Show that $6k^2-11k+4>0$.

$a=k-1$

$b=2k$

$c=7k-4$

$b^2-4ac<0$

$(2k)^2-4(k-1)(7k-4)<0$

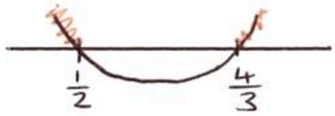
$4k^2-28k^2+44k-16<0$

$-24k^2+44k-16<0 \quad) \div -4$

$6k^2-11k+4>0$

b) Find the range of values of k satisfying this inequality.

C.V $k=\frac{4}{3}, k=\frac{1}{2}$



$k<\frac{1}{2}$ OR $k>\frac{4}{3}$

Exam question 1

The quadratic equation $(k+1)x^2+4kx+9=0$ has real roots.

(a) Show that $4k^2-9k-9 \geqslant 0$. *(3 marks)*

(b) Hence find the possible values of k. ***(4 marks)***

a) $a=k+1$

$b=4k$

$c=9$

$b^2-4ac \geq 0$

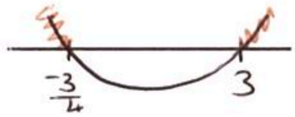
$(4k)^2-4(k+1)(9) \geq 0$

$16k^2-36k-36 \geq 0$

$4k^2-9k-9 \geq 0$

b) C.V

$k=3, -\frac{3}{4}$



$k<-\frac{3}{4}$ OR $k>3$

Exam question 2

The quadratic equation

$$(2k-3)x^2+2x+(k-1)=0$$

where k is a constant, has real roots.

(a) Show that $2k^2-5k+2 \leqslant 0$. *(3 marks)*

(b) (i) Factorise $2k^2-5k+2$. *(1 mark)*

(ii) Hence, or otherwise, solve the quadratic inequality

$$2k^2-5k+2 \leqslant 0$$ *(3 marks)*

a) $a=2k-3$

$b=2$

$c=k-1$

$b^2-4ac \geq 0$

$4-4(2k-3)(k-1) \geq 0$

$4-8k^2+20k-12 \geq 0$

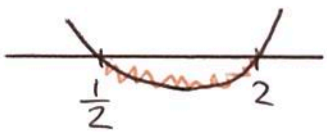
$-8k^2+20k-8 \geq 0$

$2k^2-5k+2 \leq 0$

bi) $(2k-1)(k-2)$

ii) C.V

$k=\frac{1}{2}, 2$



$\frac{1}{2}<k<2$

Simultaneous equations

Example 12

Solve the following simultaneous equations

	$x - y = 2$ $2x^2 - 3y^2 = 15$		$2x + y = 10$ $x^2 + y^2 = 25$
1. Rearrange the linear equation to make either x or y the subject.	$x = y + 2$		$y = 10 - 2x$
2. Substitute the equation from step 1 into the quadratic/circle equation	$2(y + 2)^2 - 3y^2 = 15$		$x^2 + (10 - 2x)^2 = 25$
3. Simplify the equation from step 2 (you should always end up with a quadratic in x or y)	$2(y^2 + 4y + 4) - 3y^2 = 15$ $2y^2 + 8y + 8 - 3y^2 = 15$ $8y + 8 - y^2 = 15$ $0 = y^2 - 8y + 7$		$x^2 + 100 - 40x + 4x^2 = 25$ $5x^2 - 40x + 75 = 0$ $x^2 - 8x + 15 = 0$
4. Solve the quadratic	$(y - 7)(y - 1) = 0$ $y = 7$ $y = 1$		$(x - 3)(x - 5) = 0$ $x = 3$ $x = 5$
5. Find the corresponding coordinates using the equation in step 1	$y = 7$ $x = 7 + 2 = 9$ Or $(9,7)$	$y = 1$ $x = 1 + 2 = 3$ Or $(3,1)$	$y = 10 - 6$ $y = 10 - 10$ $y = 4$ $y = 0$ $(3,4)$ $(5,0)$

Example 13

Solve the following simultaneous equations

a) $x + y = 5$ ① $x = 5 - y$
$y^2 + 2x = 18$ ②

Sub ① in ②

$y^2 + 2(5 - y) = 18$

$y^2 - 2y + 10 = 18$

$y^2 - 2y - 8 = 0$

$y = 4, -2$

$y = 4$ $y = -2$
$x = 1$ $x = 7$

b) $y = 2x$ ①
$x^2 + y^2 - 10x = 15$ ②

Sub ① in ②

$x^2 + (2x)^2 - 10x = 15$

$5x^2 - 10x - 15 = 0$

$x = 3$ $x = -1$
$y = 6$ $y = -2$

c) $y = 3 - x$ ①
$x^2 + y^2 = 5$ ②

Sub ① in ②

$x^2 + (3 - x)^2 = 5$

$x^2 + 9 - 6x + x^2 = 5$

$2x^2 - 6x + 4 = 0$

$x = 2$ $x = 1$
$y = 1$ $y = 2$

d) $y = 3x + 6$ ①
$(x - 1)^2 + (y + 1)^2 = 10$ ②

Sub ① in ②

$(x - 1)^2 + (3x + 7)^2 = 10$

$x^2 - 2x + 1 + 9x^2 + 42x + 49 = 10$

$10x^2 + 40x + 40 = 0$

$x = -2$
$y = 0$

e) $y = 2x + 6$ ①
$xy + x + 6 = 0$ ②

Sub ① in ②

$x(2x+6) + x + 6 = 0$

$2x^2 + 6x + x + 6 = 0$

$2x^2 + 7x + 6 = 0$

$x = \frac{-3}{2}$ $\quad x = -2$

$y = 3$ $\quad y = 2$

f) $2x + y = 11$ ① $\quad y = 11 - 2x$
$x^2 + y^2 - 6x = 1$ ②

Sub ① in ②

$x^2 + (11 - 2x)^2 - 6x = 1$

$x^2 + 121 - 44x + 4x^2 - 6x = 1$

$5x^2 - 50x + 120 = 0$

$x = 6$ $\quad x = 4$

$y = 1$ $\quad y = 3$

g) $y = -x + 4$ ①
$x^2 + y^2 = 10$ ②

Sub ① in ②

$x^2 + (-x+4)^2 = 10$

$x^2 + x^2 - 8x + 16 = 10$

$2x^2 - 8x + 6 = 0$

$x = 3$ $\quad x = 1$

$y = 1$ $\quad y = 3$

h) $5x + 3y = 4$ ① $\quad x = \frac{4 - 3y}{5}$
$5x^2 - 3y^2 = 8$ ②

Sub ① in ②

$5\left(\frac{4-3y}{5}\right)^2 - 3y^2 = 8$

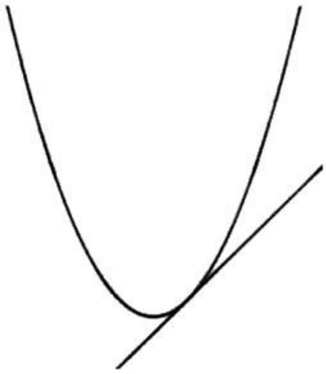
$\frac{(4-3y)^2}{5} - 3y^2 = 8$

$16 - 24y + 9y^2 - 15y^2 = 40$

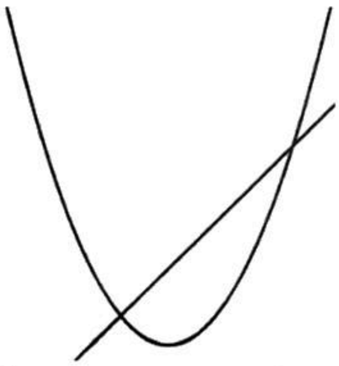
$6y^2 + 24y + 24 = 0$

$y = -2$
$x = 2$

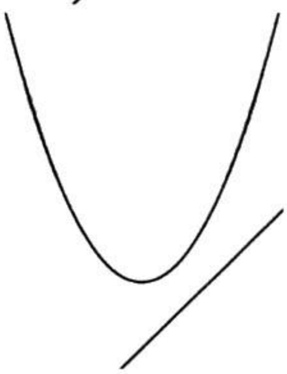
One point of intersection.
The line is tangent to the curve/circle.



Two points of intersection.



No points of intersection.



A line and a curve will intersect once, twice or not at all. The quadratic found when solving the equations simultaneously (i.e. from step 3) can be used to determine the number of points of intersection by finding the discriminant.

For $ax^2 + bx + c = 0$

- If $b^2 - 4ac > 0$ then there are two different real roots so two points of intersection.
- If $b^2 - 4ac < 0$ then there are no real roots so no points of intersection.
- If $b^2 - 4ac = 0$ then there are two equal/repeated real roots so one point of intersection. This means the line is a tangent to the curve.

Exam question 3

Determine the coordinates of the points of intersection of the line $y = x + 11$ (1) and the curve $y = x^2 + 10x + 19$ (2).

(4 marks)

$x^2 + 10x + 19 = x + 11$

$x^2 + 9x + 8 = 0$

$x = -1 \qquad x = -8$

$y = 10 \qquad y = 3$

Exam question 4

The curve C has equation $y = x^2 + 7$. The line L has equation $y = k(3x + 1)$, where k is a constant.

(a) Show that the x-coordinates of any points of intersection of the line L with the curve C **satisfy the equation**

$$x^2 - 3kx + 7 - k = 0$$

(1 mark)

(b) The curve C and the line L intersect in two distinct points. Show that

$$9k^2 + 4k - 28 > 0$$

(3 marks)

(c) Solve the inequality $9k^2 + 4k - 28 > 0$.

(4 marks)

a) $x^2 + 7 = k(3x+1)$

$x^2 + 7 = 3kx + k$

$x^2 - 3kx + 7 - k = 0$

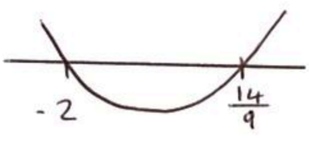
b) $a = 1 \qquad b^2 - 4ac > 0$

$b = -3k \qquad (-3k)^2 - 4(1)(7-k) > 0$

$c = 7 - k \qquad 9k^2 - 28 + 4k > 0$

$9k^2 + 4k - 28 > 0$

c) C.V $k = \frac{14}{9}, -2$



$k < -2$ OR $k > \frac{14}{9}$