Pure Sector 1: Graph Sketching

Aims:

- understand and use language and symbols associated with set theory.
- find the domain and range of a function and be able to use correct language and notation to describe functions accurately.
- understand the graphs, symmetry and periodicity of the sine, cosine and tangent functions including transformations.
- sketch and use transformations of the functions a^x, e^x and $\ln x$.
- understand and use graphs of the functions $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ and as well as transformations.
- understand and be able to use the graph of $y = |x|$and combinations of transformations of this graph, points of intersection and solutions of equations and inequalities.

Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \ldots\}$
$\{x_1, x_2, \ldots\}$	the set with the elements $x_1, x_2, \ldots$
$\{x: \ldots\}$	the set of all x such that...
$\mathbb{Z}$	the set of integers $\{0, \pm1, \pm2, \pm3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}_0^+$	the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers $\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}^+\right\}$

Range and Domain

The set of all possible inputs of a mapping or function is called the **domain** (x values).

The set of outputs for a particular set of inputs for a mapping or function is called the **range** (y values).

Example 1

The function h has domain $\{-2, -1, 0, 3, 7\}$ and is defined as $h(x) = (x-3)^2 + 2$. Find the range of h.

$h(-2) = (-2-3)^2 + 2 = 27$
$h(-1) = (-1-3)^2 + 2 = 18$
$h(0) = (0-3)^2 + 2 = 11$
$h(3) = (3-3)^2 + 2 = 2$
$h(7) = (7-3)^2 + 2 = 18$

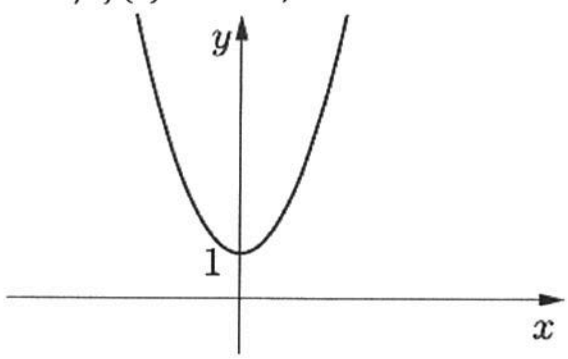
Range
$\{h(x) : h(x) = 2, 11, 18, 27\}$

If any outputs are repeated we only write the value once in the range.

Example 2

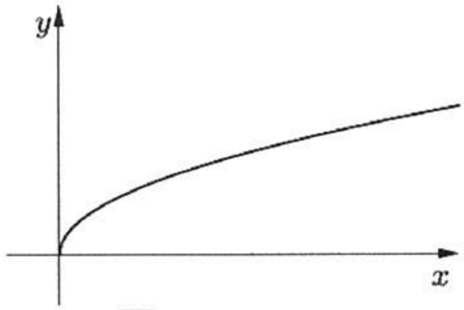
Using set notation state range of the following functions:

a) $f(x) = x^2 + 1, x \in \mathbb{R}$



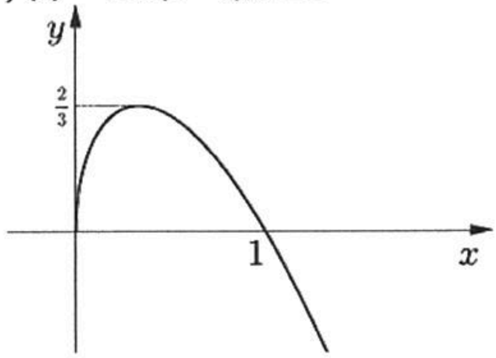
$\{f(x) : f(x) \geq 1, f(x) \in \mathbb{R}\}$

b) $f: x \to \sqrt{x}, x \geq 0$



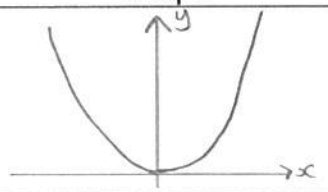
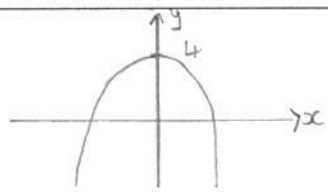
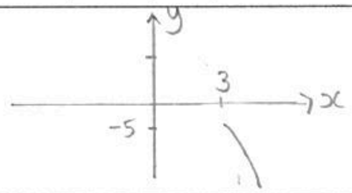
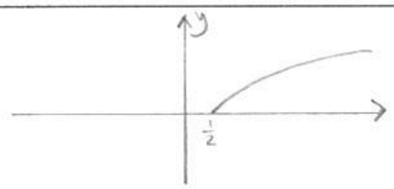
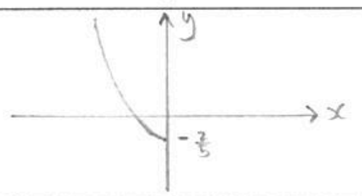
$\{f(x) : f(x) \geq 0, f(x) \in \mathbb{R}\}$

c) $f(x) = \sqrt{3x}(1 - x), x \geq 0$



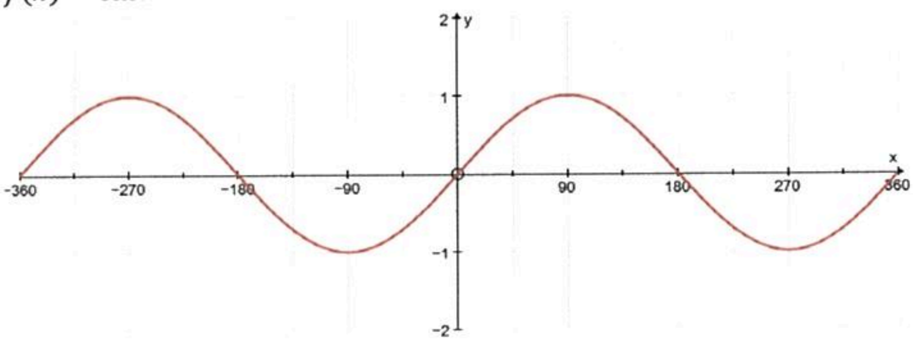
$\{f(x) : f(x) \leq \frac{2}{3}, f(x) \in \mathbb{R}\}$

Example 3

Function and Domain	Graph	Range
$f(x) = x^2$ $x \in \mathbb{R}$		$\{f(x) : f(x) \geq 0, f(x) \in \mathbb{R}\}$
$f(x) = 4 - x^2$ $x \in \mathbb{R}$		$\{f(x) : f(x) \leq 4, f(x) \in \mathbb{R}\}$
$f(x) = 4 - x^2$ $x \in \mathbb{R}, x > 3$		$\{f(x) : f(x) < -5, f(x) \in \mathbb{R}\}$
$f(x) = \sqrt{2x - 1}$ $x \geq 0.5$		$\{f(x) : f(x) \geq 0, f(x) \in \mathbb{R}\}$
$f(x) = \frac{x^2 - 2}{5}$ $x \in \mathbb{R}, x \leq 0$		$\{f(x) : f(x) \geq -\frac{2}{5}, f(x) \in \mathbb{R}\}$

Trigonometric Graphs

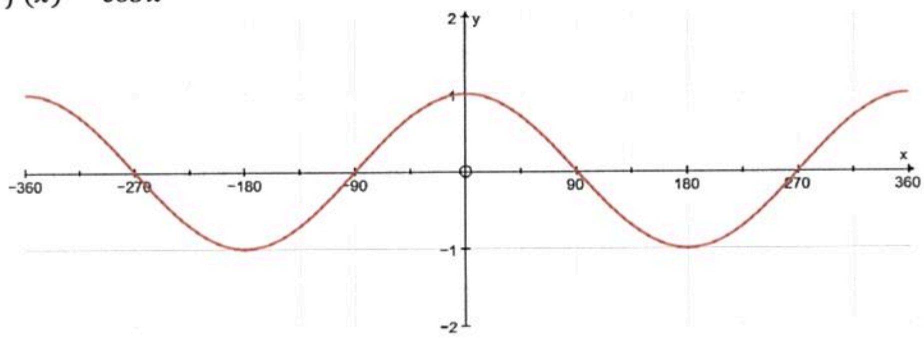
$f(x) = \sin x$



Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{f(x) : -1 \le f(x) \le 1, f(x) \in \mathbb{R}\}$

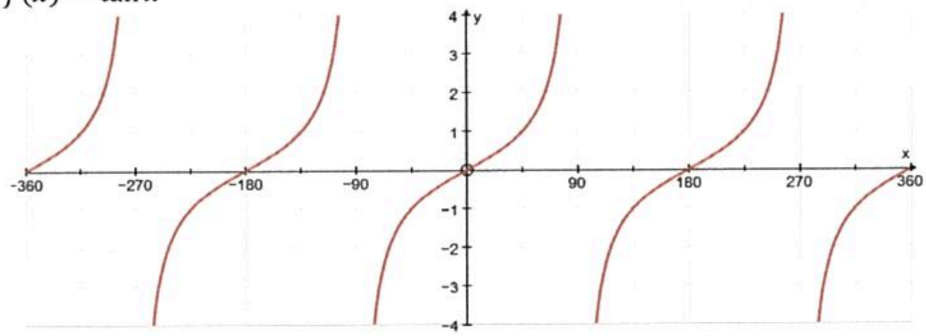
$f(x) = \cos x$



Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{f(x) : -1 \le f(x) \le 1, f(x) \in \mathbb{R}\}$

$f(x) = \tan x$



Domain: $\{x : x \in \mathbb{R}, x \neq 90(2n-1)\}$

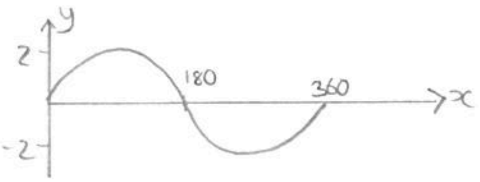
Range: $\{f(x) : f(x) \in \mathbb{R}\}$

Example 4

Domain $\{x : 0 \le x \le 360, x \in \mathbb{R}\}$

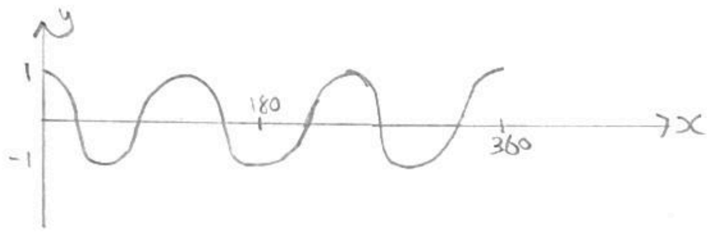
Sketch each function and state its range.

a) $f(x) = 2\sin x$



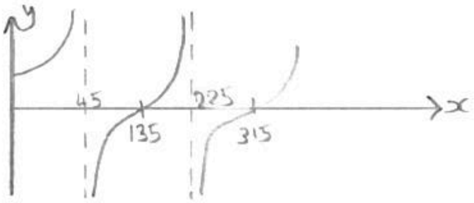
$\{f(x) : -2 \le f(x) \le 2, f(x) \in \mathbb{R}\}$

b) $f(x) = \cos(3x)$



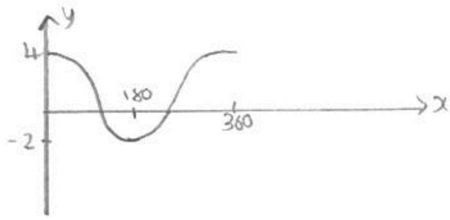
$\{f(x) : -1 \le f(x) \le 1, f(x) \in \mathbb{R}\}$

c) $f(x) = \tan(x + 45)$



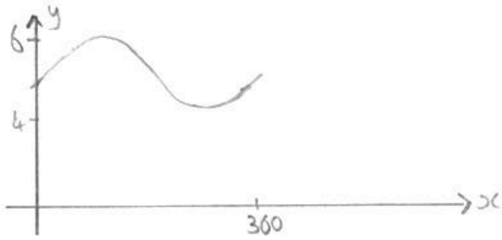
$\{f(x) : f(x) \in \mathbb{R}\}$

d) $f(x) = 3\cos x + 1$



$\{f(x) : -2 \leq f(x) \leq 4, f(x) \in \mathbb{R}\}$

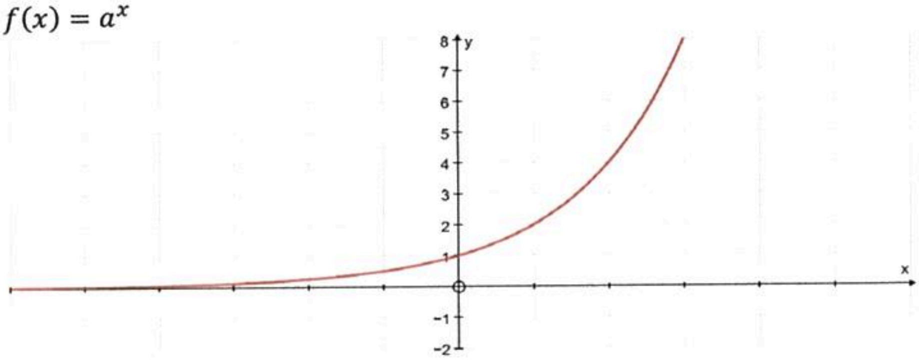
e) $f(x) = \sin(x - 10) + 5$



$\{f(x) : 4 \leq f(x) \leq 6, f(x) \in \mathbb{R}\}$

Exponential Graphs

$f(x) = a^x$



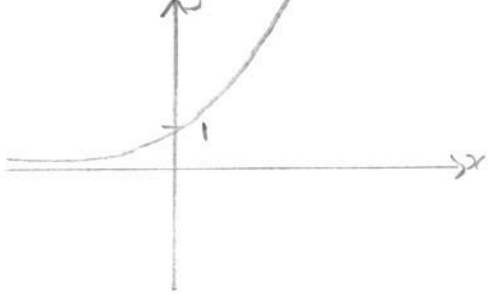
Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{f(x) : f(x) > 0, f(x) \in \mathbb{R}\}$

The constant e is the base of a natural logarithm written as $\ln x$ or $\log_e x$ (we study this in more detail later in Pure Sector 1). e is an irrational number and has the value 2.718 ...

<u>Example 5</u>

Sketch the graph of $f(x) = e^x$. State the domain and range.



Domain

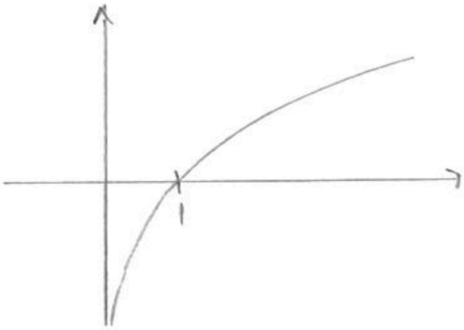
$\{x : x \in \mathbb{R}\}$

Range

$\{f(x) : f(x) > 0, f(x) \in \mathbb{R}\}$

Example 6

Sketch the graph of $f(x) = lnx$. State the domain and range.



Domain

$\{x : x > 0, x \in \mathbb{R}\}$

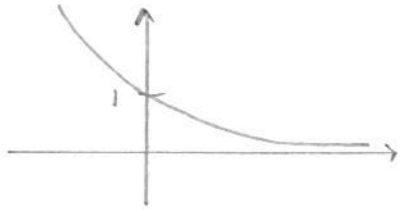
Range

$\{f(x) : f(x) \in \mathbb{R}\}$

Example 7

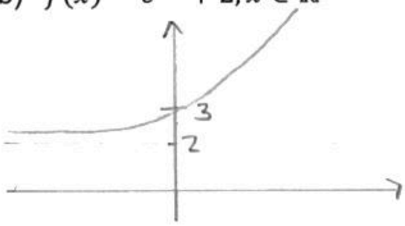
Sketch each function and state its range.

a) $f(x) = e^{-x}, x \in \mathbb{R}$



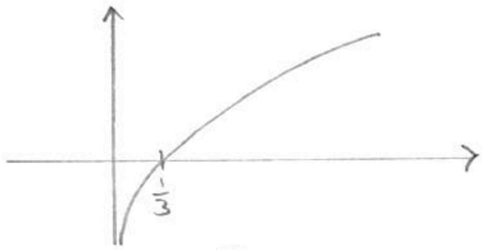
$\{f(x) : f(x) > 0, f(x) \in \mathbb{R}\}$

b) $f(x) = e^{3x} + 2, x \in \mathbb{R}$



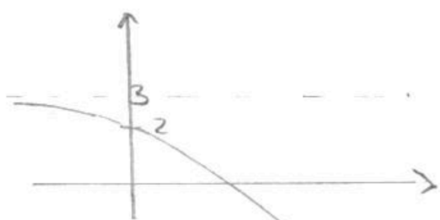
$\{f(x) : f(x) > 2, f(x) \in \mathbb{R}\}$

c) $f(x) = \ln 3x, x > 0$



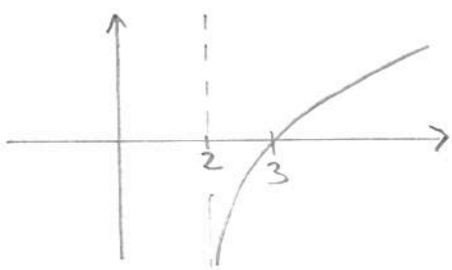
$\{f(x) : f(x) \in \mathbb{R}\}$

d) $f(x) = 3 - e^{2x}, x \in \mathbb{R}$



$\{f(x) : f(x) < 3, f(x) \in \mathbb{R}\}$

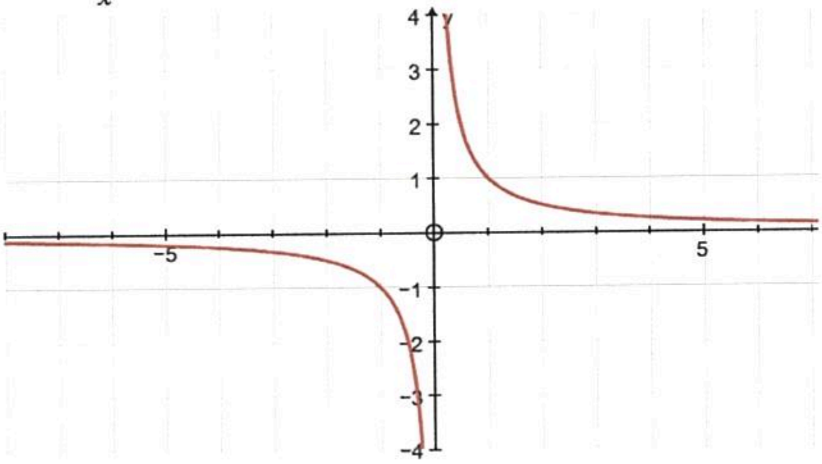
e) $f(x) = 3\ln(x-2),\ x > 2$



$\{f(x) : f(x) \in \mathbb{R}\}$

Reciprocal Graphs

$f(x) = \frac{1}{x}$



An asymptote is a line which the graph approaches but never reaches.

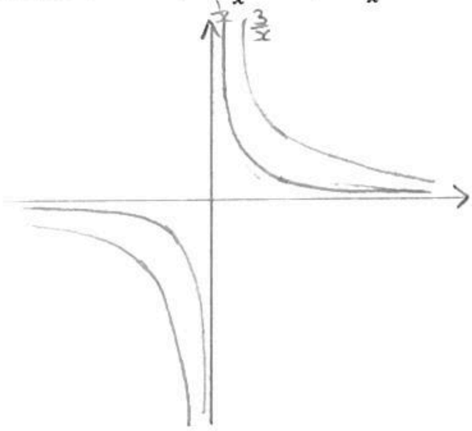
Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$

Range: $\{f(x) : f(x) \in \mathbb{R}, f(x) \neq 0\}$

Asymptotes: $x = 0, y = 0$

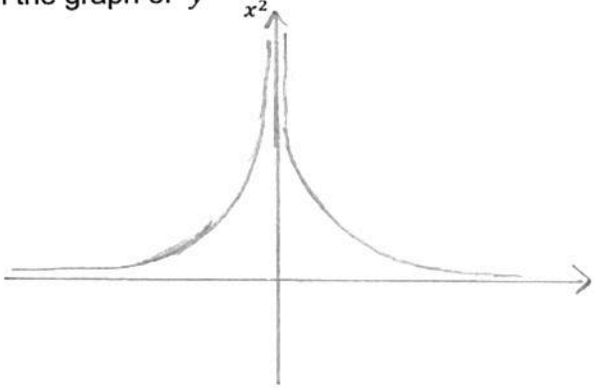
Example 8

On the same axes, sketch $y = \frac{1}{x}$ and $y = \frac{3}{x}$



Example 9

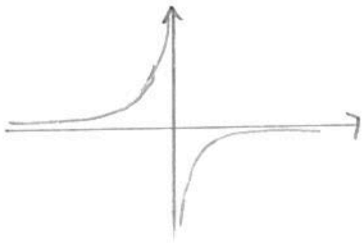
Sketch the graph of $y = \frac{1}{x^2}$



Example 10

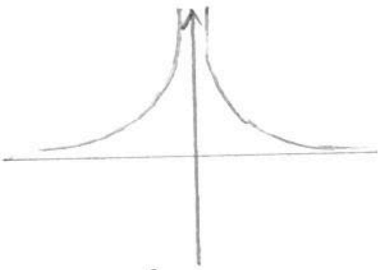
Sketch each function and state any asymptotes.

a) $f(x) = -\frac{b}{x},\ x \in \mathbb{R}, x \neq 0$



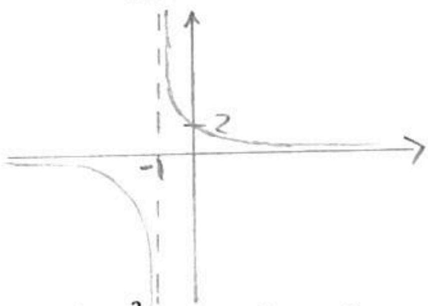
Asymptotes
$x = 0,\ y = 0$

b) $f(x) = \frac{4}{x^2}, x \in \mathbb{R}, x \neq 0$



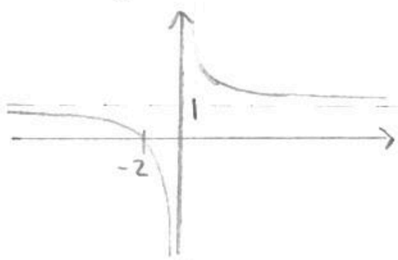
Asymptote
$x = 0,\ y = 0$

c) $f(x) = \frac{2}{x+1}, x \in \mathbb{R}, x \neq -1$



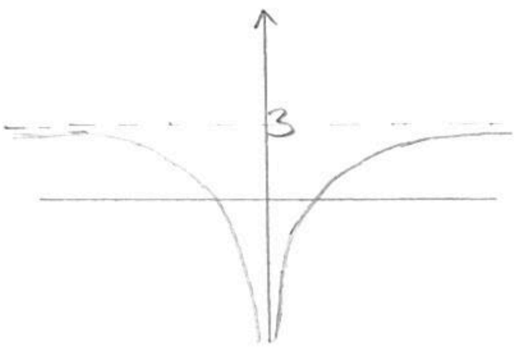
Asymptotes
$x = -1,\ y = 0$

d) $f(x) = \frac{2}{x} + 1, x \in \mathbb{R}, x \neq 0$



Asymptotes
$x = 0,\ y = 1$

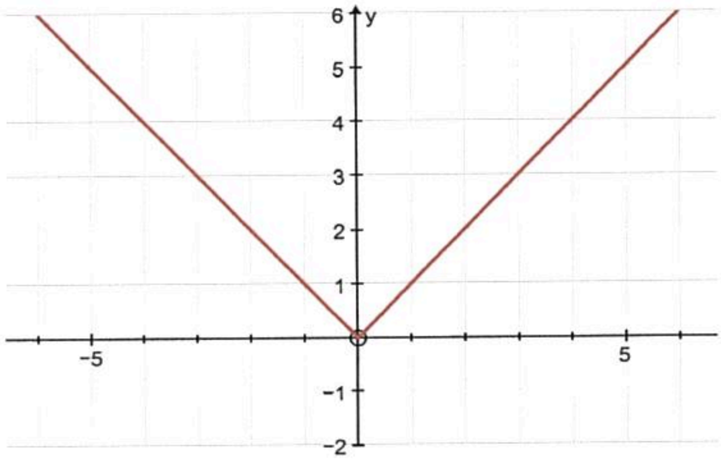
e) $f(x) = 3 - \frac{2}{x^2}, x \in \mathbb{R}, x \neq 0$



Asymptotes
$y = 3,\ x = 0$

Modulus

$f(x) = |x|$



The modulus sign $|\quad|$ means you use the positive value of whatever is inside. Using your graphics calculator, this is abs ( ).

So: $|-4| = 4$

$|2.5| = 2.5$

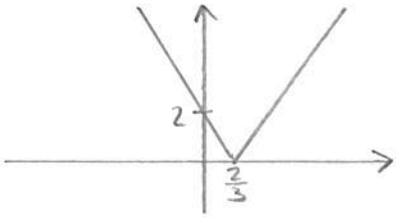
Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{f(x) : f(x) \geq 0, f(x) \in \mathbb{R}\}$

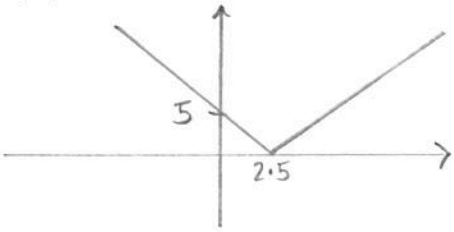
Example 11

Sketch each function and state the range.

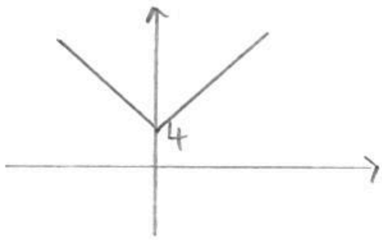
a) $f(x) = |3x - 2|$



b) $f(x) = |5 - 2x|$



c) $f(x) = |2x| + 4$

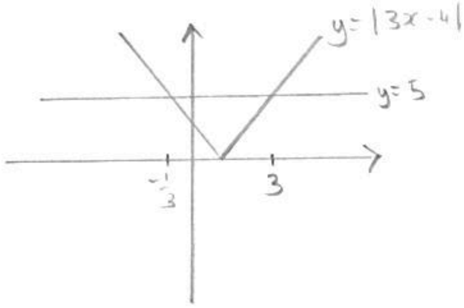


Example 12

Solve the equation $|3x - 4| = 5$

$3x - 4 = 5$
$3x = 9$
$x = 3$

$3x - 4 = -5$
$3x = -1$
$x = \frac{-1}{3}$



Example 13

Solve the equation $|x+1| = |2x-1|$

$x+1 = 2x-1$

$2 = x$

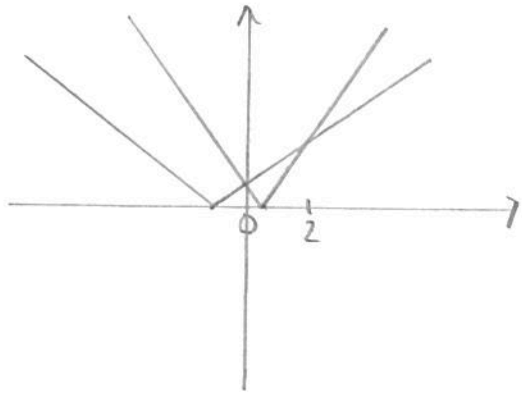
$x = 2$

$x+1 = -(2x-1)$

$x+1 = -2x+1$

$3x = 0$

$x = 0$



Example 14

Solve each equation:

a) $|x-5| = 3$

$x-5=3$

$x=8$

$x-5=-3$

$x=2$

b) $|2x-7| = 4$

$2x-7=4$

$2x=11$

$x=\frac{11}{2}$

$2x-7=-4$

$2x=3$

$x=\frac{3}{2}$

c) $|x-2| = |x+4|$

$x-2 = x+4$

No solutions

$x-2=-x-4$

$2x=-2$

$x=-1$

d) $|2x+1| = |9-2x|$

$2x+1 = 9-2x$

$4x = 8$

$x = 2$

$2x+1 = -9+2x$

No solutions

e) $|3x-4| = |2x+3|$

$3x-4 = 2x+3$

$x = 7$

$3x-4 = -2x-3$

$5x = 1$

$x = \frac{1}{5}$

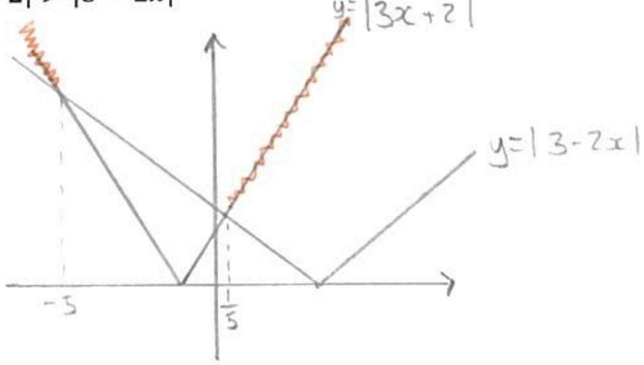
Example 15

a) Solve the equation $|3x + 2| = |3 - 2x|$
b) Hence solve the inequality $|3x + 2| > |3 - 2x|$

a) $3x + 2 = 3 - 2x$
$5x = 1$
$x = \frac{1}{5}$

$3x + 2 = -3 + 2x$
$x = -5$

b)

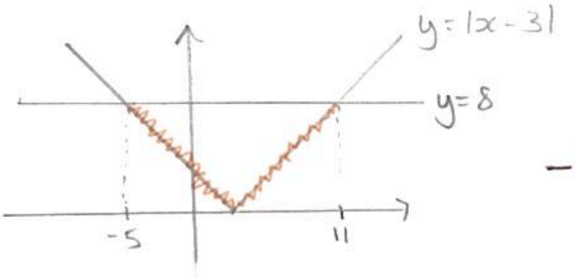


$x < -5$ OR $x > \frac{1}{5}$

Example 16

a) $|x - 3| < 8$

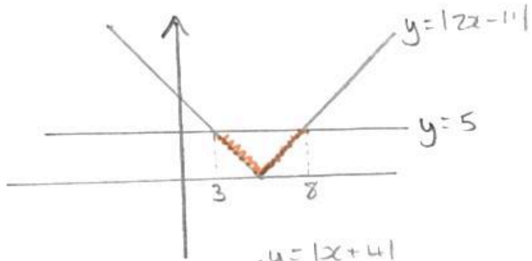
$x - 3 = 8$
$x = 11$
$x - 3 = -8$
$x = -5$



$-5 < x < 11$

b) $|2x - 11| \leq 5$

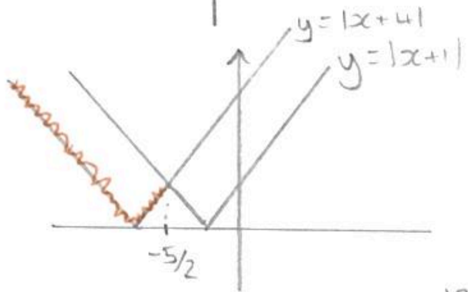
$2x - 11 = 5$
$2x = 16$
$x = 8$
$2x - 11 = -5$
$2x = 6$
$x = 3$



$3 \leq x \leq 8$

c) $|x + 4| \leq |x + 1|$

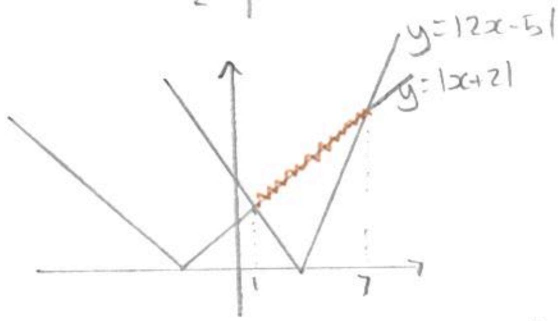
$x + 4 = x + 1$
No solutions
$x + 4 = -x - 1$
$2x = -5$
$x = \frac{-5}{2}$



$x \leq \frac{-5}{2}$

d) $|x + 2| > |2x - 5|$

$x + 2 = 2x - 5$
$-x = -7$
$x = 7$
$x + 2 = -2x + 5$
$3x = 3$
$x = 1$

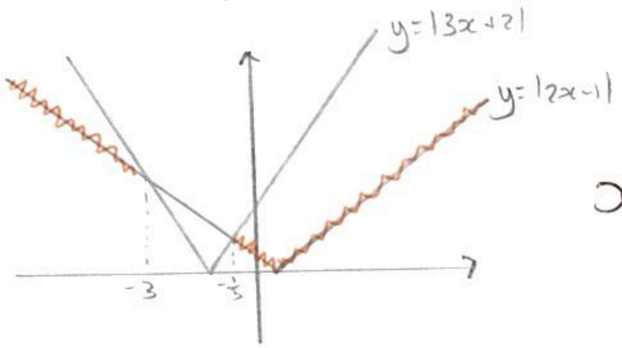


$1 < x < 7$

e) $|2x - 1| < |3x + 2|$

$2x - 1 = 3x + 2$
$-x = 3$
$x = -3$

$2x - 1 = -3x - 2$
$5x = -1$
$x = \frac{-1}{5}$



$x > \frac{-1}{5}$ OR $x < -3$

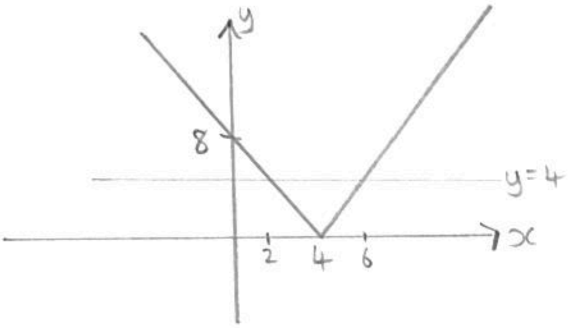
Exam Questions

4 (a) Sketch the graph of $y = |8 - 2x|$. *(2 marks)*

(b) Solve the equation $|8 - 2x| = 4$. *(2 marks)*

(c) Solve the inequality $|8 - 2x| > 4$. *(2 marks)*

a)



b) $8 - 2x = 4$

$-2x = -4$

$x = 2$

$8 - 2x = -4$

$-2x = -12$

$x = 6$

c) $x < 2$ OR $x > 6$

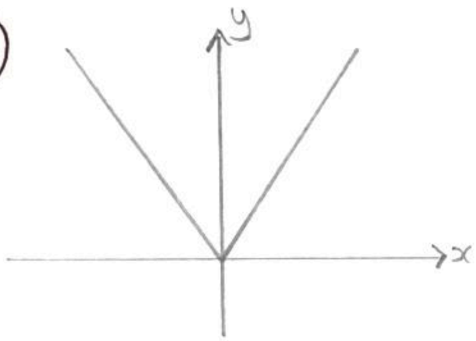
7 (a) Sketch the graph of $y = |2x|$. *(1 mark)*

(b) On a separate diagram, sketch the graph of $y = 4 - |2x|$, indicating the coordinates of the points where the graph crosses the coordinate axes. *(3 marks)*

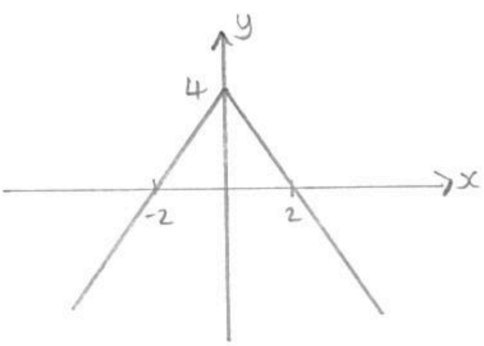
(c) Solve $4 - |2x| = x$. *(3 marks)*

(d) Hence, or otherwise, solve the inequality $4 - |2x| > x$. *(2 marks)*

a)



b)



c) $4 - |2x| = x$

$4 - 2x = x$

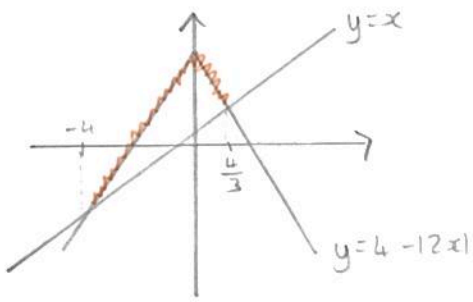
$3x = 4$

$x = \frac{4}{3}$

$4 + 2x = x$

$x = -4$

d)



$-4 < x < \frac{4}{3}$