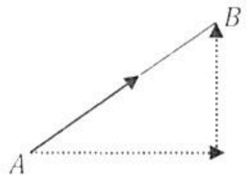
Mechanics Sector 1: Vectors and RUVAT

Scalar – quantity that is defined by **magnitude only**
Vector – quantity that is defined by a **scalar magnitude** and a **direction.**

Scalar	**Vector**
Distance	Displacement
Speed	Velocity
Time	Acceleration
Mass	Force

A movement from A to B may be represented by the vector $\overrightarrow{AB}$. The magnitude is given by the length and the direction by the arrow. Vector can also be written as using single letter in bold, $\boldsymbol{a}$ (you would write this as $\underline{a}$).

We can represent the vector $\overrightarrow{AB}$ as a column vector.



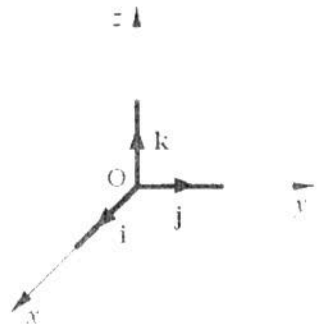
$$\overrightarrow{AB} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

This tells us the number of units moved in the x-direction

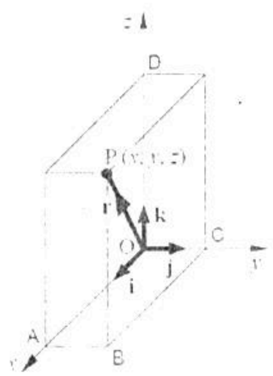
This tells us the number of units moved in the y-direction

3D Vectors

We can also represent vectors in three dimensions relative to a three dimensional coordinate grid. A third axis, the z-axis, is added at right angles to the xy-plane. Conventionally, we show the z-axis pointing vertically upwards with the xy-plane horizontal.



We can represent the vector $\overrightarrow{OP}$ as a column vector.



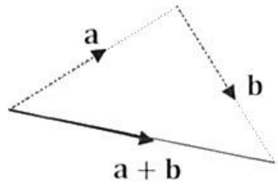
$$\overrightarrow{OP} = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$$

This tells us the number of units moved in the x-direction

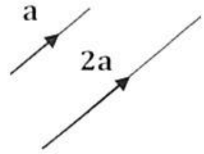
This tells us the number of units moved in the y-direction

This tells us the number of units moved in the z-direction

Adding, Subtracting and Multiplying Vectors

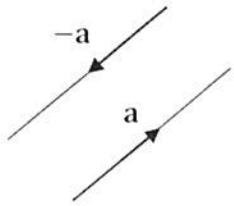


We can add and subtract vectors with same number of components by adding or subtracting the respective components. For example, when $\mathbf{a} = \begin{bmatrix}4\\3\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}2\\-7\end{bmatrix}$ then $+\mathbf{b} = \begin{bmatrix}4\\3\end{bmatrix} + \begin{bmatrix}2\\-7\end{bmatrix} = \begin{bmatrix}6\\-4\end{bmatrix}$, this can be represented geometrically.



When you multiply a vector by a scalar quantity, t, you multiply each of the components by t.

For example, when $\mathbf{a} = \begin{bmatrix}4\\3\end{bmatrix}$ then the vector $2\mathbf{a} = \begin{bmatrix}8\\6\end{bmatrix}$, the vector $2\mathbf{a}$ is in the same direction (**parallel**) as $\mathbf{a}$ but twice as long.



The vector $\mathbf{a}$ and the vector $-\mathbf{a}$ have the same length (magnitude) but have the opposite direction. We would say that the vectors are anti-parallel.

Example 1

Given that $\mathbf{a} = \begin{bmatrix}6\\2\\-1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}0\\-5\\-2\end{bmatrix}$, find

a) $\mathbf{a} - \mathbf{b}$

$$\underline{a} - \underline{b} = \begin{pmatrix}6\\2\\-1\end{pmatrix} - \begin{pmatrix}0\\-5\\-2\end{pmatrix} = \begin{pmatrix}6\\7\\1\end{pmatrix}$$

b) $2\mathbf{a} + 3\mathbf{b}$

$$2\underline{a} + 3\underline{b} = 2\begin{pmatrix}6\\2\\-1\end{pmatrix} + 3\begin{pmatrix}0\\-5\\-2\end{pmatrix} = \begin{pmatrix}12\\4\\-2\end{pmatrix} + \begin{pmatrix}0\\-15\\-6\end{pmatrix} = \begin{pmatrix}12\\-11\\-8\end{pmatrix}$$

c) $\mathbf{b} - 3\mathbf{a}$

$$\underline{b} - 3\underline{a} = \begin{pmatrix}0\\-5\\-2\end{pmatrix} - 3\begin{pmatrix}6\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\-5\\-2\end{pmatrix} - \begin{pmatrix}18\\6\\-3\end{pmatrix} = \begin{pmatrix}-18\\-11\\1\end{pmatrix}$$

Example 2

Identify which of the following vectors are parallel:

a) $\mathbf{a} = \begin{bmatrix}5\\2\\-3\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}60\\24\\-36\end{bmatrix}$

$$\begin{pmatrix}60\\24\\-36\end{pmatrix} = 12\begin{pmatrix}5\\2\\-3\end{pmatrix} \qquad \underline{b} = 12\underline{a}$$

$\underline{b}$ is a multiple of a

$\therefore$ $\underline{a}$ and $\underline{b}$ are parallel

b) $\mathbf{c} = \begin{bmatrix}2\\-4\\6\end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix}-1\\-2\\3\end{bmatrix}$,

$$\begin{pmatrix}2\\-4\\6\end{pmatrix} = 2\begin{pmatrix}1\\-2\\3\end{pmatrix}$$

$\underline{c}$ is not a multiple of $\underline{d}$

$\therefore$

Example 3

Given that the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel find the values of x and y.

$\mathbf{a} = \begin{bmatrix}2\\1\\-7\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}6\\x\\y\end{bmatrix}$

$$\begin{pmatrix}6\\x\\y\end{pmatrix} = \lambda\begin{pmatrix}2\\1\\-7\end{pmatrix}$$

$6 = 2\lambda \quad \therefore \lambda = 3$

$x = \lambda \times 1 = 3$

$y = -7\lambda = -21$

Base Unit Vectors

A unit vector is a vector that has a magnitude of 1 (see later).
The most important unit vectors are those running parallel to the x, y and z axes called **base unit vectors**. $\mathbf{i}$ gives the number of units moved in the x direction, $\mathbf{j}$ gives the number of units moved in the y direction and $\mathbf{k}$ gives the number of units moved in the z direction.

In two dimensions, the base unit vectors are **i** and **j**

$$\mathbf{i} = \begin{bmatrix}1\\0\end{bmatrix} \quad \mathbf{j} = \begin{bmatrix}0\\1\end{bmatrix}$$

In three dimensions the base unit vectors are

$$\mathbf{i} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \mathbf{j} = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \mathbf{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

For example, the vector $\begin{bmatrix}4\\5\end{bmatrix}$ can be written as $4\mathbf{i} + 5\mathbf{j}$ and the vector $\begin{bmatrix}6\\1\\-2\end{bmatrix}$ can be written as $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Example 4

Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$, find these terms of $\mathbf{i}$ and $\mathbf{j}$.

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{a} + \mathbf{b}$ **c** $2\mathbf{a} - \mathbf{b}$ **d** $2\mathbf{b} + \mathbf{a}$
e $3\mathbf{a} - 2\mathbf{b}$ **f** $\mathbf{b} - 3\mathbf{a}$ **g** $4\mathbf{b} - \mathbf{a}$ **h** $2\mathbf{a} - 3\mathbf{b}$

a) $2i + 3j + 4i - j = 6i + 2j$
b) $3(2i + 3j) + 4i - j = 10i + 8j$
c) $2(2i + 3j) - (4i - j) = 7j$
d) $2(4i - j) + (2i + 3j) = 10i + j$
e) $3(2i + 3j) - 2(4i - j) = -2i + 11j$
f) $4i - j - 3(2i + 3j) = -2i - 10j$
g) $4(4i - j) - (2i + 3j) = 14i - 7j$
h) $2(2i + 3j) - 3(4i - j) = -8i + 9j$

Example 5

Given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, find

a $2\mathbf{a}$ b $-3\mathbf{b}$ c $\mathbf{a} + \mathbf{b}$ d $2\mathbf{a} + \mathbf{b}$
e $\mathbf{a} + 2\mathbf{b}$ f $2\mathbf{a} + 3\mathbf{b}$ g $\mathbf{a} - \mathbf{b}$ h $2\mathbf{a} - 3\mathbf{b}$

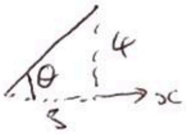
a) $2(3i + 4j - 7k) = 6i + 8j - 14k$
b) $-3(-2i - 2j + 5k) = 6i + 6j - 15k$
c) $3i + 4j - 7k + (-2i - 2j + 5k) = i + 2j - 2k$
d) $2(3i + 4j - 7k) + (-2i - 2j + 5k) = 4i + 6j - 9k$
e) $(3i + 4j - 7k) + 2(-2i - 2j + 5k) = -i + 3k$
f) $2(3i + 4j - 7k) + 3(-2i - 2j + 5k) = 2j + k$
g) $3i + 4j - 7k - (-2i - 2j + 5k) = 5i + 8j - 12k$
h) $2(3i + 4j - 7k) - 3(-2i - 2j + 5k) = 12i + 14j - 29k$

Example 6

Find the angle that each of these vectors makes with the positive x-axis.

a $3\mathbf{i} + 4\mathbf{j}$ **b** $6\mathbf{i} - 8\mathbf{j}$ **c** $5\mathbf{i} + 12\mathbf{j}$ **d** $2\mathbf{i} + 4\mathbf{j}$

a)

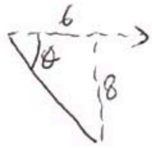


$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$= 53.1°$

above +ve x axis (or anticlockwise)

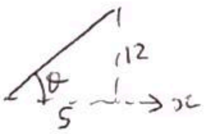
b)



$\theta = \tan^{-1}\left(\frac{8}{6}\right)$

$= 53.1°$ below +ve x axis (or clockwise)

c)

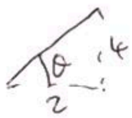


$\theta = \tan^{-1}\left(\frac{12}{5}\right)$

$= 67.4°$

above +ve x axis (or anticlockwise)

d)



$\theta = \tan^{-1}\left(\frac{4}{2}\right) = 63.4°$

above +ve x axis (or anticlockwise)

Example 7

Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$, find

a λ if $\mathbf{a} + \lambda\mathbf{b}$ is parallel to the vector $\mathbf{i}$,

// to $\underline{i}$ → $\underline{j}$ component = 0

A$\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}2\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}2+3\lambda\\5-\lambda\end{pmatrix}$

$5 - \lambda = 0$

$\therefore \lambda = 5$

b μ if $\mu\mathbf{a} + \mathbf{b}$ is parallel to the vector $\mathbf{j}$,

// to $\underline{j}$ → $\underline{i}$ component = 0

A$\begin{pmatrix}0\\1\end{pmatrix}$ $\mu\begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}2\mu+3\\5\mu-1\end{pmatrix}$

$\underline{i}$: $2\mu + 3 = 0$ $\mu = -\frac{3}{2}$

Example 8 (harder)

Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$,

b μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$,

c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$,

d t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$.

a) // to $\underline{i} + \underline{j} = \begin{pmatrix}1\\1\end{pmatrix}$ $\underline{i}$ component = $\underline{j}$ component

$\begin{pmatrix}3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}3+\lambda\\4-2\lambda\end{pmatrix}$

$\therefore 3 + \lambda = 4 - 2\lambda$

$3\lambda = 1 \Rightarrow \lambda = \frac{1}{3}$

b) // to $\underline{i} + 3\underline{j}$

$\therefore$ $\underline{j}$ component = $3\times$ $\underline{i}$ component

$\mu\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}3\mu+1\\4\mu-2\end{pmatrix}$

$3(3\mu+1) = 4\mu - 2$

$9\mu + 3 = 4\mu - 2$

$5\mu = -5$

$\mu = -1$

c) // to $2\underline{i} + \underline{j}$ $\cancel{2}$ $\underline{i}$ 2

→ $\underline{i}$ component = $2\times$ $\underline{j}$ component

$\begin{pmatrix}3\\4\end{pmatrix} - s\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}3-s\\4+2s\end{pmatrix}$

$3 - s = 2(4 + 2s)$

$3 - s = 8 + 4s$

$-5 = 5s$

$s = -1$

d) // to $-2\underline{i} + 3\underline{j}$

$3(\underline{i}\text{ component}) = -2(\underline{j}\text{ component})$

$\begin{pmatrix}1\\-2\end{pmatrix} - t\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1-3t\\-2-4t\end{pmatrix}$

$3(1 - 3t) = -2(-2 - 4t)$

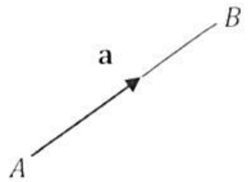
$3 - 9t = 4 + 8t$

$-1 = 17t$ $t = -\frac{1}{17}$

Magnitude of a Vector

The **magnitude** (or **modulus**) of a vector is given by the length of the line segment representing it. The magnitude of a vector is written as $|\overrightarrow{AB}|$ or $|\mathbf{a}|$.

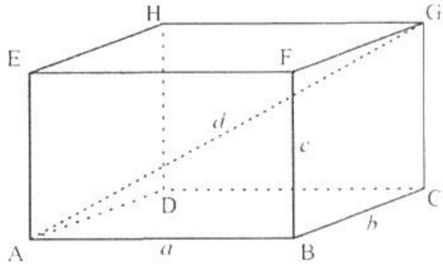
We can calculate this using Pythagoras' Theorem for the vector where $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$



$|\overrightarrow{AB}| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34}$

> In two dimensions, the magnitude of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is $\sqrt{a^2 + b^2}$

The magnitude of a three-dimensional vector can be found by applying Pythagoras's Theorem in three dimensions.



$\overrightarrow{AG} = 2\underline{i} - 4\underline{j} + \underline{k}$

$|\overrightarrow{AG}| = \sqrt{2^2 + (-4)^2 + 1^2}$

$= \sqrt{21}$

> In three dimensions, the magnitude of the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is $\sqrt{a^2 + b^2 + c^2}$

Note the vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ have different directions but the same magnitude.

Example 9

Given that $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 7 \\ -4 \\ 3 \end{bmatrix}$ find the magnitude of the vectors

a) $\mathbf{p}$ b) $\mathbf{q}$ c) $\mathbf{p} + \mathbf{q}$

a) $|\underline{p}| = \sqrt{1^2 + 2^2 + (-2)^2}$

$|\underline{p}| = \sqrt{9} = 3$

$|\underline{q}| = \sqrt{7^2 + (-4)^2 + 3^2}$

$= \sqrt{74}$

$\underline{p} + \underline{q} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix}$

$|\underline{p} + \underline{q}| = \sqrt{8^2 + (-2)^2 + 1^2}$

$= \sqrt{69}$

Example 10

Find the magnitude of each of these vectors.

a $3\mathbf{i} - 4\mathbf{j}$ **b** $6\mathbf{i} - 8\mathbf{j}$ **c** $5\mathbf{i} + 12\mathbf{j}$ **d** $2\mathbf{i} + 4\mathbf{j}$ **e** $4\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}$

a) $\sqrt{3^2 + 4^2}$ $= \sqrt{25}$ $= 5$

b) $\sqrt{6^2 + (-8)^2}$ $= \sqrt{100}$ $= 10$

c) $\sqrt{5^2 + 12^2}$ $= \sqrt{169}$ $= 13$

d) $\sqrt{2^2 + 4^2}$ $= \sqrt{20}$ $= 2\sqrt{5}$

e) $\sqrt{4^2 + 3^2 + (-12)^2}$ $= \sqrt{169}$ $= 13$

Bearings

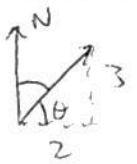
The direction of a vector can be given as a bearing.
This is the angle measured in a clockwise direction from the north line (usually represented by the positive **j** direction (vertical).
I would recommend that you draw a right angled triangle using your **i** and **j** components and resultant vector. You can then find an angle within this triangle using SOHCATOA and then use this angle to find the appropriate bearing angle.

Example 11

In this question, the horizontal unit vectors **i** and **j** are directed due east and due north respectively.

Find the magnitude and bearing of these vectors.

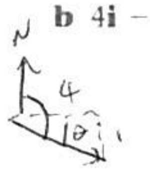
a $2\mathbf{i} + 3\mathbf{j}$



$\theta = \tan^{-1}\left(\frac{3}{2}\right)$
$= 56.3$
$\therefore$ bearing $= 90 - 56.3...$
$= 33.69...$
$= 033.7^\circ$ (3sf)

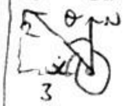
NB could also find $\tan^{-1}(2/3)$ to find bearing directly

b $4\mathbf{i} - \mathbf{j}$



$\theta = \tan^{-1}(1/4)$
$= 14.0...$
$\therefore$ bearing $= 90 + 14.0..$
$= 104^\circ$ (3sf)

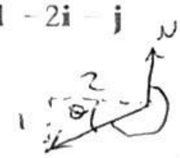
c $-3\mathbf{i} + 2\mathbf{j}$



$\alpha = \tan^{-1}\left(\frac{2}{3}\right)$
$= 33.7^\circ...$
$\therefore$ bearing $= 270 + 33.7..$
$= 303.7..$
$= 304^\circ$ (3sf)

or $\theta = \tan^{-1}(3/2) = 56.3...$
bearing $= 360 - 56.3...$
$= 304^\circ$

d $-2\mathbf{i} - \mathbf{j}$



$\theta = \tan^{-1}(1/2) = 26.56...$
bearing $= 270 - 26.56...$
$= 243.43...$
$= 243^\circ$ (3sf)

Position and Translation Vectors

In general, a vector has no specific location. However the position vector of A relative to the origin is $\overrightarrow{OA}$, this gives the position in relation to the origin. A position vector is like a co-ordinate, a translation vector is like listening to the directions on a sat-nav.
If point A has position vector $\mathbf{a}$ and point B has position vector $\mathbf{b}$, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Example 12

The points A, B and C have coordinates $(2, -5, 6)$, $(0, 2, 5)$ and $(-9, 2, 6)$ respectively.

a) Find the vector $\overrightarrow{BC}$

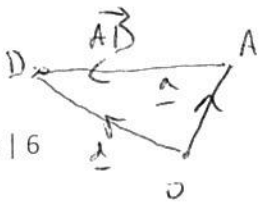
$\underline{b} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$ $\quad \underline{c} = \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix}$ $\quad \therefore \overrightarrow{BC} = \underline{c} - \underline{b}$
$= \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix}$

b) Find the vector $\overrightarrow{CA}$

$\underline{a} = \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$ $\quad \therefore \overrightarrow{CA} = \underline{a} - \underline{c} = \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} -9 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \\ 0 \end{pmatrix}$

c) The point D is such that $\overrightarrow{AD} = 2\overrightarrow{BC}$. Show that D has the coordinates $(-16, -5, 8)$.

$2\overrightarrow{BC} = 2\begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -18 \\ 0 \\ 2 \end{pmatrix} = \overrightarrow{AD}$ $\quad$ need position vector $\underline{d}$



$\underline{d} = \underline{a} + \overrightarrow{AD}$
$= \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix} + \begin{pmatrix} -18 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -16 \\ -5 \\ 8 \end{pmatrix}$

$\therefore$ co-ordinates of D are $(-16, -5, 8)$

this is vector from O (0,0,0) to D

Unit Vectors

A unit vector is a vector that has magnitude 1. For example $\begin{bmatrix}0\\-1\end{bmatrix}$, $\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$, $\begin{bmatrix}1\\0\\0\end{bmatrix}$ are all unit vectors. You can always find a unit vector in the same direction as any given vector.

It is possible to find a unit vector parallel to any given vector, **a**, by dividing the vector by its magnitude. The unit vector parallel to the vector a is denoted by $\hat{a}$
So, in general, $\hat{a} = \frac{a}{|a|}$. or $\frac{1}{|\underline{a}|}\underline{a}$

Example 16

Find unit vectors in the direction of these vectors.

a $\begin{pmatrix}5\\-12\end{pmatrix}$ **b** $\begin{pmatrix}3\\4\\-12\end{pmatrix}$ c $3\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$

d $2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ **e** $15\mathbf{i} + 20\mathbf{j}$ f $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

a) $|\underline{a}| = \sqrt{5^2+12^2}$
$= 13$
$\therefore \hat{\underline{a}} = \frac{1}{13}\begin{pmatrix}5\\-12\end{pmatrix}$

b) $|\underline{a}| = \sqrt{3^2+4^2+12^2}$
$= 13$
$\therefore \hat{\underline{a}} = \frac{1}{13}\begin{pmatrix}3\\4\\-12\end{pmatrix}$

c) $|\underline{a}| = \sqrt{3^2+6^2+12^2}$
$= 3\sqrt{21}$
$\hat{\underline{a}} = \frac{1}{3\sqrt{21}}\begin{pmatrix}3\\-6\\12\end{pmatrix} = \frac{\sqrt{21}}{63}\begin{pmatrix}3\\-6\\12\end{pmatrix}$

d) $|\underline{a}| = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$
$\therefore \hat{\underline{a}} = \frac{1}{2\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{\sqrt{3}}{6}\begin{pmatrix}2\\2\\2\end{pmatrix}$
$= \frac{\sqrt{3}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$

e) $|\underline{a}| = \sqrt{15^2+20^2}$
$= 25$
$\hat{\underline{a}} = \frac{1}{25}\begin{pmatrix}15\\20\end{pmatrix}$
$= \frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$

f) $|\underline{a}| = \sqrt{1^2+2^2+5^2}$
$= \sqrt{30}$
$\hat{\underline{a}} = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}$

Example 17

The vectors a and b are given by $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$.
Find the unit vector in the direction of

a) **b**
b) $\overrightarrow{AB}$

a) $|\underline{b}| = \sqrt{3^2+3^2+7^2}$
$= \sqrt{67}$
$\therefore \hat{\underline{b}} = \frac{1}{\sqrt{67}}\begin{pmatrix}-3\\3\\7\end{pmatrix}$
$= \frac{\sqrt{67}}{67}\begin{pmatrix}-3\\3\\7\end{pmatrix}$

b) $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}-3\\3\\7\end{pmatrix} - \begin{pmatrix}4\\-3\\1\end{pmatrix}$
$= \begin{pmatrix}-7\\6\\6\end{pmatrix}$
$|\overrightarrow{AB}| = \sqrt{7^2+6^2+6^2}$
$= 11$
$\therefore \hat{\overrightarrow{AB}} = \frac{1}{11}\begin{pmatrix}-7\\6\\6\end{pmatrix}$

Distance between two points in three dimensions

If two points have the coordinates $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the vector $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$

and the distance between the two points P and Q is equal to the **magnitude of the vector $\overrightarrow{PQ}$**.

So $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

> The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by
> $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 13

The point A and B have the coordinates $(3, 2, -1)$ and $(2, 1, 5)$ respectively. Find the distance between the points A and B.

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$$

$$\text{distance} = |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-1)^2 + 6^2}$$
$$= \sqrt{38}$$

Colinear

If three points A, B and C are collinear then the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ will all be parallel.

Example 14

Prove that the points A $(-5, 1, 3)$, B $(-2, 0, 5)$ and C $(7, -3, 11)$ are collinear.

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix} - \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 8 \end{pmatrix} \qquad \overrightarrow{AC} = 4(\overrightarrow{AB}) \qquad \overrightarrow{BC} = 3(\overrightarrow{AB})$$

$\overrightarrow{AC}$ and $\overrightarrow{BC}$ are both multiples of $\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AC}$, $\overrightarrow{BC}$ and $\overrightarrow{AB}$ are all parallel to each other

$\therefore$ A, B and C are collinear

Example 15 (Harder)

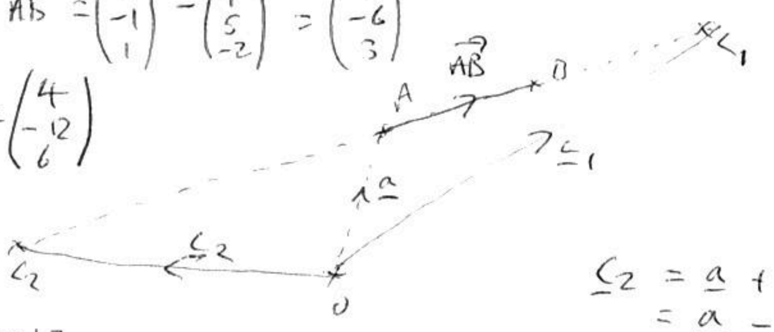
Three points A, B and C are collinear with $\overline{OA} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ and $\overline{OB} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

magnitude of $\overrightarrow{AC}$ magnitude of $\overrightarrow{AB}$

Given that $AC = 2AB$ find the coordinates of the two possible positions of point C.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$

$$2\overrightarrow{AB} = \begin{pmatrix} 4 \\ -12 \\ 6 \end{pmatrix}$$



$$\underline{c}_1 = \underline{a} + \cancel{2\overrightarrow{AB}}$$
$$= \underline{a} + 2\overrightarrow{AB}$$
$$= \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ -12 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}$$
$$\therefore C_1 = (5, -7, 4)$$

$$\underline{c}_2 = \underline{a} + \overrightarrow{AC_2}$$
$$= \underline{a} - 2\overrightarrow{AB}$$
$$= \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ -12 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 17 \\ -8 \end{pmatrix}$$
$$\therefore C_2 = (-3, 17, -8)$$

General Vector Problems

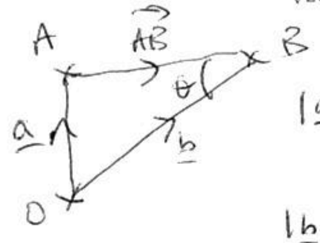
Example 18

The points A and B have coordinates $(2, 1, -2)$ and $(3, 4, -1)$ respectively. The acute angle OBA is θ, where O is the origin.

a) Find the vector $\overrightarrow{AB}$

b) Show that $\cos\theta = \frac{14}{\sqrt{286}}$

a) $\overrightarrow{AB} = \begin{pmatrix}3\\4\\-1\end{pmatrix} - \begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$



b) use cosine rule where lengths of triangle = magnitude of vectors

$|\underline{a}| = \sqrt{2^2+1^2+2^2}$
$= 3$

$|\underline{b}| = \sqrt{3^2+4^2+1^2}$
$= \sqrt{26}$

$|\overrightarrow{AB}| = \sqrt{1^2+3^2+1^2} = \sqrt{11}$

$a^2 = b^2 + AB^2 - 2(b)(AB)\cos\theta$

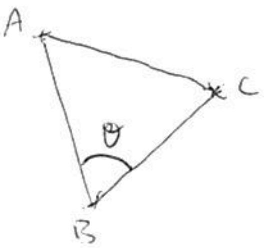
$\cos\theta = \frac{b^2 + AB^2 - a^2}{2(b)(AB)}$

$= \frac{(\sqrt{26})^2 + (\sqrt{11})^2 - 3^2}{2\sqrt{26}\sqrt{11}}$

$= \frac{28}{2\sqrt{286}} = \frac{14}{\sqrt{286}}$ as required

Example 19

The points A, B and C have coordinates $(2, 4, 3)$, $(3, 1, -1)$ and $(1, -2, 4)$ respectively. Find the size of angle ABC, giving your answer to the nearest degree.



$\overrightarrow{AB} = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\-4\end{pmatrix}$ $\quad |\overrightarrow{AB}| = \sqrt{1^2+3^2+4^2} = \sqrt{26}$

$\overrightarrow{AC} = \begin{pmatrix}1\\-2\\4\end{pmatrix} - \begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}-1\\-6\\1\end{pmatrix}$ $\quad |\overrightarrow{AC}| = \sqrt{1^2+6^2+1^2} = \sqrt{38}$

$\overrightarrow{BC} = \begin{pmatrix}1\\-2\\4\end{pmatrix} - \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$

$|\overrightarrow{BC}| = \sqrt{2^2+3^2+5^2} = \sqrt{38}$

$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta$

$38 = 26 + 26 - 2\sqrt{26}\sqrt{26}\cos\theta$

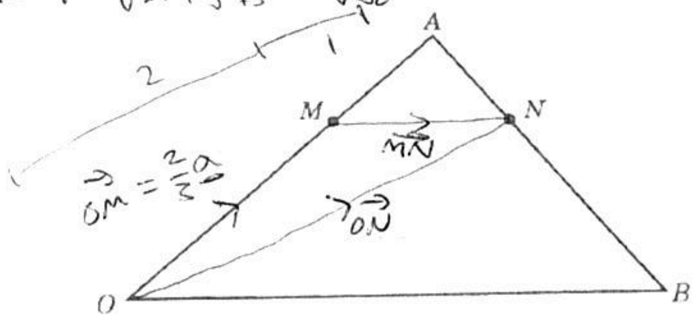
$\cos\theta = \frac{26 + 26 - 38}{2\sqrt{26}\sqrt{26}} = 0.2692\ldots$

$\theta = \cos^{-1}(0.2692\ldots)$

$= 74.3815\ldots$

$\therefore \theta = 74°$ (nearest degree)

Example 20



OAB is a triangle. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point M divides OA in the ratio $2:1$. MN is parallel to OB. Express the vector $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

triangle AMN is similar triangle to OAB $\rightarrow$ each side is // (so angles are same) + sides ~~of~~ of AMN are $\frac{1}{3}$ of sides of OAB

$\overrightarrow{OM} : \overrightarrow{MA}$ is also $2:1$

$\therefore \overrightarrow{OM} = \frac{2}{3}\underline{a}$

$\rightarrow \overrightarrow{MN} = \frac{1}{3}\overrightarrow{OB}$

$= \frac{1}{3}\underline{b}$

$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$\overrightarrow{ON} = \frac{2}{3}\underline{a} + \frac{1}{3}\underline{b}$

Exam Questions

1. Edexcel June 18 AS Pure Paper

Given that the point A has position vector $4\mathbf{i} - 5\mathbf{j}$ and the point B has position vector $-5\mathbf{i} - 2\mathbf{j}$,

(a) find the vector $\overrightarrow{AB}$, $\quad \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$

(b) find $|\overrightarrow{AB}|$. $\quad |\overrightarrow{AB}| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$

Give your answer as a simplified surd.

2. AQA Practice Set 1 AS Paper 1

The position vector of point A is $7\mathbf{i} + 9\mathbf{j}$

The position vector of the midpoint of the line joining point A to point B is $3\mathbf{i} + 6\mathbf{j}$

$\overrightarrow{AM} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \overrightarrow{MB}$

(a) Find the position vector of the point B.

$\underline{b} = \underline{m} + \overrightarrow{MB} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

or
$\underline{b} = \underline{a} + 2\overrightarrow{AM}$
$= \begin{pmatrix} 7 \\ 9 \end{pmatrix} + 2\begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

[2 marks]

(b) Find $|\overline{AB}|$

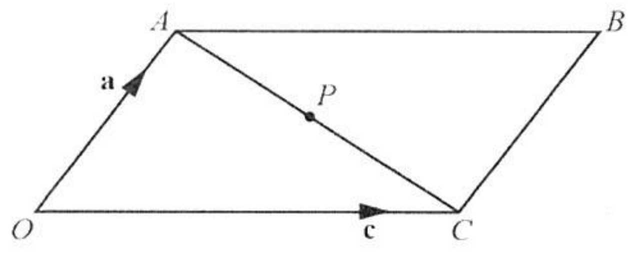
$\overrightarrow{AB} = 2\overrightarrow{AM} = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{8^2 + 6^2}$
$= 10$

[2 marks]

3. OCR June 18 AS Paper 1

$OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. P is the midpoint of AC.



$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$

(i) Find the following in terms of **a** and **c**, simplifying your answers.

(a) $\overrightarrow{AC}$ $\quad \overrightarrow{AC} = \underline{c} - \underline{a}$ [1]

(b) $\overrightarrow{OP}$ $\quad \overrightarrow{OP} = \underline{a} + \overrightarrow{AP} = \underline{a} + \frac{1}{2}\overrightarrow{AC} = \underline{a} + \frac{1}{2}(\underline{c} - \underline{a})$ [2]

$= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{c}$

(ii) Hence prove that the diagonals of a parallelogram bisect one another. [4]

$\overrightarrow{OB} = \underline{a} + \underline{c}$

$\overrightarrow{OP} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{c} = \frac{1}{2}\overrightarrow{OB}$

P is the midpoint of OB and of AC

$\therefore$ diagonals bisect one another

4. OCR Practice Set 2 Paper 2

Points A, B and C have position vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\-1\\5\end{pmatrix}$ and $\begin{pmatrix}-4\\0\\3\end{pmatrix}$ respectively.

(i) Find the exact distance between the midpoint of AB and the midpoint of BC. **[4]**

Point D has position vector $\begin{pmatrix}x\\-6\\z\end{pmatrix}$ and the line CD is parallel to the line AB.

(ii) Find all the possible pairs of x and z. **[4]**

5. Edexcel June 18 A2 Paper 2

Relative to a fixed origin O,

the point A has position vector $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$,

the point B has position vector $(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$,

and the point C has position vector $(a\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$, where a is a constant and $a < 0$

D is the point such that $\overrightarrow{AB} = \overrightarrow{BD}$.

(a) Find the position vector of D.

(2)

Given $|\overrightarrow{AC}| = 4$

(b) find the value of a.

(3)

6. OCR June 18 A2 Paper 2

The points A and B have position vectors $\begin{pmatrix}1\\-2\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\-1\\2\end{pmatrix}$ respectively.

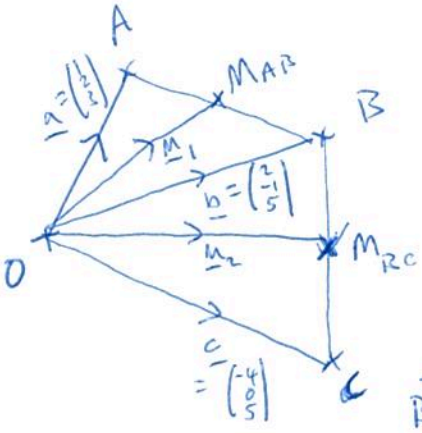
(i) Find the exact length of AB. **[2]**

(ii) Find the position vector of the midpoint of AB. **[1]**

The points P and Q have position vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}5\\1\\3\end{pmatrix}$ respectively.

(iii) Show that $ABPQ$ is a parallelogram. **[3]**

4. OCR Practice Set 2 Paper 2



Method 1

$$\overrightarrow{AB} = \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$$

$$\therefore \underline{m}_1 = \underline{a} + \tfrac{1}{2}\overrightarrow{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}3/2\\1/2\\4\end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix}-4\\0\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\+5\end{pmatrix} = \begin{pmatrix}-6\\1\\-2\end{pmatrix}$$

$$\therefore \underline{m}_2 = \underline{b} + \tfrac{1}{2}\overrightarrow{BC}$$
$$= \begin{pmatrix}2\\-1\\5\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}-6\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\-1/2\\4\end{pmatrix}$$

$$\overrightarrow{M_{AB}M_{BC}} = \underline{m}_2 - \underline{m}_1$$
$$= \begin{pmatrix}-1\\-1/2\\4\end{pmatrix} - \begin{pmatrix}3/2\\1/2\\4\end{pmatrix} = \begin{pmatrix}-5/2\\-1\\0\end{pmatrix}$$

$$\therefore \text{distance} = |\overrightarrow{M_{AB}M_{BC}}|$$
$$= \sqrt{\left(\tfrac{5}{2}\right)^2 + (1)^2 + 0^2}$$
$$= \sqrt{\tfrac{29}{4}} = \frac{\sqrt{29}}{2}$$

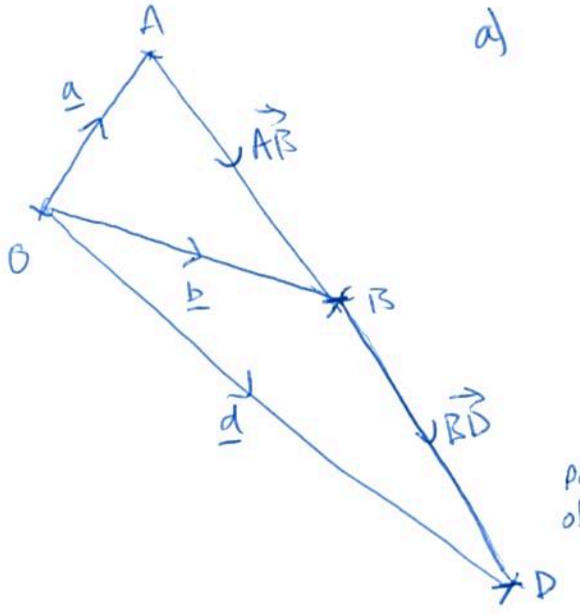
Method 2

$$\overrightarrow{AB} = \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix}-4\\0\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}-6\\1\\-2\end{pmatrix}$$

$$\overrightarrow{M_{AB}M_{BC}} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}$$
$$= \tfrac{1}{2}\begin{pmatrix}1\\-3\\2\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}-6\\1\\-2\end{pmatrix}$$
$$= \begin{pmatrix}-5/2\\-1\\0\end{pmatrix}$$

$$\therefore \text{distance} = \sqrt{\left(\tfrac{5}{2}\right)^2 + (1)^2 + 0^2}$$
$$= \frac{\sqrt{29}}{2}$$

5. Edexcel June 18 A2 Paper 2



a) $\vec{AB} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$

$\therefore \vec{BD} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$

position vector of D, $\underline{d} = \underline{b} + \vec{BD}$

$= \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$

$\underline{d} = \begin{pmatrix} 6 \\ -7 \\ 10 \end{pmatrix}$

b) $\vec{AC} = \begin{pmatrix} a \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} a-2 \\ 2 \\ 2 \end{pmatrix}$

$|\vec{AC}| = 4$

$\sqrt{(a-2)^2 + 2^2 + 2^2} = 4$

$\sqrt{(a-2)^2 + 8} = 4$

$(a-2)^2 + 8 = 16$

$(a-2)^2 = 8$

$a = 2 \pm 2\sqrt{2}$

$a < 0$

$\therefore a = 2 - 2\sqrt{2}$

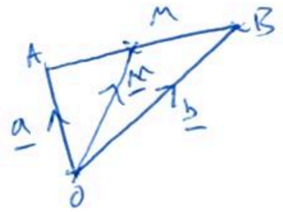
6. OCR June 18 A2 Paper 2

i) $\vec{AB} = \begin{pmatrix}-3\\-1\\2\end{pmatrix} - \begin{pmatrix}1\\-2\\5\end{pmatrix} = \begin{pmatrix}-4\\1\\-3\end{pmatrix}$

$\therefore$ length $AB = |\vec{AB}| = \sqrt{4^2+1^2+3^2}$

$= \sqrt{26}$

ii)



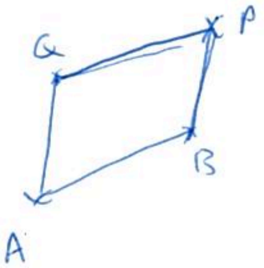
$\underline{m} = \underline{a} + \frac{1}{2}\vec{AB}$

NB or $\underline{m} = \underline{b} - \frac{1}{2}\vec{AB}$

$= \begin{pmatrix}1\\-2\\5\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-4\\1\\-3\end{pmatrix}$

$\underline{m} = \begin{pmatrix}-1\\-3/2\\7/2\end{pmatrix}$

iii)



if parallelogram

$\left.\begin{array}{l}\vec{AQ} = \vec{BP}\\ \vec{AB} = \vec{QP}\end{array}\right\}$ this means the opposite sides are the same length + in same direction (parallel)

$\vec{AQ} = \begin{pmatrix}5\\1\\3\end{pmatrix} - \begin{pmatrix}1\\-2\\5\end{pmatrix} = \begin{pmatrix}4\\3\\-2\end{pmatrix}$

$\therefore \vec{AQ} = \vec{BP}$

$\vec{BP} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}-3\\-1\\2\end{pmatrix} = \begin{pmatrix}4\\3\\-2\end{pmatrix}$

$\vec{AB} = \begin{pmatrix}-4\\1\\-3\end{pmatrix}$ $\quad \vec{QP} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}5\\1\\3\end{pmatrix} = \begin{pmatrix}-4\\1\\-3\end{pmatrix}$

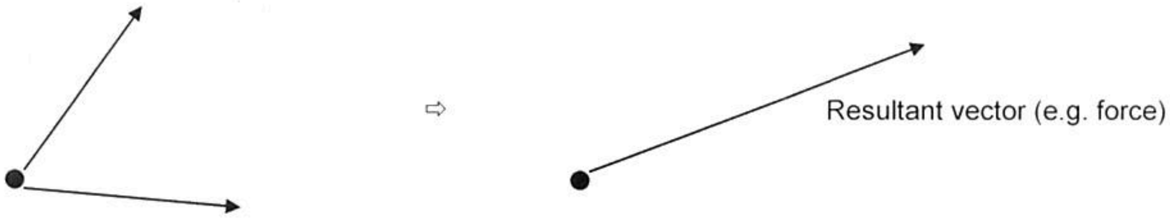
$\vec{AB} = \vec{QP}$

$\therefore$ ABPQ is a parallelogram.

Resultant vectors

The resultant vector is defined as the single vector that would have the same effect on an object as all the vectors that are acting on the object.

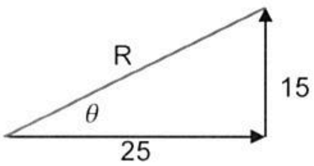
i.e.



Finding the resultant vector of two vectors that are acting in perpendicular directions

1. Move the vertical vector so that it is now at the end of the horizontal vector

2. Draw a line from the start of the horizontal vector to the end of the vertical vector to form a right angled triangle



3. The magnitude of the resultant vector can be found by finding the 'length' of the hypotenuse using Pythagoras' Theorem.

$$R^2 = 25^2 + 15^2 = 850$$

$$R = 29.1 \text{ N}$$

4. The direction of the resultant vector to the horizontal direction can be found by using

$$\boldsymbol{\theta = \tan^{-1}\left(\frac{vertical}{horizontal}\right)}$$

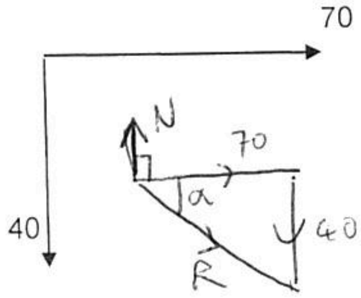
In this case

$$\theta = \tan^{-1}\left(\frac{15}{25}\right) = 31° \text{ to the horizontal}$$

Example 21

Find the magnitude and direction of the resultant vector of the following sets of perpendicular vectors. Give the directions as a bearing.

a)



$R = \sqrt{70^2 + 40^2}$

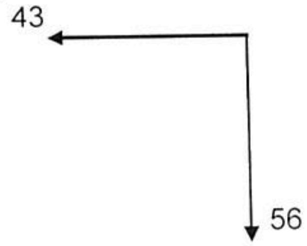
$=$

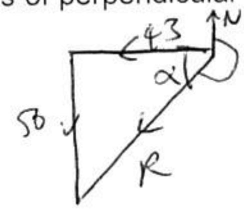
$\alpha = \tan^{-1}\left(\frac{40}{70}\right) =$

$\therefore$ bearing $= 90 + \alpha$

$=$

b)





$R = \sqrt{56^2 + 43^2}$

$=$

$\alpha = \tan^{-1}\left(\frac{56}{43}\right) =$

bearing $= 270 - \alpha$

Finding the resultant vector when the vectors are given in column or base vector form

In this case, the resultant vector is the sum of the individual vectors

Example 22

Find the resultant vector of the following sets of vectors

a) $4\mathbf{i} + 2\mathbf{j}$, $7\mathbf{i} - 3\mathbf{j}$ and $-\mathbf{i} + 13\mathbf{j}$

$\underline{R} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 7 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 13 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$

b) $\begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\underline{R} = \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -4 \end{pmatrix}$

Vector Components

Finding the components of a vector (resolving the vector) is the reverse process of finding the resultant; you are splitting the vector into two parts, which would combine back to form your original vector.

Each part of the original resultant vector is called a **component** of the vector.

Usually vectors are resolved into horizontal and vertical directions.
The exception is when an object is on an inclined plane. In this case we resolve parallel and perpendicular to the plane.

NB as displacement, velocity and acceleration are all vectors, we could resolve them in a similar way.

Rules

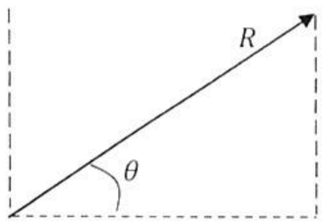
When resolving **into an angle**, multiply the magnitude of the force by the **cosine** of the angle.

$$R \cos \theta$$

When resolving **away from an angle**, multiply the magnitude of the force by the **sine** of the angle.

$$R \sin \theta$$

These quick rules come from considering a right angled triangle and using SOHCAHTOA



This side equivalent to the component when you have resolved away from the angle.

Considering SOHCAHTOA, the side of the triangle is equal to $R \sin \theta$

This side is the component when you have resolved into the angle.

Considering SOHCAHTOA, the side of the triangle is equal to $R \cos \theta$

Resolving vectors is used when considering

- Converting a vector with given magnitude and direction into either column or base vector form
- Finding resultant vectors when there are more than two vectors and they are not given in column or base vector form (such as in a force diagram – see the mechanics sector 1 notes on forces)
- Forces in equilibrium (see the mechanics sector 1 notes on forces)

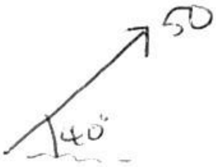
NB

- If the vector is acting **in the direction** that you are resolving in, then the component in that direction is **equal to the vector**
- If the vector is acting **perpendicular to the direction** that you are resolving in, then the component in that direction is **equal to zero**.

- You use the same method when resolving vectors on an inclined plane, except that the directions you are resolving in are **parallel** and **perpendicular** to the plane, rather than horizontally and vertically (see the mechanics sector 1 notes on forces)

Example 23

The vector **p** has a magnitude of 50 and is inclined at an angle of 40° to the x-axis. Find the vector **p**, given that it lies entirely in the x-y plane.



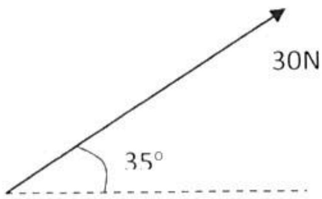
$\underline{i}$ component $(\rightarrow)$ $= 50\cos 40$ $\qquad$ $\underline{j}$ component $(\uparrow)$ $= 50\sin 40$

$\therefore \underline{p} = \begin{pmatrix} 50\cos 40 \\ 50\sin 40 \end{pmatrix}$ or $(50\cos 40)\underline{i} + (50\sin 40)\underline{j}$

Example 24

Find the components of the following forces

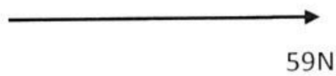
a)



$\rightarrow$ $30\cos 35$

$\uparrow$ $30\sin 35$

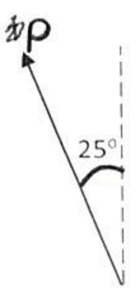
b)



$\rightarrow$ 59

$\uparrow$ 0

c)



$\rightarrow$ $p\sin 25$ to the left

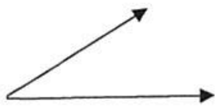
or $-p\sin 25$

$\uparrow$ $p\cos 35$

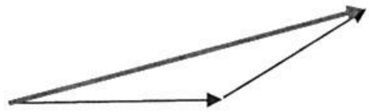
Finding the resultant vector when the vectors under consideration are not perpendicular

There are a number of different methods that can be used to find the resultant vector when the forces you are considering are not perpendicular.

- If there are just two vectors (or three force vectors in equilibrium), you can form a vector triangle by moving one vector so that it's start is at the end of the other vector and then joining the start of the first vector with the end of the second. You can then use the sine and cosine rule to find any unknown quantities.



becomes

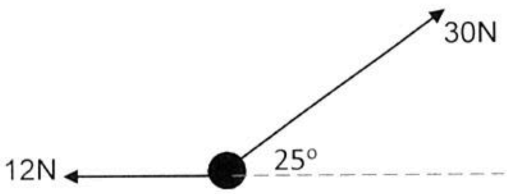


- If there are more than two vectors (or more than three forces in equilibrium), you are expected to resolve the vectors into horizontal and vertical components to find the **resultant horizontal and vertical components**, which can then be used as shown above to find the overall resultant vector.

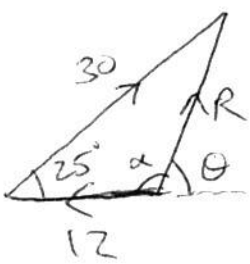
- Note that the latter method also works when there are just two vectors

<u>Example 25</u>

Find the resultant force (form a vector triangle and use the sine or cosine rule)



cosine rule



$$R^2 = 12^2 + 30^2 - 2\times12\times30\cos25$$

$$R = 19.8\text{ N}$$

sine rule

$$\frac{\sin\alpha}{30} = \frac{\sin25}{R}$$

$30\cos25 > 12$

$\therefore$ α is obtuse

$$\alpha = \sin^{-1}\left(\frac{30\sin25}{19.79\ldots}\right) = 39.9^\circ \text{ or } 140^\circ$$

$\rightarrow \alpha = 140^\circ$

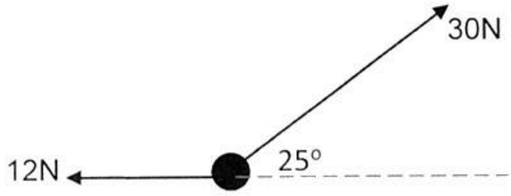
$$\therefore \theta = 180 - \alpha$$

$$= \cancel{140.14}\ldots$$

$$= \cancel{140}^\circ \text{ (3sf)}$$

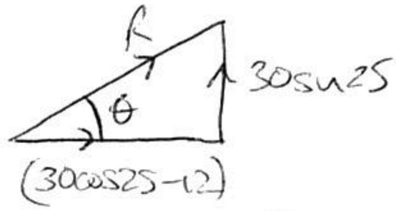
Example 26

Find the resultant force (use the components method)



$\rightarrow \quad 30\cos25 - 12$

$= 15.2...$



$\uparrow \quad 30\sin25$

$= 12.7..$

$R = (30\sin25)^2 + (30\cos25-12)^2$

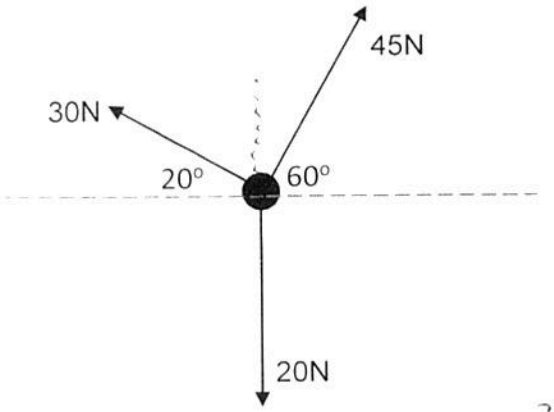
$= 19.8N \quad (3sf)$

$\theta = \tan^{-1}\left(\dfrac{30\sin25}{30\cos25-12}\right) = 39.85...$

$= 39.9° \quad (3sf)$

Example 27

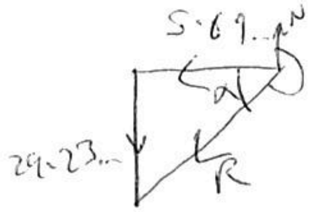
Find the resultant force



$\rightarrow \quad 45\cos60 - 30\cos20 = -5.690...$

5.69 to left.

$\uparrow \quad 45\sin60 + 30\sin20 - 20 = 29.23...$



$R = \sqrt{(29.23..)^2 + (5.69..)^2}$

$= 29.8N \quad (3sf)$

$\alpha = \tan^{-1}\left(\dfrac{29.23..}{5.69..}\right) = 78.98...$

$\alpha = 79.0°$

or bearing $= 270 - \alpha$

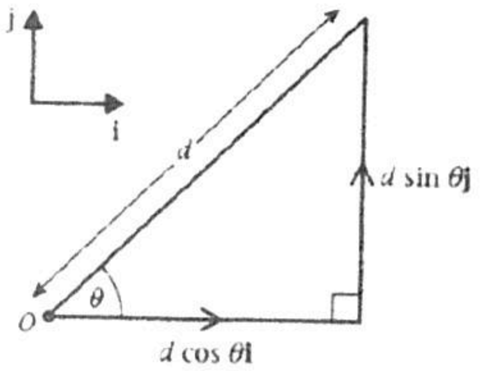
$= 191° \quad (3sf)$

Expressing quantities as vectors

As mentioned, quantities such as displacement, velocity and acceleration are vector quantities.

In one dimension, this is taken into account with a negative sign, as explained in the SUVAT section.

In **two dimensions**, we need to express the quantity either in **i and j form**, or as a **column vector**. We can work out each component by **resolving** the vector quantity in the horizontal and vertical direction, using the angle between the quantity and the horizontal.



$$\boxed{\mathbf{r} = d\cos\theta\,\mathbf{i} + d\sin\theta\,\mathbf{j}}$$

$$\boxed{\mathbf{r} = \begin{pmatrix} d\cos\theta \\ d\sin\theta \end{pmatrix}}$$

NB
Remember that if a **bearing** is used to describe the direction of a vector, the bearing is an angle measured from the **north line** in a **clockwise** direction.

Example 28

An aeroplane is travelling at a speed of 150ms^{-1} on a bearing of 300°. Express the velocity of the aeroplane in the form $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ and as a column vector, where **i** and **j** are unit vectors that are directed east and north, respectively.

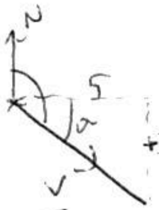
$$\underline{v} = \begin{pmatrix} -150\sin 60 \\ 150\cos 60 \end{pmatrix} \quad \text{or} \quad \underline{v} = \begin{pmatrix} -150\cos 30 \\ 150\sin 30 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} -75\sqrt{3} \\ 75 \end{pmatrix}$$

in $\underline{i}$ and $\underline{j}$ form $\quad \underline{v} = -75\sqrt{3}\,\underline{i} + 75\,\underline{j}$

Example 29

A toy car moves in a straight line with a velocity of $5\mathbf{i} - 3\mathbf{j}$. Find the speed and direction of the toy car, expressing the direction as a bearing.



$= \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Speed $= |\underline{v}|$

$= \sqrt{5^2 + 3^2}$

$= \sqrt{34}$

$\alpha = \tan^{-1}\left(\frac{3}{5}\right)$

$= 30.96\ldots$

$\therefore$ bearing $= 90 + \alpha$

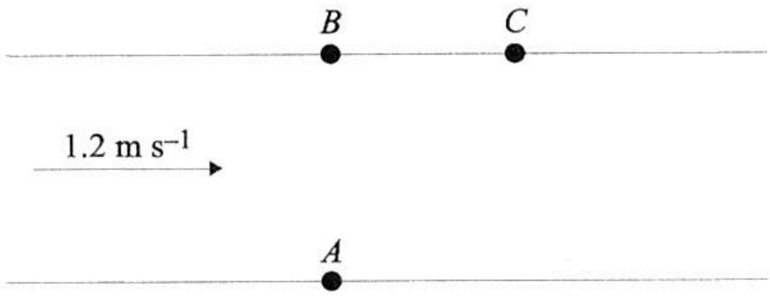
$= 120.96\ldots$

$= 121^\circ$ (3sf)

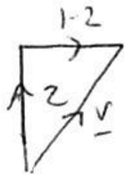
Exam Question

AQA M1 June 2017

3 The water in a river flows at $1.2\ \mathrm{m\ s^{-1}}$ between two parallel banks that are 30 metres apart. The point A is on one bank. The point B is on the other bank directly opposite A, and the point C is downstream from B. The points are shown in the diagram below.



A boat, which moves at $2\ \mathrm{m\ s^{-1}}$ relative to the water, crosses the river.



Model the boat as a particle.

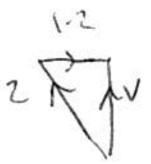
(a) On one crossing the boat travels so that its velocity, relative to the water, is perpendicular to the bank. It travels directly from the point A to the point C. Find the distance between B and C. time taken to cross river $= \frac{30}{2} = 15$s

[3 marks]

$\therefore BC = 15 \times 1.2 = 18$m

(b) On another crossing, the boat heads in a different direction, so that it travels directly from A to B. Find the time it takes the boat to cross from A to B.

[4 marks]



$v = \sqrt{2^2 - 1.2^2}$

$= 1.6\ \mathrm{ms^{-1}}$

time $= \frac{d}{v} = \frac{30}{1.6} = 18.75$s

Vectors in Mechanics – RUVAT

If the object is accelerating, we must use the equations of motion (SUVAT equations) to determine unknown quantities such as displacement and velocity.
These can be applied to vectors as well as numerical values:

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

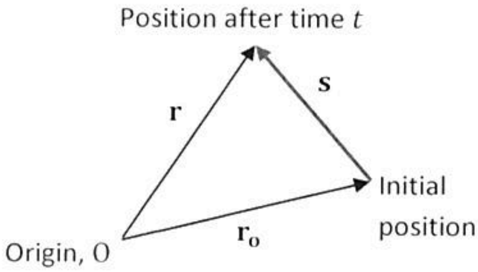
$$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

NB
We **can't use** the last equation ($v^2 = u^2 + 2as$), as multiplying vectors together involves methods with vectors beyond the syllabus.

In many situations, the object will not start at the origin, O. The start and end position relative to the origin, A and B, will be given as **position vectors**.

The displacement of the object is the vector connecting the start and end position.



$\mathbf{r_o}$ is the initial position vector

$\mathbf{r}$ is the position vector after a certain time t

$\mathbf{s}$ is the displacement vector between the initial position and the position after a certain time t

The displacement is linked to the position vectors via the following equations:

$$\mathbf{r} = \mathbf{r_o} + \mathbf{s}$$

or

$$\mathbf{s} = \mathbf{r} - \mathbf{r_o}$$

The displacement vector can also be found using equations of motion:

- If the object is travelling at a **constant velocity**, use

$$\mathbf{s} = \mathbf{v}t$$

- If the object is **accelerating** (or decelerating) then use SUVAT equations with vectors

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

The velocity after a certain time t (if the object is accelerating) can be found using

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

NB

- The speed of an object at a certain time is the magnitude of the velocity vector at that time
- The direction of motion of an object is determined by the velocity vector
- The distance moved by an object is the magnitude of the displacement vector $\mathbf{s}$.
- The distance between an object and the origin is the magnitude of the final position vector $\mathbf{r}$.

Example 30 - RUVAT

A particle has initial velocity $3\mathbf{i} - 5\mathbf{j}$ and acceleration $-\mathbf{i} + 2\mathbf{j}$. It is initially positioned at the origin. Find

a) The velocity of the particle after 5 seconds
b) The position of the particle after this time.

$\underline{u} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \underline{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \underline{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{s} = \underline{r}$

if object initially positioned at origin, the position vector at time t will be the same as the displacement vector

a) $\underline{v} = \underline{u} + \underline{a}t \qquad t = 5$

$\underline{v} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} 5$

$\underline{v} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \text{ms}^{-1}$

b) $\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$=$

Example 31 - RUVAT

A particle is initially at the origin with a velocity of $-\mathbf{i}+2\mathbf{j}$. After 4 seconds, the particle is at the point A, with position vector $4\mathbf{i}-6\mathbf{j}$.

a) Find the acceleration of the particle

b) Find the speed of the particle when it reaches A.

starts at origin

$\underline{r}=\underline{s} \quad \therefore \underline{s}=\begin{pmatrix}4\\-6\end{pmatrix} \qquad u=\begin{pmatrix}-1\\2\end{pmatrix} \qquad t=4$

a) $\underline{s}=\underline{u}t+\frac{1}{2}\underline{a}t^2$

$\begin{pmatrix}4\\-6\end{pmatrix}=4\begin{pmatrix}-1\\2\end{pmatrix}+\frac{1}{2}\underline{a}(4)^2$

$\begin{pmatrix}4\\-6\end{pmatrix}=\begin{pmatrix}-4\\8\end{pmatrix}+8\underline{a}$

$8\underline{a}=\begin{pmatrix}4\\-6\end{pmatrix}-\begin{pmatrix}-4\\8\end{pmatrix}=\begin{pmatrix}8\\-14\end{pmatrix}$

$\therefore \underline{a}=\frac{1}{8}\begin{pmatrix}8\\-14\end{pmatrix}=\begin{pmatrix}1\\-7/4\end{pmatrix}\text{ ms}^{-2}$

b) $\underline{v}=\underline{u}+\underline{a}t$

$=\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}1\\-7/4\end{pmatrix}4$

$\underline{v}=\begin{pmatrix}3\\-5\end{pmatrix}$

Speed $=|\underline{v}|$

$=\sqrt{3^2+(-5)^2}$

$=\sqrt{34}\text{ ms}^{-1}$

$=5.83\text{ ms}^{-1}$ (3sf)

Example 32 – Constant Velocity

A boat is travelling at a constant velocity of $4\mathrm{i}-2\mathrm{j}$ ms^{-1}. Initially, the boat is at a point A, which is at position $-3\mathrm{i}+10\mathrm{j}$ relative to the origin.

Find:

a) The speed of the boat

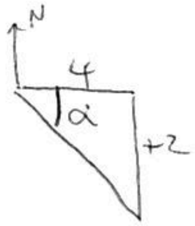
b) The direction of motion that the boat is travelling in, giving your answer as a bearing

c) An expression for the position of the boat after t seconds

d) The distance between the boat and the origin after 4 seconds

a) speed $=|\underline{v}|=\sqrt{4^2+2^2}=2\sqrt{5}\text{ ms}^{-1}$

$=4.47\text{ms}^{-1}$ (3sf)

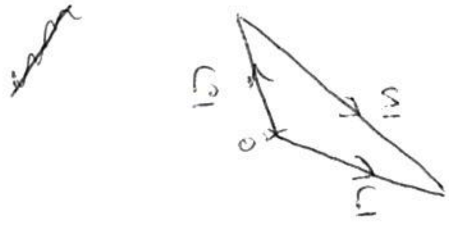
b) 

$\alpha=\tan^{-1}\left(\frac{2}{4}\right)$

$=26.6°$

$\therefore$ bearing $=90+26.6=116.6°=117°$ (3sf)

c) constant velocity $\therefore \underline{s}=\underline{v}t$



$\underline{r}=\underline{r}_0+\underline{s}$

$\underline{r}=\begin{pmatrix}-3\\10\end{pmatrix}+\begin{pmatrix}4\\-2\end{pmatrix}t \qquad \underline{r}=\begin{pmatrix}-3+4t\\10-2t\end{pmatrix}$

d) $t=4 \qquad \underline{r}=\begin{pmatrix}-3+4(4)\\10-2(4)\end{pmatrix}=\begin{pmatrix}13\\2\end{pmatrix}$

distance from origin to boat position $=|\underline{r}|=\sqrt{13^2+2^2}=\sqrt{173}$

$=13.2$m (3sf)

Example 33 – RUVAT

A ball is moving on a horizontal plane at a constant acceleration of $3\mathbf{i}\ \mathrm{ms}^{-2}$. Its initial position and velocity are $\mathbf{i} + 5\mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.

Find:

a) The velocity of the ball after 4 seconds.
b) The position of the ball after 7 seconds.
c) The bearing from the origin to the ball's position after 7 seconds.

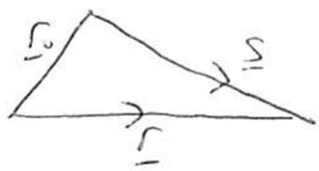
$$\underline{r}_0 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \underline{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad \underline{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

a) $t = 4$

$$\underline{v} = \underline{u} + \underline{a}t$$
$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} 4$$
$$\underline{v} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

b) $t = 7$

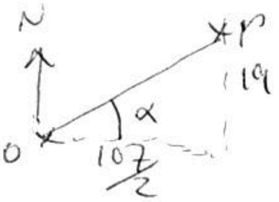
$$\underline{s} = \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$
$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix} 7 + \tfrac{1}{2}\begin{pmatrix} 3 \\ 0 \end{pmatrix} 7^2$$
$$= \begin{pmatrix} -21 \\ 14 \end{pmatrix} + \begin{pmatrix} \frac{147}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{105}{2} \\ 14 \end{pmatrix}$$



position vector

$$\underline{r} = \underline{r}_0 + \underline{s}$$
$$= \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 105/2 \\ 14 \end{pmatrix} = \begin{pmatrix} \frac{107}{2} \\ 19 \end{pmatrix} \text{m}$$

c)



$$\alpha = \tan^{-1}\left(\frac{19}{\left(107/2\right)}\right)$$
$$= 19.55\ldots$$

$$\text{bearing} = 90 - \alpha$$
$$= 70.4^\circ$$

Example 34 – RUVAT

A particle is moving on a horizontal plane. Initially the particle has a velocity of $12\mathbf{i} + 4\mathbf{j}$ and is at a position of $-2\mathbf{i} + 7\mathbf{j}$ relative to the origin . After 5 seconds, the velocity of the particle is $15\mathbf{i} - 3\mathbf{j}$.
Find

a) The acceleration of the particle.
b) An expression of the velocity of the particle after t seconds.
c) An expression for the position of the particle after t seconds.
d) The time at which the direction of motion of the particle is parallel to the vector $\mathbf{i}$

$\underline{r}_0 = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 15 \\ -3 \end{pmatrix}$

$t = 5$

a)

$\underline{v} = \underline{u} + \underline{a}t$

$\begin{pmatrix} 15 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \end{pmatrix} + 5\underline{a}$ $\therefore 5\underline{a} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} 0.6 \\ -1.4 \end{pmatrix} ms^{-2}$

If the direction of motion of the particle is parallel to the vector i (or due east) then the j component of the velocity vector is equal to zero

If the direction of motion of the particle is parallel to the vector j (or due north) then the i component of the velocity vector is equal to zero

b)

$\underline{v} = \underline{u} + \underline{a}t$

$\underline{v} = \begin{pmatrix} 12 \\ 4 \end{pmatrix} + \begin{pmatrix} 0.6 \\ -1.4 \end{pmatrix} t$

$\underline{v} = \begin{pmatrix} 12 + 0.6t \\ 4 - 1.4t \end{pmatrix}$

c)

$\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$= \begin{pmatrix} 12 \\ 4 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0.6 \\ -1.4 \end{pmatrix} t^2$ $\therefore \underline{s} = \begin{pmatrix} 12t + 0.3t^2 \\ 4t - 0.7t^2 \end{pmatrix}$

$\underline{r} = \underline{r}_0 + \underline{s}$ $\therefore \underline{r} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} + \begin{pmatrix} 12t + 0.3t^2 \\ 4t - 0.7t^2 \end{pmatrix}$

$= \begin{pmatrix} -2 + 12t + 0.3t^2 \\ 7 + 4t - 0.7t^2 \end{pmatrix}$

d) direction of motion = direction of $\underline{v}$

// to $\underline{i}$ $\Rightarrow$ $\underline{j}$ component of $\underline{v} = 0$

$4 - 1.4t = 0$

$4 = 1.4t$

$t = \frac{4}{1.4} = \frac{20}{7} s$

$= 2.86 s$ (3sf)

Exam Questions

AQA June 2009

7 A particle moves on a smooth horizontal plane. It is initially at the point A, with position vector $(9\mathbf{i} + 7\mathbf{j})$ m, and has velocity $(-2\mathbf{i} + 2\mathbf{j})\ \mathrm{m\,s^{-1}}$. The particle moves with a constant acceleration of $(0.25\mathbf{i} + 0.3\mathbf{j})\ \mathrm{m\,s^{-2}}$ for 20 seconds until it reaches the point B. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

(a) Find the velocity of the particle at the point B. *(3 marks)*

(b) Find the velocity of the particle when it is travelling due north. *(4 marks)*

(c) Find the position vector of the point B. *(3 marks)*

(d) Find the average velocity of the particle as it moves from A to B. *(2 marks)*

AQA January 2008

8 A Jet Ski is at the origin and is travelling due north at $5\ \mathrm{m\,s^{-1}}$ when it begins to accelerate uniformly. After accelerating for 40 seconds, it is travelling due east at $4\ \mathrm{m\,s^{-1}}$. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

(a) Show that the acceleration of the Jet Ski is $(0.1\mathbf{i} - 0.125\mathbf{j})\ \mathrm{m\,s^{-2}}$. *(4 marks)*

(b) Find the position vector of the Jet Ski at the end of the 40 second period. *(3 marks)*

(c) The Jet Ski is travelling southeast t seconds after it leaves the origin.

(i) Find t. *(5 marks)*

(ii) Find the velocity of the Jet Ski at this time. *(2 marks)*

AQA June 2012

7 A particle moves with a constant acceleration of $(0.1\mathbf{i} - 0.2\mathbf{j})\ \mathrm{m\,s^{-2}}$. It is initially at the origin where it has velocity $(-\mathbf{i} + 3\mathbf{j})\ \mathrm{m\,s^{-1}}$. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

(a) Find an expression for the position vector of the particle t seconds after it has left the origin. *(2 marks)*

(b) Find the time that it takes for the particle to reach the point where it is due east of the origin. *(3 marks)*

(c) Find the speed of the particle when it is travelling south-east. *(6 marks)*

AQA June 2017

7 A jet ski moves on a lake, with an acceleration of $(0.25\mathbf{i} + 1.2\mathbf{j})\ \mathrm{m\ s^{-2}}$. At the point A, the jet ski has velocity $(4\mathbf{i} - 1.6\mathbf{j})\ \mathrm{m\ s^{-1}}$.

The unit vectors **i** and **j** are directed east and north respectively.

(a) Find the speed of the jet ski 2 seconds after it leaves A.

[4 marks]

(b) At the point B, the jet ski has speed $10\ \mathrm{m\ s^{-1}}$. Find the average velocity of the jet ski as it travels from A to B.

[9 marks]

AQA June 2014

7 Two particles, A and B, move on a horizontal surface with constant accelerations of $-0.4\mathbf{i}\ \mathrm{m\ s^{-2}}$ and $0.2\mathbf{j}\ \mathrm{m\ s^{-2}}$ respectively. At time $t = 0$, particle A starts at the origin with velocity $(4\mathbf{i} + 2\mathbf{j})\ \mathrm{m\ s^{-1}}$. At time $t = 0$, particle B starts at the point with position vector $11.2\mathbf{i}$ metres, with velocity $(0.4\mathbf{i} + 0.6\mathbf{j})\ \mathrm{m\ s^{-1}}$.

(a) Find the position vector of A, 10 seconds after it leaves the origin.

[2 marks]

(b) Show that the two particles collide, and find the position vector of the point where they collide.

[9 marks]

AQA New Spec Practice Paper 2

10 Relative to a fixed point, O, a lighthouse has position vector $(30\mathbf{i} + 50\mathbf{j})$ km

At noon, a boat has position vector $(85\mathbf{i} - 45\mathbf{j})$ km

The unit vectors **i** and **j** represent east and north respectively.

10 (a) Find the distance between the boat and the lighthouse at noon.

[2 marks]

10 (b) Find the bearing of the lighthouse from the boat at noon.

[2 marks]

Edexcel New Spec Specimen Paper

6. A particle, P, moves with constant acceleration $(\mathbf{i} - 2\mathbf{j})$ m s^{-2}.

At time t = 0 seconds, the particle is at the point A with position vector $(2\mathbf{i} + 5\mathbf{j})$ m and is moving with velocity $\mathbf{u}$ m s^{-1}.

At time t = 3 seconds, P is at the point B with position vector $(-2.5\mathbf{i} + 8\mathbf{j})$ m.

Find $\mathbf{u}$.

(4)

Solomon Paper E (Edexcel)

7. A particle has an initial velocity of $(\mathbf{i} - 5\mathbf{j})$ m s^{-1} and is accelerating uniformly in the direction $(2\mathbf{i} + \mathbf{j})$ where $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors.

Given that the magnitude of the acceleration is $3\sqrt{5}$ m s^{-2},

(a) show that, after t seconds, the velocity vector of the particle is

$$[(6t + 1)\mathbf{i} + (3t - 5)\mathbf{j}] \quad \text{m s}^{-1}.$$

(6 marks)

(b) Using your answer to part *(a)*, or otherwise, find the value of t for which the speed of the particle is at its minimum.

(5 marks)

Solomon Paper G (Edexcel)

5. Two dogs, Fido and Growler, are playing in a field. Fido is moving in a straight line so that at time t his position vector relative to a fixed origin, O, is given by $[(2t - 3)\mathbf{i} + t\mathbf{j}]$ metres. Growler is stationary at the point with position vector $(2\mathbf{i} + 5\mathbf{j})$ metres, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal perpendicular unit vectors.

(a) Find the displacement vector of Fido from Growler in terms of t. **(2 marks)**

(b) Find the value of t for which the two dogs are closest. **(6 marks)**

(c) Find the minimum distance between the two dogs. **(3 marks)**

AQA June 2009 M1

a) $\underline{u} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ $\quad \underline{a} = \begin{pmatrix} 0.25 \\ 0.3 \end{pmatrix}$ $\quad t = 20$

$$\underline{v} = \underline{u} + \underline{a}t = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 0.3 \end{pmatrix} 20$$

$$\underline{v} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \text{ ms}^{-1}$$

b) travelling due north $\rightarrow$ $\underline{i}$ component of $\underline{v} = 0$

$$\underline{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 0.3 \end{pmatrix} t = \begin{pmatrix} -2 + 0.25t \\ 2 + 0.3t \end{pmatrix}$$

$$\therefore \; -2 + 0.25t = 0 \qquad t = 8\text{s}$$

$$\underline{v} = \begin{pmatrix} -2 + 0.25(8) \\ 2 + 0.3(8) \end{pmatrix} = \begin{pmatrix} 0 \\ 4.4 \end{pmatrix} \text{ ms}^{-1}$$

c) $\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$$\underline{s} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} 20 + \frac{1}{2}\begin{pmatrix} 0.25 \\ 0.3 \end{pmatrix} \times 20^2$$

$$= \begin{pmatrix} 10 \\ 100 \end{pmatrix}$$

$\therefore$ position vector

$$\underline{r} = \underline{r}_0 + \underline{s} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} + \begin{pmatrix} 10 \\ 100 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 19 \\ 107 \end{pmatrix} \text{ m}$$

d) $$\underline{v}_{av.} = \frac{\text{total } \underline{s}}{\text{total time}} = \frac{\begin{pmatrix} 10 \\ 100 \end{pmatrix}}{20} \quad \text{or} \quad \frac{1}{20}\begin{pmatrix} 10 \\ 100 \end{pmatrix}$$

$$\underline{v}_{av.} = \begin{pmatrix} 0.5 \\ 5 \end{pmatrix} \text{ ms}^{-1}$$

AQA January 2008 M1

8 a) travelling due north initially at $5\,ms^{-1}$ $\rightarrow$ $\underline{i}$ component of $\underline{u} = 0$

$$\therefore \quad \underline{u} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$t = 40$ travelling due east at $4\,ms^{-1}$ $\rightarrow$ $\underline{j}$ component of $\underline{v} = 0$

$$\therefore \underline{v} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\underline{v} = \underline{u} + \underline{a}t$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + 40\underline{a} \quad \longrightarrow \quad 40\underline{a} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$40\underline{a} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

$$\therefore \underline{a} = \frac{1}{40}\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.125 \end{pmatrix}$$

$$= 0.1\underline{i} - 0.125\underline{j} \ ms^{-2}$$

as required

b) $\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$$= \begin{pmatrix} 0 \\ 5 \end{pmatrix} 40 + \frac{1}{2}\begin{pmatrix} 0.1 \\ -0.125 \end{pmatrix} 40^2$$

$$= \begin{pmatrix} 80 \\ 100 \end{pmatrix}$$

Starts at origin

$\rightarrow \underline{s} = \underline{r}$

$\therefore$ position vector $\underline{r} = \begin{pmatrix} 80 \\ 100 \end{pmatrix}$

c) i) travelling Southeast

$$\underline{v} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 0.1 \\ -0.125 \end{pmatrix} t \qquad \underline{v} = \begin{pmatrix} 0.1t \\ 5 - 0.125t \end{pmatrix}$$

$\downarrow$ for $\underline{v}$

$\underline{i}$ component $= -\underline{j}$ component

$$0.1t = -(5 - 0.125t)$$

$$0.1t = -5 + 0.125t$$

$$5 = 0.025t \qquad \therefore t = 200\,s$$

ii) $\underline{v} = \begin{pmatrix} 0.1 \times 200 \\ 5 - 0.125(200) \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \end{pmatrix} ms^{-1}$

AQA June 2012 M1

$\underline{a} = \begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} \quad \underline{u} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

initially at origin

$\therefore \underline{r} = \underline{s}$

a) $\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} t^2$

$= \begin{pmatrix} -t + 0.05t^2 \\ 3t \quad 0.1t^2 \end{pmatrix}$

$\therefore \underline{r} = \begin{pmatrix} -t + 0.05t^2 \\ 3t - 0.1t^2 \end{pmatrix}$

b) particle's <u>position</u> is due east of origin

$\rightarrow$ $\underline{j}$ component of $\underline{r}$ is zero (and $\underline{i}$ component is +ve)

$3t - 0.1t^2 = 0$

$t(3 - 0.1t) = 0$

$t = 0$

$3 - 0.1t = 0$

$0.1t = 3$

$t = 30s$

c) particle travelling south-east

$\underline{v} = \underline{u} + \underline{a}t$

$\underline{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix} t$

$\underline{v} = \begin{pmatrix} -1 + 0.1t \\ 3 - 0.2t \end{pmatrix}$

SE $\rightarrow$ $\underline{i}$ component of $\underline{v}$ $= -\underline{j}$ component of $\underline{v}$

$-1 + 0.1t = 3 - 0.2t$

$0.3t = 4$

$t = \frac{40}{3}s = 13.3s$

AQA June 2017 M1

$\underline{a} = \begin{pmatrix} 0.25 \\ 1.2 \end{pmatrix} \qquad \underline{u} = \begin{pmatrix} 4 \\ -1.6 \end{pmatrix}$

a) $t = 2$

$$\underline{v} = \underline{u} + \underline{a}t$$
$$= \begin{pmatrix} 4 \\ -1.6 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 1.2 \end{pmatrix} 2$$
$$= \begin{pmatrix} 4 \\ -1.6 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 2.4 \end{pmatrix}$$
$$= \begin{pmatrix} 4.5 \\ 0.8 \end{pmatrix}$$

$\therefore$ speed $= |\underline{v}|$
$$= \sqrt{4.5^2 + 0.8^2}$$
$$= 4.57 \text{ms}^{-1}$$

b)

$$\underline{v} = \underline{u} + \underline{a}t$$
$$\underline{v} = \begin{pmatrix} 4 \\ -1.6 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 1.2 \end{pmatrix} t$$
$$\underline{v} = \begin{pmatrix} 4 + 0.25t \\ -1.6 + 1.2t \end{pmatrix}$$

speed $= 10$ at B

$\therefore |\underline{v}| = 10$

$$\sqrt{(4 + 0.25t)^2 + (-1.6 + 1.2t)^2} = 10$$
$$16 + 2t + \frac{1}{16}t^2 + \frac{36}{25}t^2 - \frac{96}{25}t + \frac{64}{25} = 100$$

$\times 400$
$$6400 + 800t + 25t^2 + 416t^2 - 1536t + 1024 = 40000$$
$$441t^2 - 736t - 32576 = 0$$
$$t = \frac{736 \pm \sqrt{(-736)^2 - 4 \times -32576 \times 441}}{2 \times 441}$$

$t = 9.47\text{s}$ or -7.80
ignore as $t \geq 0$

$t = 9.47\text{s}$

$$\underline{v}_B = \begin{pmatrix} 4 + 0.75(9.469\ldots) \\ -1.6 + 1.2(9.469\ldots) \end{pmatrix} = \begin{matrix} 6.3673\ldots \\ 9.7634\ldots \end{matrix} = \begin{pmatrix} 6.37 \\ 9.76 \end{pmatrix}$$

~~av vel = (6.37, 9.76)~~

$$\underline{s} = \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$
$$= \begin{pmatrix} 4 \\ -1.6 \end{pmatrix} 9.469\ldots + \frac{1}{2}\begin{pmatrix} 0.25 \\ 1.2 \end{pmatrix}(9.469\ldots)^2$$
$$= \begin{pmatrix} 49.087\ldots \\ 38.652\ldots \end{pmatrix}$$

$$\therefore \text{av vel} = \frac{\underline{s}}{t} = \begin{pmatrix} \frac{49.087}{9.469\ldots} \\ \frac{38.652\ldots}{9.469} \end{pmatrix} = \begin{pmatrix} 5.18 \\ 4.08 \end{pmatrix} \text{ms}^{-1}$$

AQA June 2014 M1

a) $\underline{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\underline{a} = \begin{pmatrix} -0.4 \\ 0 \end{pmatrix}$ $t = 10$

A starts at origin $\rightarrow \underline{r} = \underline{s}$

$\therefore \underline{r} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\underline{r} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 10 + \frac{1}{2}\begin{pmatrix} -0.4 \\ 0 \end{pmatrix} 10^2$

$= \begin{pmatrix} 40 \\ 20 \end{pmatrix} + \begin{pmatrix} -20 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$

b)

If A and B collide, $\underline{r}_A = \underline{r}_B$ at same time t

$\underline{r}_A = \begin{pmatrix} 4 \\ 2 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} -0.4 \\ 0 \end{pmatrix} t^2$

$\underline{r}_A = \begin{pmatrix} 4t - 0.2t^2 \\ 2t \end{pmatrix}$

$\underline{r}_B = \underline{r}_0 + \underline{s}$ for B, $\underline{r}_0 = \begin{pmatrix} 11.2 \\ 0 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$ $\underline{a} = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$

$\underline{r}_B = \begin{pmatrix} 11.2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ 0.2 \end{pmatrix} t^2$

$\underline{r}_B = \begin{pmatrix} 11.2 + 0.4t \\ 0.6t + 0.1t^2 \end{pmatrix}$

$\underline{r}_A = \underline{r}_B$

$\begin{pmatrix} 4t - 0.2t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} 11.2 + 0.4t \\ 0.6t + 0.1t^2 \end{pmatrix}$

$4t - 0.2t^2 = 11.2 + 0.4t$ and $2t = 0.6t + 0.1t^2$

$0 = 0.2t^2 - 3.6t + 11.2$ $0 = 0.1t^2 - 1.4t$

$\times 5$ $0 = t^2 - 18t + 56$ $\times 10$ $0 = t^2 - 14t$

$= (t - 14)(t - 4)$ $= t(t - 14)$

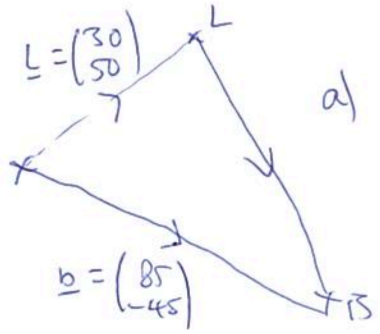
$\underline{t = 14}$ $t = 4$ $t = 0$ $\underline{t = 14}$

A and B have same position vector when $t = 14$ $\rightarrow$ A + B will collide at $t = 14$

$t = 14$ $\underline{r} = \begin{pmatrix} 4(14) - 0.2(14)^2 \\ 2(14) \end{pmatrix}$

$= \begin{pmatrix} 16.8 \\ 28 \end{pmatrix}$

AQA new spec practice paper 2



a) $\overrightarrow{LB} = \underline{b} - \underline{l}$

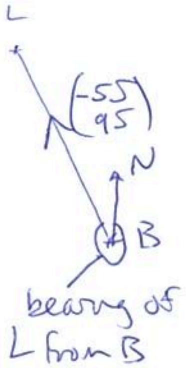
$= \begin{pmatrix}85\\-45\end{pmatrix} - \begin{pmatrix}30\\50\end{pmatrix} = \begin{pmatrix}55\\-95\end{pmatrix}$

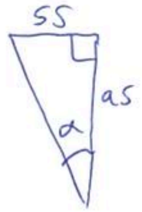
distance $= \sqrt{55^2 + 95^2}$

$= 109.772\ldots$

$= 110\text{ km}$

b)





$\alpha = \tan^{-1}\left(\frac{55}{95}\right) = 30.1\ldots$

$\therefore$ bearing $= 360 - \alpha$

$= 329.93\ldots$

$\therefore$ bearing $= 330°$ (3sf)

Edexcel New Spec Specimen Paper

$$\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \underline{u} = ? \quad \underline{r}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$t = 3 \quad \underline{r} = \begin{pmatrix} -2.5 \\ 8 \end{pmatrix}$$

$$\underline{s} = \underline{r} - \underline{r}_0$$
$$= \begin{pmatrix} -2.5 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$\underline{s} = \begin{pmatrix} -4.5 \\ 3 \end{pmatrix}$$

$$\underline{s} = \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$
$$\begin{pmatrix} -4.5 \\ 3 \end{pmatrix} = 3\underline{u} + \tfrac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix}3^2$$
$$\begin{pmatrix} -4.5 \\ 3 \end{pmatrix} = 3\underline{u} + \begin{pmatrix} 9/2 \\ -9 \end{pmatrix}$$
$$3\underline{u} = \begin{pmatrix} -4.5 \\ 3 \end{pmatrix} - \begin{pmatrix} 4.5 \\ -9 \end{pmatrix}$$
$$3\underline{u} = \begin{pmatrix} -9 \\ 12 \end{pmatrix}$$
$$\underline{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \text{ ms}^{-1}$$

Solomon Paper E (Edexcel)

a) $\underline{u} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ $\qquad$ $\underline{d}$ of $\underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{a}$ is in direction $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad |\underline{d}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

unit vector of direction vector $\hat{\underline{d}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ← this has magnitude = 1

$$\therefore \underline{a} = \hat{\underline{d}} \times |\underline{a}|$$
$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \times 3\sqrt{5}$$
$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\underline{v} = \underline{u} + \underline{a}t$$
$$= \begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix}t = \begin{pmatrix} 1+6t \\ -5+3t \end{pmatrix} = (1+6t)\underline{i} + (-5+3t)\underline{j}$$
$$= (6t+1)\underline{i} + (3t-5)\underline{j}$$ as required

b) min speed

$$\text{speed} = \sqrt{(6t+1)^2 + (3t-5)^2}$$
$$= \sqrt{36t^2 + 12t + 1 + 9t^2 - 30t + 25}$$
$$= \sqrt{45t^2 - 18t + 26}$$

min speed when quad for t is min

method 1 → 1st derivative of quad = 0

1st derivative $= 90t - 18$

$$90t - 18 = 0$$
$$90t = 18$$
$$t = 0.2\text{s}$$

Method 2 → complete the square

$$45(t^2 - 0.4t) + 26$$
$$45\left((t-0.2)^2 - \tfrac{1}{25}\right) + 26$$
$$45(t-0.2)^2 - \tfrac{9}{5} + 26$$
$$45(t-0.2)^2 + \tfrac{121}{5}$$

$t = 0.2$ at min point

Solomon Paper G (Edexcel)

a) displacement vector $\overrightarrow{GF} = \begin{pmatrix} 2t-3 \\ t \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$\overrightarrow{GF} = \begin{pmatrix} 2t-5 \\ t-5 \end{pmatrix}$$

b) closest when $|\overrightarrow{GF}|$ is min.

$$|\overrightarrow{GF}| = \sqrt{(2t-5)^2 + (t-5)^2}$$

$$= \sqrt{4t^2 - 20t + 25 + t^2 - 10t + 25}$$

$$= \sqrt{5t^2 - 30t + 50}$$

$|\overrightarrow{GF}|$ is min when quad is min

$$5t^2 - 30t + 50$$

1st derivative = 0

$5t^2 - 30t + 50$

1st derivative $= 10t - 30$

$10t - 30 = 0$

$10t = 30$

$t = 3s$

or

complete the square

$5(t^2 - 6t + 10)$

$5((t-3)^2 - 9 + 10)$

$5(t-3)^2 + 5$

$\rightarrow t = 3$ at min.

c) $\therefore$ min distance $= |\overrightarrow{GF}|$ at $t=3$

$$= \sqrt{5(3)^2 - 30(3) + 50}$$

$$= \sqrt{5}$$

$$= 2.24m$$