Pure Sector 2: Trigonometry 2

Aims:

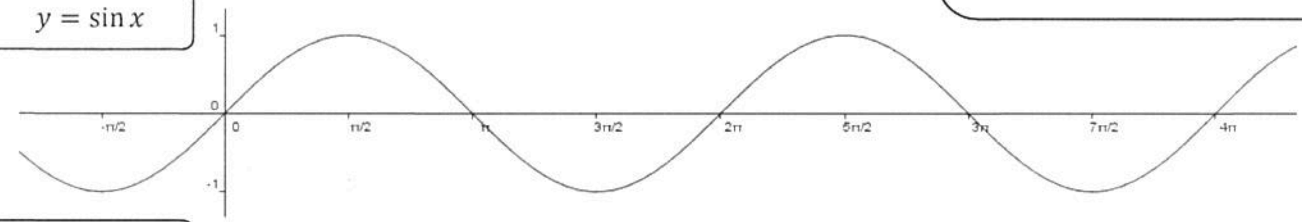
- To understand and use the sine, cosine and tangent functions
- To recognise and use common exact values of sine, cosine and tangent
- To understand and use inverse trigonometric functions
- To understand and use the definitions of secant, cosecant and cotangent
- To solve simple trigonometric equations, finding all solutions in a given interval
- To solve more complex trigonometric equations including quadratics

Sine, Cosine and Tangent Graphs

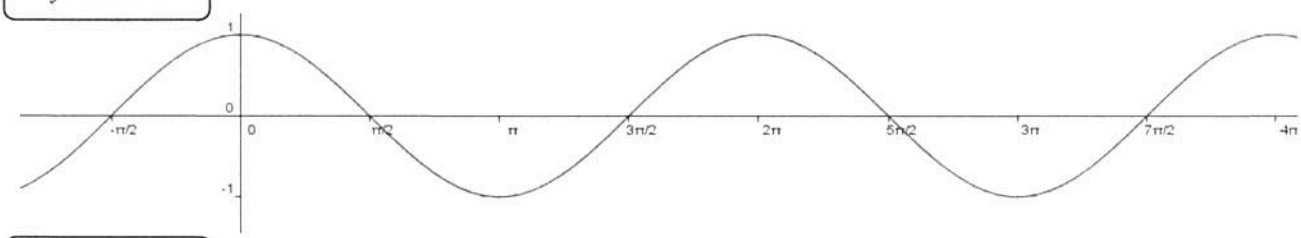
You need to be able to sketch the graphs of sine, cosine and tangent including any points of intersection with the axes and for tan graphs any asymptotes.

If you are using a graphical calculator make sure you have checked whether it is in degrees or radians and used a suitable interval on the axes.

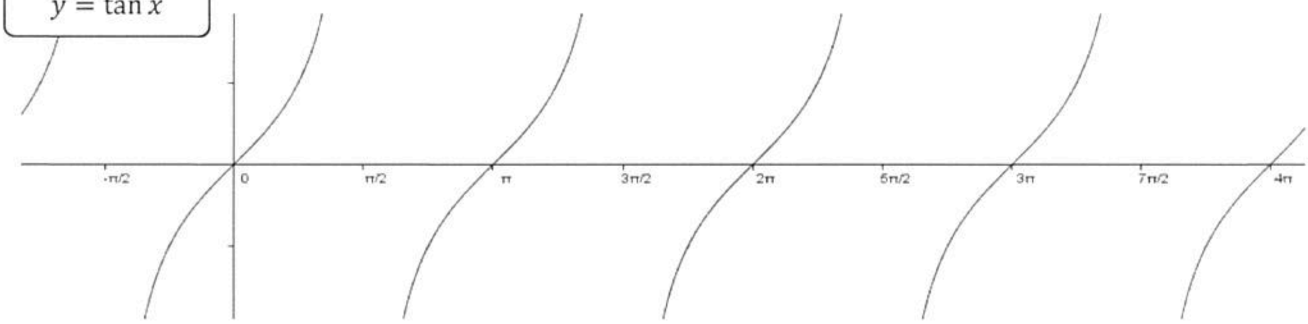
$y = \sin x$



$y = \cos x$



$y = \tan x$

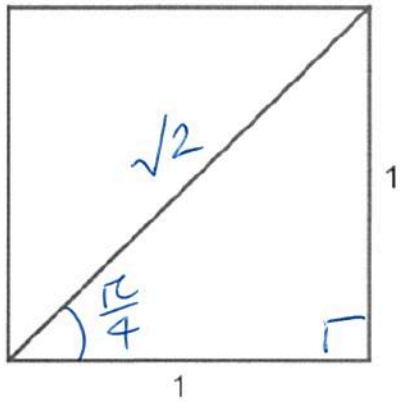


Exact Values

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{\pi}{4} = 1$



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

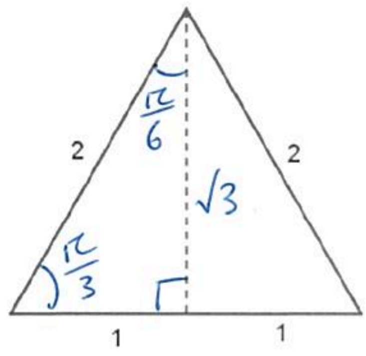
$\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \sqrt{3}$

$\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$



Secant, cosecant and cotangent

So far we have used sine, cosine and tangent. The remaining three trigonometric ratios are secant, cosecant and cotangent (written as $\sec x$, $\operatorname{cosec} x$ and $\cot x$). They are defined as:

$$\sec x = \frac{1}{\cos x}$$

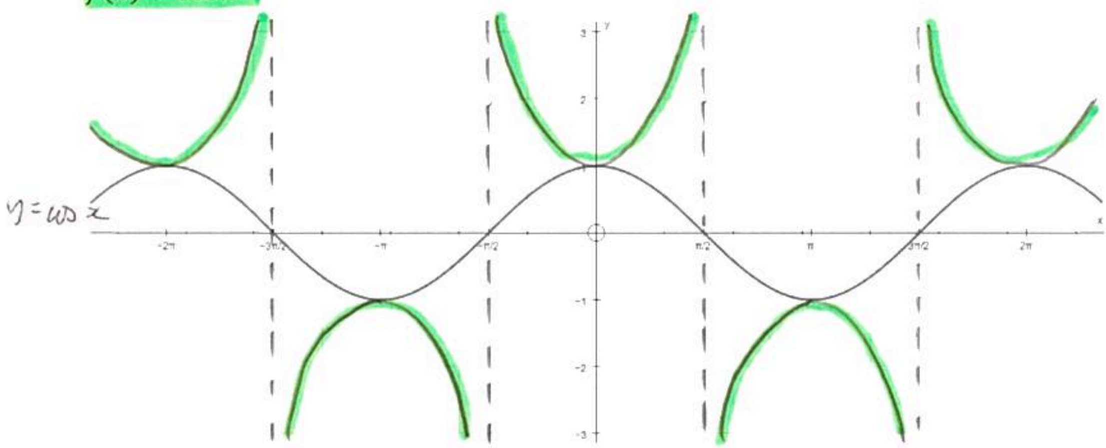
$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

You must remember these! Use the third letter rule. E.g. $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$ and $\cot x = \frac{1}{\tan x}$.

Graphs

$f(x) = \sec x$



Domain:

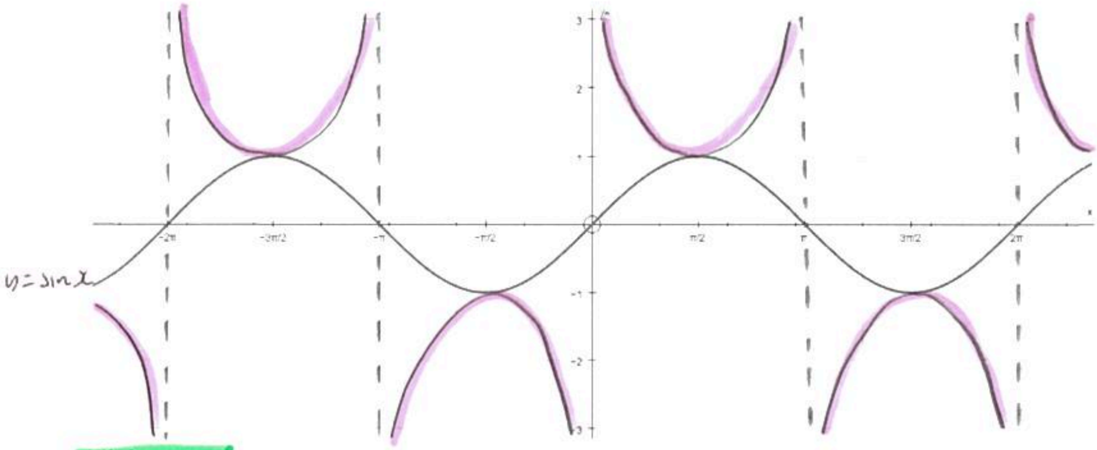
$\{x : x \in \mathbb{R}, x \neq (2n-1)\frac{\pi}{2}, n \in \mathbb{Z}\}$

Range:

$\{f(x) : f(x) \in \mathbb{R}, f(x) \geq 1\}$

$\cup \{f(x) : f(x) \in \mathbb{R}, f(x) \leq -1\}$

$f(x) = \operatorname{cosec} x$



Domain:

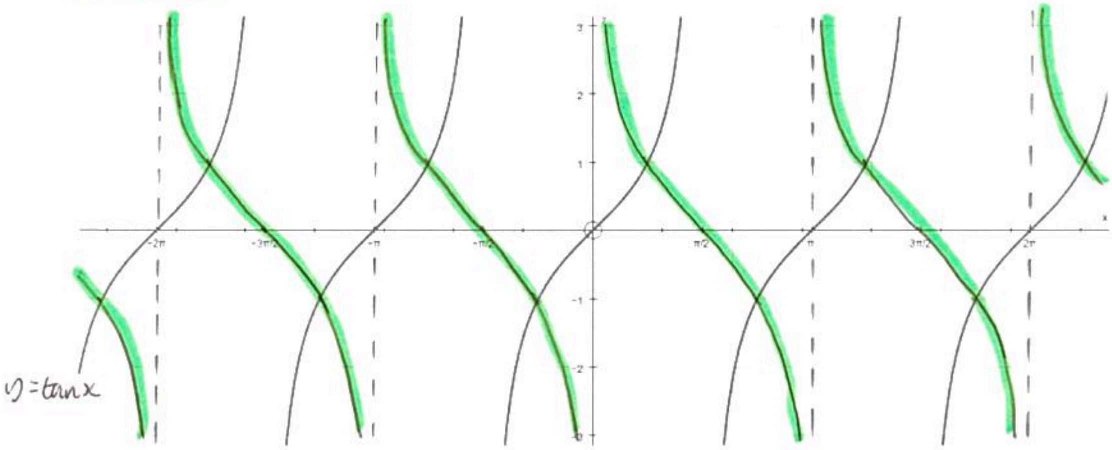
$\{x : x \in R, x \neq n\pi, n \in Z\}$

Range:

$\{f(x) : f(x) \in R, f(x) \geq 1\}$

$\cup \{f(x) : f(x) \in R, f(x) \leq -1\}$

$f(x) = \cot x$



Domain:

$\{x : x \in R, x \neq n\pi, n \in Z\}$

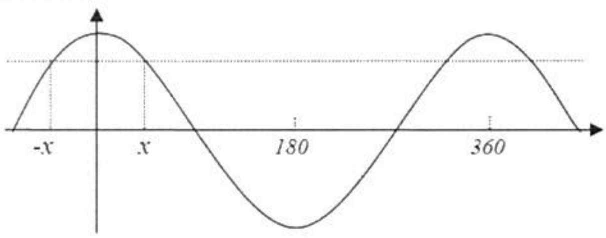
Range:

$\{f(x) : f(x) \in R\}$

Solving Simple Equations

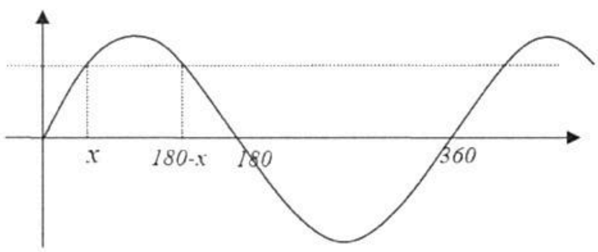
When you solve a trigonometric equation on your calculator you are only given the principal value (for example $\sin^{-1} 0.5 = 30°$). As trigonometric graphs are infinite and repetitive we can use symmetry to find all the solutions to an equation within a given interval.
To solve trigonometric equations you should always sketch and graph to find all the solutions.

Cosine



When solving $\cos x = ?$ then the first two solutions are x and $-x$ then $\pm 360°/2\pi^c$.

Sine



When solving $\sin x = ?$ then the first two solutions are x and $180° - x$ or $\pi^c - x$ then $\pm 360°/2\pi^c$.

Tangent

When solving $\tan x = ?$ then the first solution is x and then $\pm 180°/\pi^c$. You do not always need to sketch the tangent graph.

Example 1

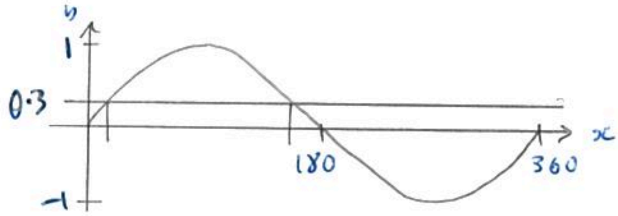
Solve the equation $\sin x = 0.3$ giving all solutions in the interval $0° \leq x \leq 360°$.

This tells you your calculator must be in degrees!

$$x = \sin^{-1}(0.3)$$

$$= 17.457\ldots,\ 180 - 17.457\ldots$$

$$= 17.5,\ 162.5$$



Example 2

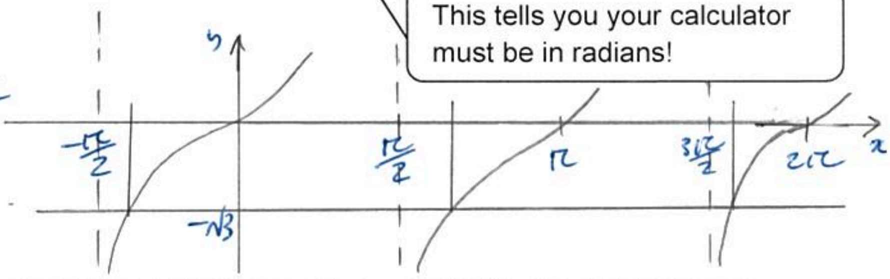
Solve the equation $\tan x = -\sqrt{3}$ giving all solutions in the interval $0 \leq x \leq 2\pi$ giving your answers in radians to three significant figures.

This tells you your calculator must be in radians!

$$x = \tan^{-1}(-\sqrt{3})$$

$$= \left(\frac{-\pi}{3}\right),\ \frac{-\pi}{3} + \pi,\ \frac{-\pi}{3} + 2\pi$$

$$= \frac{2\pi}{3},\ \frac{5\pi}{3}$$



Example 3

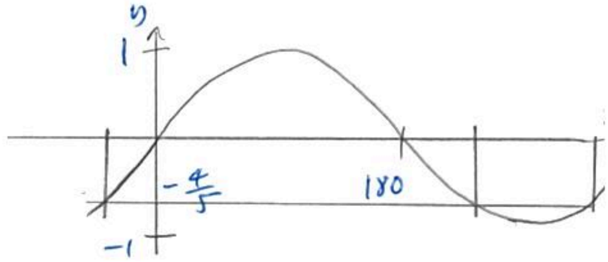
Solve the equation $\operatorname{cosec} x = -1.25$ giving all solutions in the interval $0° \leq x < 360°$ to one decimal place.

$$\sin x = \frac{-1}{1.25} = \frac{-4}{5}$$

$$x = \sin^{-1}\left(\frac{-4}{5}\right)$$

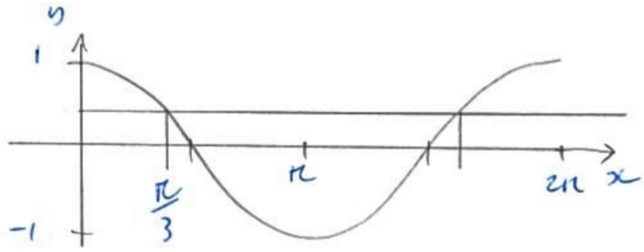
$$= (-53.130\ldots),\ 180 + 53.130,\ 360 - 53.130$$

$$= 23.1,\ 306.9$$



Example 4

Solve the equation $\cos x = \cos\frac{\pi}{3}$ giving all solutions in the interval $0 \leq x \leq 2\pi$ giving your answers in radians in terms of π.



$$x = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$$

$$= \frac{\pi}{3},\ \frac{5\pi}{3}$$

Further Solving Equations

Example 5

Solve the equation $4\cos x + 3 = 0$ giving all solutions in the interval $0° \le x \le 360°$, giving your answers to the nearest $0.1°$.

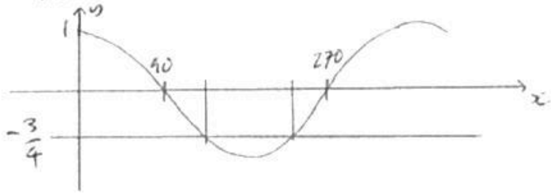
$4\cos x = -3$

$\cos x = \frac{-3}{4}$

$x = \cos^{-1}\left(\frac{-3}{4}\right)$

$= 138.590\ldots,\ 360 - 138.590\ldots$

$= \underline{183.6,\ 221.4}$



Example 6

a) Solve the equation $\sin 3x = \frac{1}{3}$ giving all solutions in the interval $0° \le x \le 180°$, giving your answers to the nearest degree.

We must change the interval as we are using $3x$.

$\boxed{0 \le 3x \le 540}$

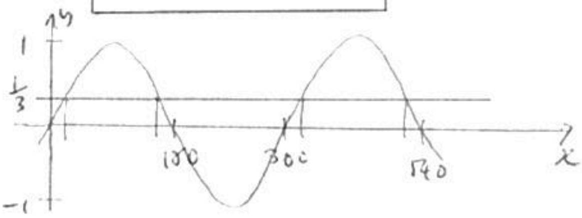
$3x = \sin^{-1}\left(\frac{1}{3}\right)$

$= 19.471\ldots,\ 180 - 19.4\ldots,\ 360 + 19.4\ldots,\ 540 - 19.4\ldots$

$x = \frac{19.47\ldots}{3},\ \frac{160.52\ldots}{3},\ \frac{379.47\ldots}{3},\ \frac{520.52\ldots}{3}$

$= 6.49\ldots,\ 53.50\ldots,\ 126.49\ldots,\ 173.50$

$= \underline{6,\ 54,\ 126,\ 174}$



b) Describe the transformation that maps the graph $y = \sin x$ to the graph $y = \sin 3x$.

STRETCH IN x-DIRECTION, SCALE FACTOR $\frac{1}{3}$

Example 7

a) Solve the equation $\cos(\theta - 80) = \frac{2}{5}$ giving all solutions in the interval $0 \le \theta \le 360$, giving your answer to four significant figures.

$\boxed{-80 \le (\theta - 80) \le 280}$

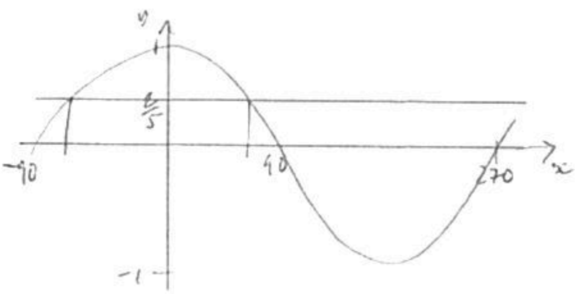
$\theta - 80 = \cos^{-1}\left(\frac{2}{5}\right)$

$= 66.421\ldots,\ -66.421\ldots$

$\theta = (-66.42\ldots + 80),\ (66.42\ldots + 80)$

$= 13.5781\ldots,\ 146.421\ldots$

$= \underline{13.58,\ 146.4}$



b) Describe the transformation that maps graph $y = \cos(\theta - 80)$ to the graph $y = \cos\theta$.

TRANSLATION $\begin{bmatrix} -80 \\ 0 \end{bmatrix}$

Example 8

Solve the equation $\cot 2x = 3.4$ giving all solutions in the interval $0 \leq x \leq \pi$ to one decimal place.

$\tan 2x = \frac{1}{3.4}$

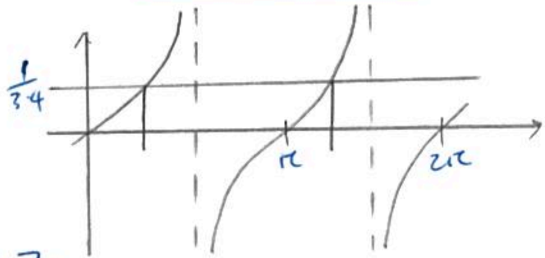
$\boxed{0 \leq 2x \leq 2\pi}$

$2x = \tan^{-1}\left(\frac{1}{3.4}\right)$

$= 0.28605, \quad \pi + 0.28605$

$x = \frac{0.286\ldots}{2}, \quad \frac{\pi + 0.286\ldots}{2}$

$= 0.1430\ldots, 1.7138\ldots = \underline{0.1, 1.7}$



Solving Quadratic Trigonometric Equations

Example 9

Solve the equation $\operatorname{cosec}^2 \theta = \frac{9}{4}$ for $0 \leq \theta \leq 360°$, giving your answers to 1 decimal place.

$\operatorname{cosec} \theta = \pm\sqrt{\frac{9}{4}} = \pm \frac{3}{2}$

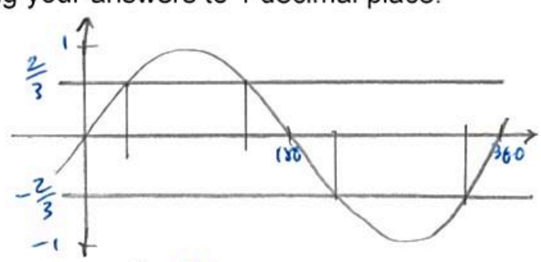
$\sin \theta = \pm \frac{2}{3}$

$\theta = \sin^{-1}\left(\frac{2}{3}\right)$

$= 41.81\ldots, 180 - 41.81, 180 + 41.81, 360 - 41.81$

$= 41.81\ldots, 138.18\ldots, 221.81\ldots, 318.18$

$= \underline{41.8, 138.2, 221.8, 318.2}$



Example 10

Solve the equation $2\cos^2\theta + 5\cos\theta - 3 = 0$ for $0 \leq \theta \leq 360°$.

let $\cos\theta = X$

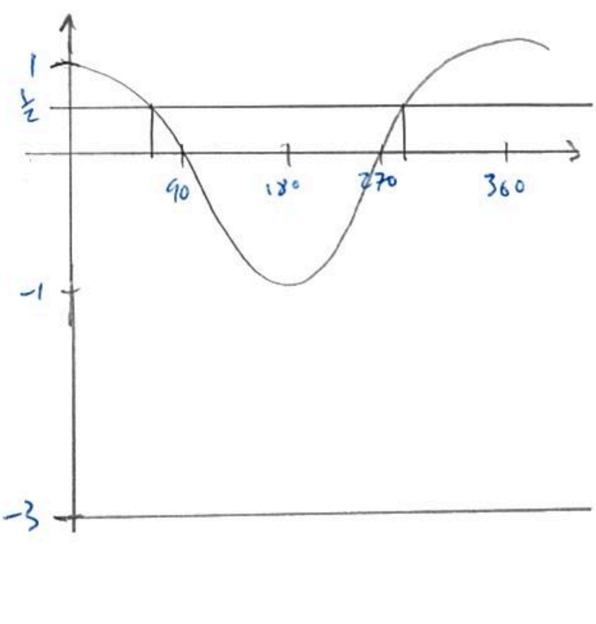
$2X^2 + 5X - 3 = 0$

$(2X - 1)(X + 3) = 0$

$\cos\theta = \frac{1}{2}$ OR $\cos\theta = -3$ (NO ROOTS)

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$= \underline{60, 300}$



Example 11

Solve the equation $\sin x \tan x - 3\sin x = 0$ giving answers in the interval $0 \leq \theta \leq 360°$.

$\sin x(\tan x - 3) = 0$

$\sin x = 0$ or $\tan x = 3$

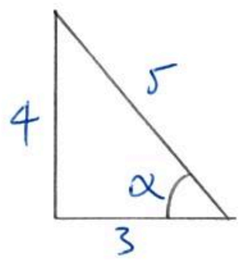
$x = 0, 180, 360$ $x = 71.56..., 251.56..$

$\underline{x = 0, 71.6, 180, 251.6, 360}$

Finding the Exact Value of Trigonometric Ratios

Example 12

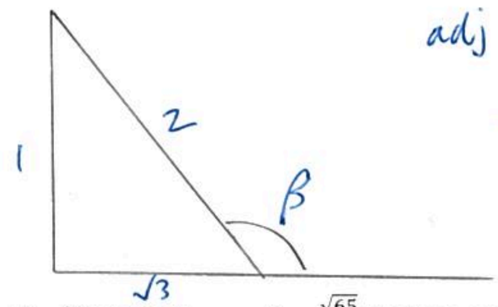
a) Given that angle α is acute and $\cos\alpha = \frac{3}{5}$ find the exact value of $\sin\alpha$.



$opp = \sqrt{5^2 - 3^2}$
$= 4$

$\underline{\sin\alpha = \frac{4}{5}}$

b) Given that angle β is obtuse and $\sin\beta = \frac{1}{2}$ find the exact value of $\tan\beta$.

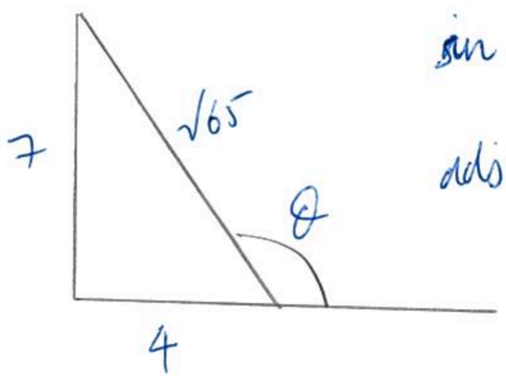


$adj = \sqrt{2^2 - 1^2}$
$= \sqrt{3}$

$\tan\theta$ -VE FOR $90 < \theta < 180$

$\underline{\tan\beta = \frac{-1}{\sqrt{3}}}$

c) Given that $\operatorname{cosec}\theta = \frac{\sqrt{65}}{7}$ find the exact value of $\sec\theta$ when θ is obtuse.



$\sin\theta = \frac{7}{\sqrt{65}}$

$adj = \sqrt{65 - 7^2}$
$= 4$

$\cos\theta$ -VE FOR $90 < \theta < 180$

$\cos\theta = \frac{-4}{\sqrt{65}}$

$\underline{\sec\theta = \frac{-\sqrt{65}}{4}}$

Exam Style Questions

2 It is given that $(\tan\theta + 1)(\tan^2\theta - 3) = 0$

(a) Find the possible values of $\tan\theta$

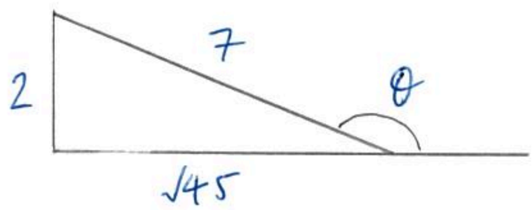
$\tan\theta = -1$ or $\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$

3 Given that $\sin\theta = \frac{2}{7}$ and θ is obtuse, find:

(a) the exact value of $\cos\theta$

(b) the exact value of $\tan\theta$



$adj = \sqrt{7^2 - 2^2}$

$= \sqrt{45}$

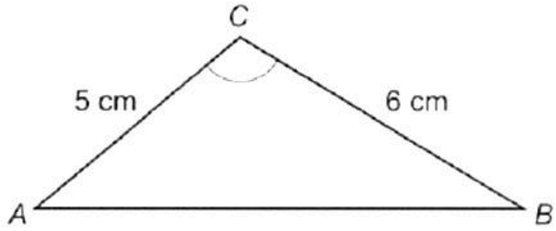
a) $\cos\theta$ -VE FOR $90 < \theta < 180$

$\cos\theta = \frac{-\sqrt{45}}{7}$

b) $\tan\theta$ -VE FOR $90 < \theta < 180$

$\tan\theta = \frac{-2}{\sqrt{45}}$

3 The diagram shows a triangle ABC.



The lengths of the sides AC and BC are 5 cm and 6 cm respectively.
The area of triangle ABC is 12.5 cm^2, and angle ACB is obtuse.

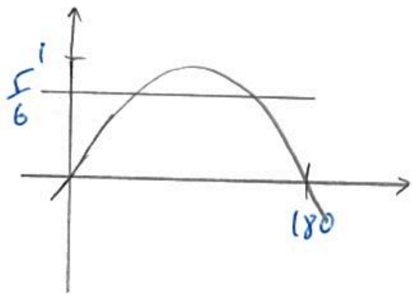
(a) Find the size of angle ACB, giving your answer to the nearest 0.1°

(b) Find the length of AB, giving your answer to two significant figures.

a) $\frac{1}{2}(5)(6)\sin\theta = 12.5$

$\sin\theta = \frac{12.5}{15} = \frac{5}{6}$

$\theta = 56.442\ldots$

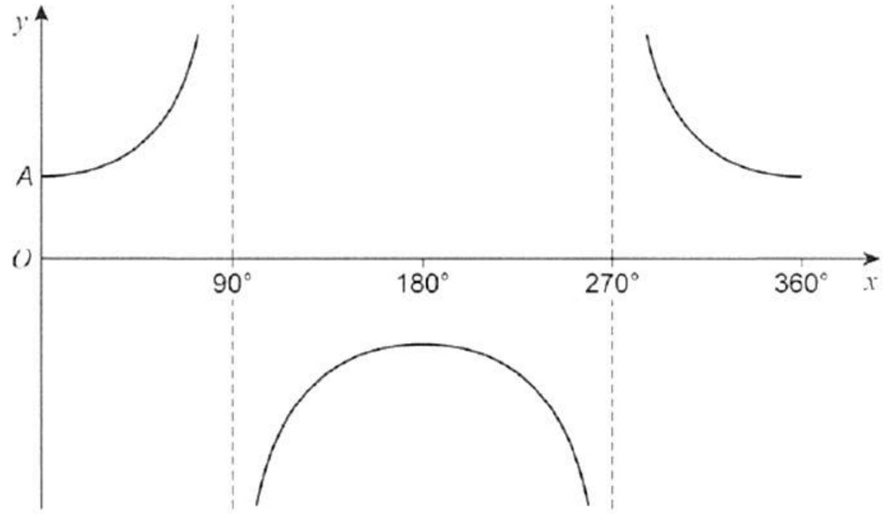


$\angle ACB = 180 - 56.4\ldots$

$= 123.557\ldots$

$= \underline{123.6°}$

5 (a) The diagram shows the graph of $y = \sec x$ for $0° \leq x \leq 360°$



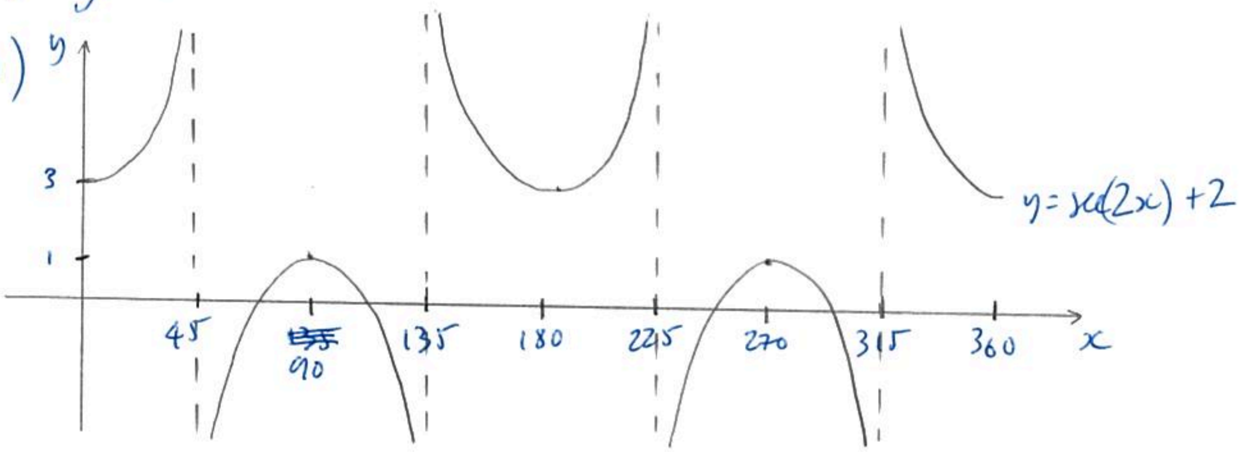
(i) The point A on the curve is where $x = 0$. State the y-coordinate of A.

(ii) Sketch the graph of $y = \sec(2x) + 2$ for $0° \leq x \leq 360°$

(b) Solve the equation $\sec x = 2$ giving all the values of x in degrees in the interval $0° \leq x \leq 360°$

a)(i) $y = 1$

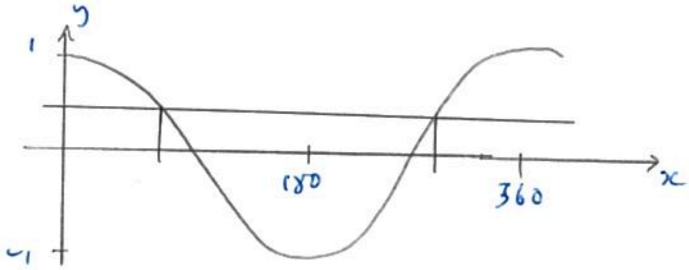
(ii)



b) $\sec x = 2$

$\cos x = \frac{1}{2}$

$\underline{x = 60, 300}$



Small-angle Approximations

Trigonometry: small angles

For small angle θ,

$\sin\theta \approx \theta$

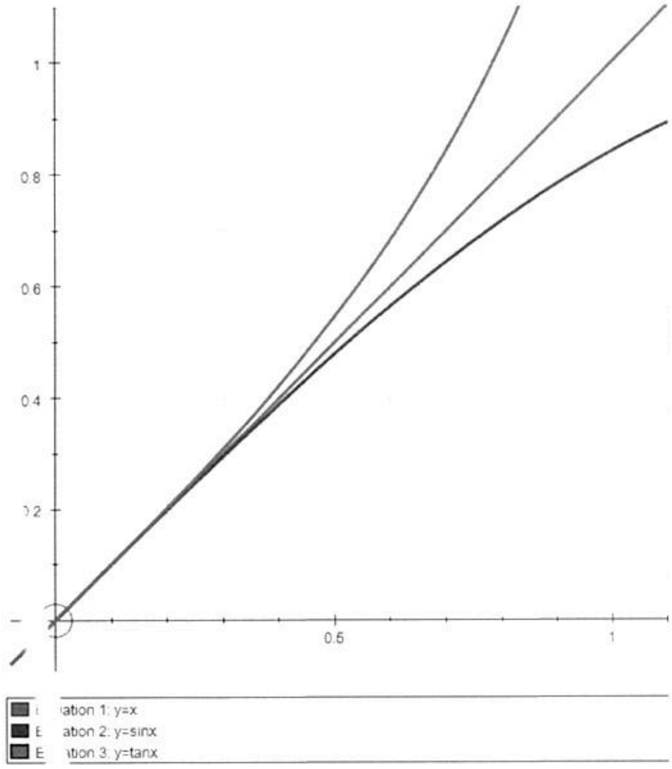
$\cos\theta \approx 1 - \dfrac{\theta^2}{2}$

$\tan\theta \approx \theta$

From the AQA Formula Booklet

Using the double angle formula you will meet next year

Where θ is in radians. Why?



All of these approximations are very good for $-0.1 \le \theta \le 0.1$ radians.

<u>Example 13</u>

a) Show that substituting $\theta = 0$ into the expression $\frac{\cos\theta - \cos 2\theta}{\theta^2}$ gives $\frac{0}{0}$ which is undefined.

$$\frac{\cos 0 - \cos 2(0)}{0^2} = \frac{1-1}{0} = \frac{0}{0}$$

b) Find an approximation for $\cos\theta - \cos 2\theta$ when θ and 2θ are both small.

$$\cos\theta - \cos 2\theta \approx 1 - \frac{\theta^2}{2} - \left(1 - \frac{(2\theta)^2}{2}\right)$$
$$\approx 1 - \frac{\theta^2}{2} - 1 + \frac{4\theta^2}{2}$$
$$\approx 2\theta^2 - \frac{\theta^2}{2}$$
$$\approx \frac{3\theta^2}{2}$$

c) Hence find $\lim\limits_{\theta \to 0} \frac{\cos\theta - \cos 2\theta}{\theta^2}$.

$$\lim_{\theta \to 0} \frac{\cos\theta - \cos 2\theta}{\theta^2} = \frac{3\cancel{\theta^2}}{2\cancel{\theta^2}} = \frac{3}{2}$$

Example 14

a) Find an approximate expression for $\sin 2\theta + \tan 3\theta$ when θ is small enough for 3θ to be considered as small.

$$\sin 2\theta + \tan 3\theta \approx 2\theta + 3\theta$$
$$\approx 5\theta$$

b) Hence find $\lim_{\theta \to 0} \frac{\sin 2\theta + \tan 3\theta}{\theta}$.

$$\lim_{\theta \to 0} \frac{\sin 2\theta + \tan 3\theta}{\theta} = \frac{5\cancel{\theta}}{\cancel{\theta}} = 5$$

Often applications of the small angle approximations will involve binomial expansions or the compound angle formulae. You will meet both of these later in the course.

Exam Question – AQA Specimen Paper 1

3 When θ is small, find an approximation for $\cos 3\theta + \theta \sin 2\theta$, giving your answer in the form $a + b\theta^2$

[3 marks]

ASSUMING 3θ IS SMALL:

$$\cos 3\theta + \theta \sin 2\theta \approx 1 - \frac{(3\theta)^2}{2} + \theta(2\theta)$$
$$\approx 1 - \frac{9\theta^2}{2} + 2\theta^2$$
$$\approx 1 - \frac{5}{2}\theta^2$$