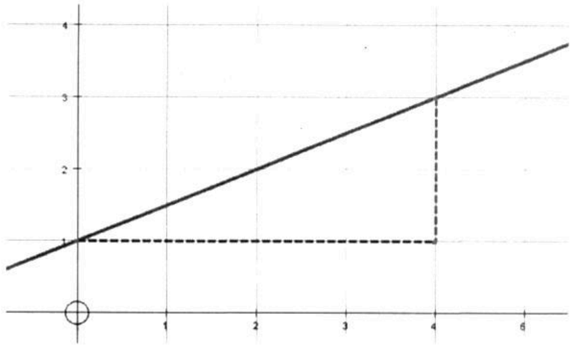
Pure Sector 1: Straight Lines

Aims:

- Calculate the gradient of a line given two points or the equation of a line.
- Find the equation of straight lines in the required form.
- Find the midpoint of a line segment.
- Find the distance between two points.
- Understand parallel and perpendicular lines.
- Find the intersection of two lines using simultaneous equations.

Gradient of a Line

The gradient tells you how steep something is. To find the gradient of a line you can use any two points on the line (x_1, y_1) and (x_2, y_2) then divide the difference in the y values by the difference in the x values. We use the letter m to represent gradient of a line.



$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Example 1

Find the gradient of the line segment joining each pair of points:

a) $A(3,4)$ and $B(7,18)$ $\quad m = \frac{18-4}{7-3} = \frac{14}{4} = \frac{7}{2}$

b) $A(-4,0)$ and $(2,7)$ $\quad m = \frac{7-0}{2+4} = \frac{7}{6}$

Example 2

A line has the gradient of $-\frac{4}{3}$ and passes through the point $(1,6)$. The point $(k, 2)$ is also on the line, find the value of k.

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$

$-\frac{4}{3} = \frac{2-6}{k-1}$

$-\frac{4}{3} = \frac{-4}{k-1}$

$\therefore k-1 = 3$ so $\underline{\underline{k=4}}$

You can also find the gradient of a straight line if you are given the equation of the line.
Remember from GCSE:

$$\boxed{y = mx + c}$$

!!!!!! MUST be $y =$!!!!

Where m is the **gradient** and c is the y **intercept**.

Example 3

A line has the equation $3x - 4y = 7$ and passes through the points A and B.

a) Find the gradient of AB.

$3x - 4y = 7$
$3x - 7 = 4y$
$\frac{3}{4}x - \frac{7}{4} = y$ so $m = \frac{3}{4}$

b) Find the coordinates where the line crosses the coordinate axes. both x and y!

x-axis $y = 0$ $3x = 7$ $x = \frac{7}{3}$ $(\frac{7}{3}, 0)$

y-axis $x = 0$ $-4y = 7$ $y = -\frac{7}{4}$ $(0, -\frac{7}{4})$

Equation of a Straight Line

To find the equation of a straight line you need to know the gradient, m, and the coordinates of a point (x_1, y_1) on the line then we can use:

$$\boxed{y - y_1 = m(x - x_1)}$$

Example 4

Find the equation of the line that passes through the point $(-2,3)$ and has a gradient of 3.

$m = 3$ $x_1 = -2$ $y_1 = 3$

$y - 3 = 3(x + 2)$
$y = 3x + 6 + 3$ $y = 3x + 9$

Example 5

a) Find the equation of the line that passes through the point $(5, -3)$ and has a gradient of $-\frac{2}{3}$, give your answer in the form of $ax + by + c = 0$ where a, b and c are integers.

$y + 3 = -\frac{2}{3}(x - 5)$ $3y + 9 = -2x + 10$
$3(y + 3) = -2(x - 5)$ $2x + 3y - 1 = 0$

b) Verify that the point $P(-1,2)$ lies on the line. $(-1, 2)$ does not lie on the line, sorry.

$(-1, 1)$ does.

$2(-1) + 3(1) - 1 = -2 + 3 - 1 = 0$ $\therefore$ does lie on line.

Example 6

Find the equation of the line that passes through the points $A(0,3)$ and $(-2,-6)$.
Put your answer in the form of $ax + by = c$ where a, b and c are integers.

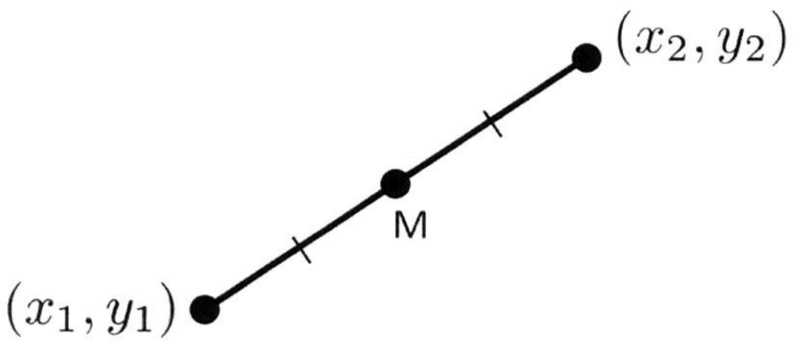
$y - y_1 = m(x - x_1)$ (x_1, y_1) can be either point, let's use $(0,3)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3+6}{0+2} = \frac{9}{2}$

$y - 3 = \frac{9}{2}(x - 0)$ $2y - 6 = 9x$

$$\boxed{\begin{array}{l} 9x - 2y + 6 = 0 \\ \text{or } -9x + 2y - 6 = 0 \end{array}}$$

Midpoint of a Line Segment



The midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$\boxed{\boldsymbol{M \text{ is} \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}}$$

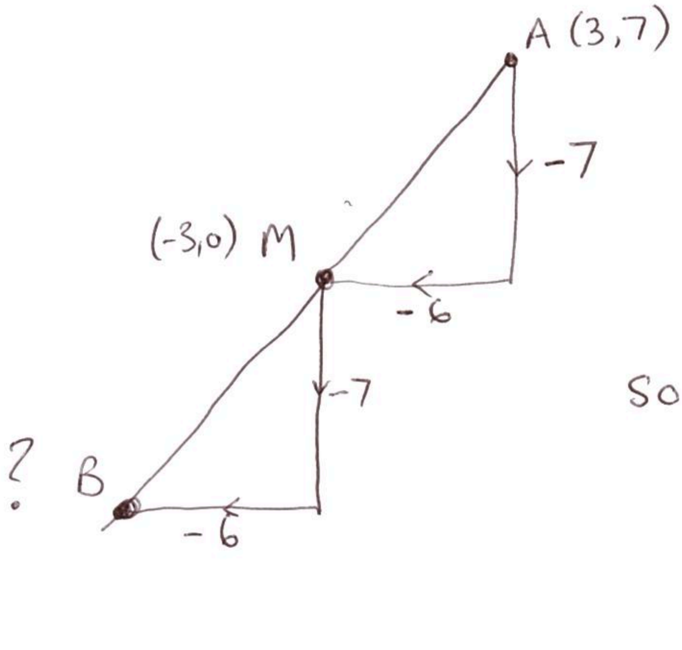
Example 7

Find the midpoint of AB where $A(3,-4)$ and $B(1,8)$.

$$\left(\frac{3+1}{2}, \frac{-4+8}{2}\right) = (2,2)$$

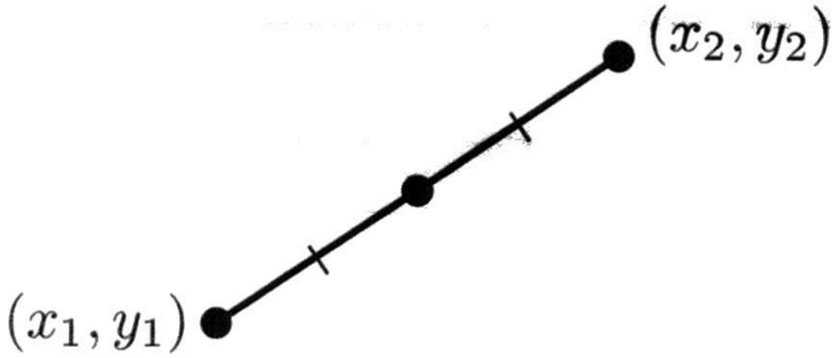
Example 8

Find the coordinates of B given that $A(3,7)$ and the midpoint of AB is $(-3,0)$.



so B is $(-9,-7)$

Distance Between Two Points



The distance between two points or length of the line segment between $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$\boxed{\boldsymbol{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

AB is the distance from A to B

Example 9

Find the distance between the two points $A(4,3)$ and $B(-1,5)$

$$AB = \sqrt{(4+1)^2 + (5-3)^2} = \sqrt{25+4}$$
$$= \underline{\underline{\sqrt{29}}}$$

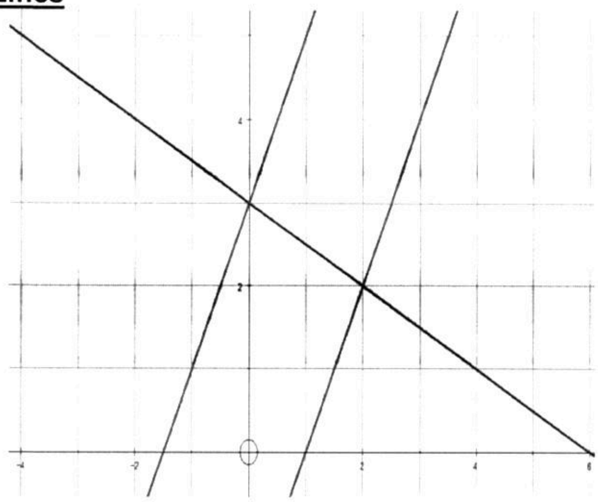
Example 10

The point C has the coordinates $(5, k)$ and the point A has the coordinates $(3, -2)$.
The distance from AC is $5\sqrt{5}$, find the possible values of k.

$$AC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$(5\sqrt{5})^2 = (\cancel{k}5-3)^2 + (k+2)^2$$
$$125 = 4 + (k+2)^2$$
$$(k+2)^2 = 121$$
$$k+2 = \pm 11$$
$$k = -2 \pm 11$$

$$\underline{\underline{k = 9 \text{ and } -13}}$$

Parallel and Perpendicular Lines



When two lines are parallel they have equal gradients. When two lines are perpendicular they meet at right angles and the product of their gradients is -1.

> Two lines with gradients m_1 and m_2 are:
>
> Parallel $\Leftrightarrow \boldsymbol{m_1 = m_2}$
> Perpendicular $\Leftrightarrow \boldsymbol{m_1 \times m_2 = -1}$
>
> Perpendicular $\Leftrightarrow \boldsymbol{m_2 = -\frac{1}{m_1}}$

Example 11

Find in the form of $ax + by + c = 0$ where *a* , *b* and *c* are integers, the equation of the line that is perpendicular to $3y - 4x = 7$ and passes through the point $(3, -1)$.

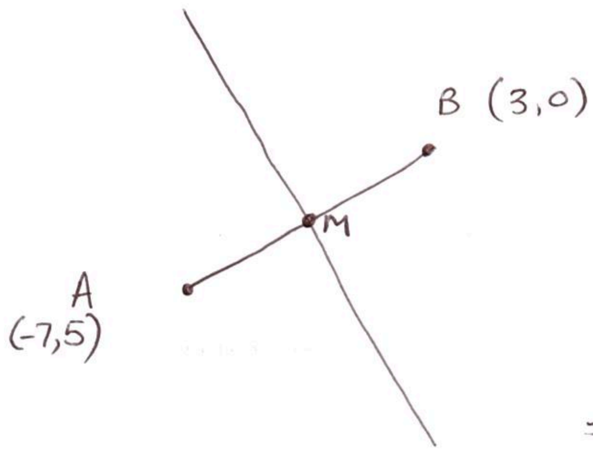
$3y - 4x = 7$
$3y = 4x + 7$
$y = \frac{4}{3}x + \frac{7}{3}$
$m = \frac{4}{3}$

new $m = -\frac{3}{4}$

$y + 1 = -\frac{3}{4}(x - 3)$
$4y + 4 = -3x + 9$
$\underline{3x + 4y - 5 = 0}$

Example 12

Find the equation of the perpendicular bisector of AB where A and B are $(-7,5)$ and $(3,0)$ respectively.



• Find M $\left(\frac{-7+3}{2}, \frac{5+0}{2}\right)$ $\left(-2, \frac{5}{2}\right)$

• Find grad AB $\frac{5-0}{-7-3} = -\frac{1}{2}$

• Perpendicularise gradient 2

• find equation of new line $y - y_1 = m(x - x_1)$
$x_1 = -2$, $y = \frac{5}{2}$, $m = 2$

$y - \frac{5}{2} = 2(x + 2)$ no need to tidy up!

but if you did $2y = 4x + 13$

Intersection of Lines

To find the coordinates of the point of intersection of two lines we use simultaneous equations.
The lines AB and BC will intersect at the point B.

Example 13

Find where the following two lines intersect $5x + 3y = 7$ and $3x - 7y = 13$.

Level off x's or y's I choose x

$5x + 3y = 7 \quad \times 3 \qquad 15x + 9y = 21$

$3x - 7y = 13 \quad \times 5 \qquad 15x - 35y = 65$

subtract $44y = -44$

$y = -1$

subst $y = -1$ into an equation $3x - 7(-1) = 13$

$3x + 7 = 13$

$3x = 6$

$x = 2$

$(2, -1)$

Check in other line

$5(2) + 3(-1) = 10 - 3 = 7$ ✓

Ans: $(2, -1)$

Example 14

Find the coordinates of C given that the equation AC is $y - 3x = 3$ and BC is $2y = 9 + 5x$.

C is the point of intersection so solve simultaneously

This time make y the subject of the first equation.

$y = (3x + 3)$ replace y in second equation.

$2y = 9 + 5x$

$2(3x + 3) = 9 + 5x$

$6x + 6 = 9 + 5x$

$x = 3$

$y = 3(3) + 3 = 12$

$(3, 12)$

check in other eqn

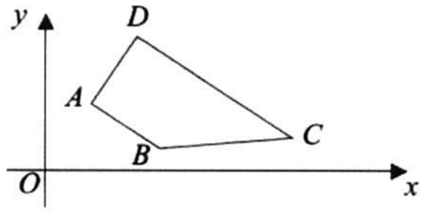
$2(12) = 9 + 5(3)$ ✓

$24 = 24$

C is $(3, 12)$

Exam Questions

1 The trapezium $ABCD$ is shown below.



The line AB has equation $2x + 3y = 14$ and DC is parallel to AB.

(a) Find the gradient of AB. *(2 marks)*

(b) The point D has coordinates $(3, 7)$.

(i) Find an equation of the line DC. *(2 marks)*

(ii) The angle BAD is a right angle. Find an equation of the line AD, giving your answer in the form $mx + ny + p = 0$, where m, n and p are integers. *(4 marks)*

(c) The line BC has equation $5y - x = 6$. Find the coordinates of B. *(3 marks)*

(a) Put $2x+3y=14$ into the form $y=mx+c$

$3y=-2x+14$ $\quad y=-\frac{2}{3}x+\frac{14}{3}$ $\quad$ gradient $=-\frac{2}{3}$

(b)(i) DC is parallel to AB $\therefore$ gradient $= -\frac{2}{3}$ $\quad$ D is $(3,7)$

$y-y_1=m(x-x_1)$ $\quad \underline{y-7=-\frac{2}{3}(x-3)}$

(ii) AD is perpendicular to AB $\therefore$ gradient $\frac{3}{2}$ $\quad$ D is $(3,7)$

$y-y_1=m(x-x_1)$ $\quad y-7=\frac{3}{2}(x-3)$ $\quad 2y-14=3x-9$

$3x-2y+5=0$ $\quad$ or $\quad -3x+2y-5=0$

(c) B is intersection of AB and BC so solve equations simultaneously

$2x+3y=14$ $\quad 5y-x=6$

$x=(5y-6)$ $\quad\quad x=5(2)-6=4$

$2(5y-6)+3y=14$ $\quad\quad$ check $2(4)+3(2)$

$8+6$

$10y-12+3y=14$ $\quad\quad 14$ ✓

$13y=26$

$y=2$ $\quad\quad$ B is $(4,2)$

2 The triangle ABC has vertices $A\,(1,\,3)$, $B\,(3,\,7)$ and $C\,(-1,\,9)$.

(a) (i) Find the gradient of AB. *(2 marks)*

(ii) **Hence show that angle ABC is a right angle.** ***(2 marks)***

(b) (i) Find the coordinates of M, the mid-point of AC. *(2 marks)*

(ii) Show that the lengths of AB and BC are equal. *(3 marks)*

(iii) Hence find an equation of the line of symmetry of the triangle ABC. *(3 marks)*

a) i) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-3}{3-1} = \frac{4}{2} = 2$

ii) right angle is at B so grad AB × grad BC = −1

grad BC $= \frac{9-7}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ $\quad 2 \times -\frac{1}{2} = -1 \therefore A\hat{B}C = 90^\circ$

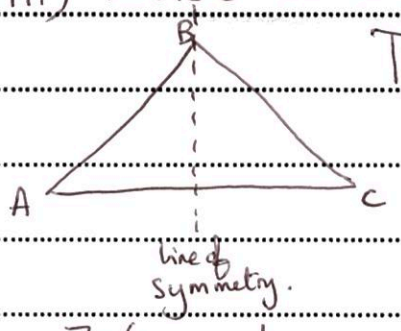
must make last statement

b) i) $M = \left(\frac{1+-1}{2}, \frac{3+9}{2}\right) = (0, 6)$

ii) $AB = \sqrt{(3-1)^2 + (7-3)^2} = \sqrt{4+16} = \sqrt{20}$

$BC = \sqrt{(3+1)^2 + (9-7)^2} = \sqrt{16+4} = \sqrt{20}$

iii) $\triangle ABC$ is a right-angled isosceles triangle



The line of symmetry goes through B and mid point of AC

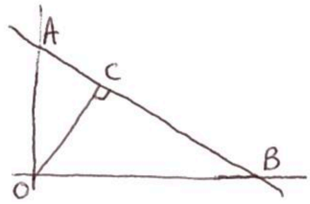
Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $y - y_1 = m(x - x_1)$

B (3, 7) M (0, 6)

$m = \frac{7-6}{3-0} = \frac{1}{3}$

$y - 6 = \frac{1}{3}(x - 0)$ not need to tidy up.

Extension questions



1) Find the shortest distance from the line $3x + 4y = 25$ to the origin.

Grad $AB = -\frac{3}{4}$

So grad $OC = \frac{4}{3}$

Solve $y = \frac{4}{3}x$ with $3x+4y=25$

$3x + \frac{16}{3}x = 25$

$x = 3$

$\therefore\ y = 4$

Coords of C $(3,4)$

$\therefore$ dist $\underline{OC = 5}$

$(3,4,5\ \Delta)$

ΔAOB is similar to ΔACO

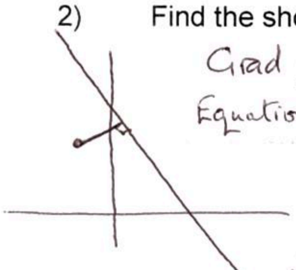
Ratio of ΔAOB sides is $\underset{AO}{3} : \underset{OB}{4} : \underset{AB}{5}$

OB is $\frac{25}{3}$ long.

So $\frac{OC}{OB} = \frac{OA}{AB}$

$OC = \frac{OA}{AB} \times OB = \frac{3}{5} \times \frac{25}{3} = \underline{\underline{5}}$

2) Find the shortest distance form the point $(-1, 2)$ to the line $2y + 3x = 14$.



Grad of $2y+3x=14$ is $-3/2$ $\therefore$ grad of shortest dist line is $2/3$

Equation of shortest dist line is $y-2 = \frac{2}{3}(x+1)$ $\quad y = \frac{2}{3}x + \frac{8}{3}$

Solve simultaneously with $2y+3x=14$

$2\left(\frac{2}{3}x + \frac{8}{3}\right) + 3x = 14$

Multiply out bracket and $\times 3$

$4x + 16 + 9x = 42$

$x = 2$

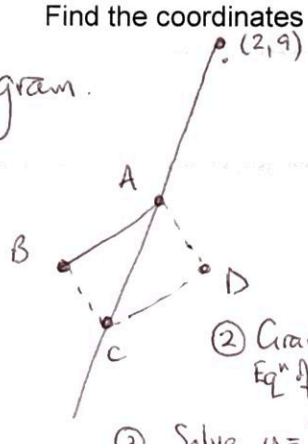
$\therefore\ y = 4$

$\text{Dist} = \sqrt{(4-2)^2 + (2+1)^2}$

$= \underline{\underline{\sqrt{13}}}$

3) The points A(1,2) and B(–2,1) are two vertices of a rectangle ABCD.
The diagonal CA produced passes through the point (2,9).
Find the coordinates of C and D

Diagram.



Idea: ① Find equation of line thro A and (2,9)

Find equation of BC

Solve simultaneously to find C

Use vectors to find D.

① $y - 2 = \left(\frac{9-2}{2-1}\right)(x-1)$ $\quad y-2 = 7(x-1)$ $\quad y = 7x - 5$

② Grad $AB = \frac{2-1}{1+2} = \frac{1}{3}$ $\therefore$ grad $BC = -3$

Eq^n of BC : $y - 1 = -3(x+2)$ $\quad y = -3x - 5$

③ Solve $y = 7x - 5$ $\quad y = -3x - 5$

$7x - 5 = -3x - 5$

$x = 0$ $\quad$ C is $(0,-5)$

$\therefore\ y = -5$

④ Vector $\overrightarrow{BC} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$ $\therefore$ coords of D are $(1+2, 2-6)$

$(3,-4)$

$\underline{C \text{ is } (0,-5) \quad D \text{ is } (3,-4)}$