4

Einstein–Cartan Theory : Retrospective and Novelties

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1. PANORAMIC ASCENT : BIRD'S EYE VIEW

A. Basic Rudiments

The Einstein–Cartan theory is but one of the generalizations of Einstein's general relativity [36, pp. 844-845][37, § 4] by means of spaces that are defined with an affine connection elaborated by É. Cartan [9][11] [14] = [16, pp. 23-193] [12] [15] starting from 1923 but whose root is found [9, p. 325] in H. Weyl [129, § 14. *Affin zusammenhängende Mannigfaltigkeit*] = [130, § 14. *Affinely Connected Manifolds*].

Below are some outlines.

Affine Connection, and Parallel Transport of the Levi-Civita Connection

Before talking about Cartan, let us say what an affine connection is, according to the canons of Riemann. An *affine connection*, in the context of differential (pseudo-)Riemannian geometry, is understood as being an operation on a smooth manifold connecting spaces that are tangent. The first consequence of this operation is the possibility of differentiating

· tangent vector fields, d.v.s. linear constructions belonging to a generally Euclidean space in which each point corresponds to a vector,

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· any section of vector bundles, more distinctly called *fiber bundles* in topology, which are objects linked to vector spaces, whereas a section is simply a continuous map of a vector/fiber bundle.

From the voice of Cartan [12, pp. 205-206] :

> It is the notion of parallelism that gives a Euclidean connection to the surface, to quote the words of H. Weyl [. . .]. In fact, what is essential in the idea of Levi-Civita [78] is that it allows to connect [*raccorder*] two small pieces of a manifold, which are infinitely close to each other, and it is this idea of *connection* that is fruitful. We can therefore imagine, by developing this idea, the possibility of arriving at a general theory of manifolds with an *affine*, *conformal*, or *projective* connection.

Cartan refers to the parallel transport of the Levi-Civita connection. What is this about, exactly? The clearest definition comes from Levi-Civita himself [78, p. 175] :

> The parallel transport, along any path, of two concurrent directions preserves their angle. It clearly means that the angle formed by two generic directions through the same point is also the angle formed by their parallels through another point. [1]

The notion of parallel transport is firmly related to that of affine connection. Indeed it is reasonable to mathematically manage the parallel transport rules starting from the concept of connection, and vice versa. This is because the Levi-Civita connection, in (pseudo-)Riemannian geometry, is the distinctive *affine and torsion free connection* on the collection of the tangent spaces—called *tangent bundle*—for each point on a manifold. The specificity of Levi-Civita connection consists in preserving the (pseudo-)Riemannian metric.

This happens thanks to a simple but ingenious procedure. It starts with an infinitesimal field, having the aim of depicting the parallel transport of two directions through two very close points, that is, from one point to an infinitely close point. (The angle between two tangents to a manifold at a point is equal to the angle between two parallel tangents at another point very close to the first).

1. Original It. version : «Il trasporto per parallelismo, lungo un cammino qualsiasi, di due direzioni concorrenti ne conserva l'angolo. Con ciò si vuol dire evidentemente che l'angolo formato da due generiche direzioni uscenti da un medesimo punto è anche l'angolo formato dalle loro parallele in un altro punto qualunque».

In mathematical language, the Levi-Civita theorem can be uttered like this :

if $(\mathcal{M}, g)$ is a (pseudo-)Riemannian manifold, there is a unique affine connection ∇ for $\mathcal{M}$ which is symmetric and compatible with the metric g such that

$$\nabla g = 0, \tag{4.1}$$

das heißt, the torsion of ∇ vanishes identically, and

$$\begin{aligned} \langle \nabla_{\vec{X}} \vec{Y}, \vec{Z} \rangle = & \\ & \frac{1}{2} \Big\{ \vec{X} \langle \vec{Y}, \vec{Z} \rangle + \vec{Y} \langle \vec{Z}, \vec{X} \rangle - \\ & \vec{Z} \langle \vec{X}, \vec{Y} \rangle + \langle [\vec{X}, \vec{Y}], \vec{Z} \rangle - \\ & \langle [\vec{Y}, \vec{Z}], \vec{X} \rangle + \langle [\vec{Z}, \vec{X}], \vec{Y} \rangle \Big\}, \end{aligned} \tag{4.2}$$

for each vector fields $\vec{X}, \vec{Y}, \vec{Z} \in \mathfrak{X}(\mathcal{M})$, where $[\cdot\ ,\cdot]$ is the Lie bracket of vector fields, and $\mathfrak{X}(\mathcal{M})$ is the vector space of vector fields on $\mathcal{M}$.

Relevant observations. Behind Eq. (4.1) there is the free torsional distinctiveness of the Civitian invention. Let τ be the torsion $\binom{1}{2}$-tensor field, or tensor of type $(1, 2)$. If

$$\tau \in \mathfrak{X}^{\nabla}(\mathcal{M}) = 0, \tag{4.3}$$

the connection ∇ is *torsion free*, and par conséquent *symmetric*. (Put otherwise, the symmetry of the connection relies on the condition in which τ vanishes identically, and ∇ is torsion free).

Gravitational Suggestions

A pivotal element is provided by the fact that Levi-Civita's mathematical solutions for his parallelism arise from a revision of Einstein's theory of gravitation, which finds its algorithmic foundations in tensor calculus (or absolute differential calculus), created by G. Ricci Curbastro [110] [111] [112] [113] [114], and re-elaborated by T. Levi-Civita [117] [79] (Section A.). This is a brilliant extension of vector calculus that takes advantage of the Riemannian metric and the Christoffel symbols [22]. See Levi-Civita [79, p. 5] = [80, p. vii] :

> Riemann's general metric and a formula of Christoffel constitute the premises of the absolute differential calculus. Its development as a systematic branch of

mathematics was a later process, the credit for which is due to Ricci, who during the ten years 1887-1896 elaborated the theory and worked out the elegant and comprehensive notation which enables it to be easily adapted to a wide variety of questions of analysis, geometry, and physics.

As everybody knows, within the general relativistic theory, Einstein employs the curvature of a 4-dimensional manifold (space-time), the geometrical structure of which is dependent on the physical phenomena taking place in it. Levi-Civita [78, p. 173] highlights it well :

> Einstein's theory of relativity [. . .] considers the geometrical structure of space as very tenuously, but also intimately, dependent on the physical phenomena taking place in it ; differently from classical theories, which assume the whole physical space as given a priori. The mathematical development of Einstein's grandiose conception (which finds in Ricci's absolute differential calculus its natural algorithmic tool) draws on the curvature of a certain 4-dimensional manifold as an essential element and the related Riemann symbols. Meeting these symbols, or continuously using them [. . .] led me to investigate whether it would be possible to somewhat reduce the formal apparatus, which serves commonly to introduce them and to establish their covariant behaviour.

A Trick to Reduce Curved Space into Small Euclidean-like Spaces

The parallel transport of the Levi-Civita connection is a method for comparing tangent spaces at different points on a manifold with its own metric ; and it is an analytic approach for examining a Riemannian space [118] as an infinity of small pieces of Euclidean space. Against this backdrop, a Riemannian space is a set that locally resembles Euclidean space ; thereby a manifold with a metric revealing a certain curvature, for every neighborhood of each of its points, looks like—and can be treated as—a tessellation of arbitrarily small flat spaces.

This is how Cartan [10, p. 297] puts it :

> Levi-Civita [78], with his definition of parallelism, was the first to succeed in making the *false* metric *spaces* of Riemann, if not true Euclidean spaces, which is impossible, at least *spaces with a Euclidean connection*, considered as collections of small pieces of Euclidean space, *oriented with respect to each other in going from point to point.* [2]

2. Original Fr. version : « C'est M. Levi-Civita qui le premier, par sa définition du parallélisme, réussit à faire des *faux espaces* métriques de Riemann, non pas de vrais espaces

In [13, p. 2] he repeats it even more tersely :

> A Riemannian space is ultimately formed by an infinity of small pieces of Euclidean spaces. [3]

Cartan's Contribution

If general relativity is inserted into (pseudo-)Riemannian geometry, the modification made by Cartan to this theory lies mainly in the use of Riemann–Cartan geometry (Section B.). Mathematically, and physico-mathematically, this involves

(1) the intrusion of *torsion (tensor)* associated with the notion of spin (Sections B., C.), which is absent in the Einsteinian model,

(2) the replacement of the Einstein–Hilbert (gravitational) action (4.66) describing the Einstein (or better, Einstein–Levi-Civita) field equations (4.4) by way of the stationary-action principle, with the *Palatini variational principle*, [4] as shown in Eq. (4.70),

(3) the assumption of *Lorentz symmetry as a local gauge symmetry* (Section 4.), and

(4) the introduction of the *spinor into a 4-dimensional curved manifold* (Section B.) ; on the construct of spinors, I refer to my [94].

As regards point (1), there is the elaboration of a further rumination. As an element pertaining to gravity, the torsion tensor can be extended to every fiber in the fabric of space-time, detectable however only at very high gravitational forces : for this reason, over the years, the Einstein–Cartan theory was applied to the dynamics of black holes (Sections D., E.).

2. DESCENT THEMES

In the second part, the topics outlined above will be covered, but also the news and possible progress in Einstein–Cartan theory.

euclidiens, ce qui est impossible, mais du moins des *espaces à connexion euclidienne*, considérés comme des collections de petits morceaux d'espaces euclidiens, *orientés de proche en proche les uns par rapport aux autres* ».

3. Original Fr. version : « [U]n espace de Riemann est, au fond, formé d'une infinité de petits morceaux d'espaces euclidiens ».

4. A. Palatini was a student of Ricci Curbastro and Levi-Civita.

Some points saillants of the theory under examination will be autonomously revisited, focusing wholly on the mathematical aspect, c.-à-d. underscoring the differential geometry underlying the theory under examination, without the burden of «sensible experiences» (experiments) of Galilean heritage. The Pisan talks [48, p. 38] about «[. . .] esperienze sensate», and [48, p. 325] «[. . .] sensate esperienze».

More precisely, we will show that it is possible to derive an Einsteinian-like gravitational field from a Cartan $\mathfrak{h}$-subalgebra, and thence create a couple of formulæ for a torsioning in a $(1 + 3)$-dimensional manifold. Some Cartan k-forms and $\mathcal{J}$-bundles, along with other Clifford bundles, and a Clifford k-form field, will help to encompass a 4D torsional spin-space. Follows an overview of quantum Yang–Mills gravity according to a geometro-topological schema. This opens up the exciting issue, not addressed here, of the emergence of space-time, indicating a manifolded-structure including its spin plus torsional foundations.

3. PRELUDE

This Section serves as a preparatory introduction to the most innovative part of this article.

A. Einstein Field Equations : betwixt Geometry (Space-Time) and Physics (Matter)

The general relativity represents the core of union between geometry and physics, viz. between space-time and matter, within the Ricci calculus for tensors (and tensor fields) ; its formulation comes out through the Einstein field equations :

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\mathrm{s}} = \varkappa \mathrm{T}_{\mu\nu}{}^{5} \tag{4.4a}$$

$$= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\mathrm{s}} = \frac{8\pi G_{\mathrm{N}}}{c^4} \mathrm{T}_{\mu\nu}, \tag{4.4b}$$

$$= R_{\mu\nu} = \varkappa \left(\mathrm{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathrm{T} \right), \tag{4.4c}$$

$$= -\frac{\partial}{\partial_{x_\xi}} \begin{Bmatrix} \mu\nu \\ \xi \end{Bmatrix} + \begin{Bmatrix} \mu\xi \\ \varrho \end{Bmatrix} \begin{Bmatrix} \nu\varrho \\ \xi \end{Bmatrix} + \frac{\partial^2 \log \sqrt{-g}}{\partial_{x_\mu} \partial_{x_\nu}} - \begin{Bmatrix} \mu\nu \\ \xi \end{Bmatrix} \frac{\partial \log \sqrt{-g}}{\partial_{x_\xi}}. \tag{4.4d}$$

Let us look at some specifics.

(1) $G_{\mu\nu} \overset{\text{viz}}{=} G_{[\mu\nu]} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\text{s}}$, $G_{[\mu\nu]} = G_{\mu\nu=\nu\mu}$, is the Einstein tensor, nay, the Ricci–Einstein tensor. [6]

(2) $R_{\mu\nu}$ is the Ricci curvature tensor, a symmetric tensor of rank 2, which can be described in four basic ways :

(i) the first form is

$$\text{Ric} \overset{\text{ιδ}}{=} \begin{cases} R_\nu{}^\mu \vec{E}_\mu \otimes \vartheta^\nu \text{ as a } \binom{1}{1}\text{-tensor}, & (4.5a) \\ R_{\nu\xi}\vartheta^\nu \otimes \vartheta^\xi = g_{\nu\mu}R_\xi{}^\mu \vartheta^\nu \otimes \vartheta^\xi \text{ as a } \binom{0}{2}\text{-tensor}, & (4.5b) \\ R^{\mu\xi}\vec{E}_\mu \otimes \vec{E}_\xi = g^{\mu\nu}\vec{E}_\mu \otimes \vec{E}_\xi \text{ as a } \binom{2}{0}\text{-tensor}, & (4.5c) \end{cases}$$

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$$\text{Ric} \in \begin{cases} \mathrm{T}^1_1(\mathcal{M}), & (4.6a) \\ \mathrm{T}^0_2(\mathcal{M}), & (4.6b) \\ \mathrm{T}^2_0(\mathcal{M}); & (4.6c) \end{cases}$$

(ii) the Ricci tensor is congruent with a contraction of the Riemann curvature tensor, so

$$\text{Ric} \overset{\text{ιδ}}{=} \begin{cases} R_\nu{}^\mu \vec{E}_\mu \otimes \vartheta^\nu = R_{\mu\xi}{}^{\xi\nu}\vec{E}_\mu \otimes \vartheta^\nu = R_{\mu\xi\varsigma}{}^\nu g^{\varsigma\xi}\vec{E}_\mu \otimes \vartheta^\nu, & (4.7a) \\ R_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu = g^{\xi\varrho}R_{\mu\xi\varrho\nu}\vartheta^\mu \otimes \vartheta^\nu; & (4.7b) \end{cases}$$

(iii) via Christoffel symbols, one has an explicit solution,

$$\text{Ric} \overset{\text{ιδ}}{=} R_{\mu\nu} = \partial_\xi \mathit{\Gamma}_{\mu\nu}{}^\xi - \partial_\nu \mathit{\Gamma}_{\mu\xi}{}^\xi + \mathit{\Gamma}_{\mu\nu}{}^\xi \mathit{\Gamma}_{\xi\varrho}{}^\varrho - \mathit{\Gamma}_{\mu\xi}{}^\varrho \mathit{\Gamma}_{\nu\varrho}{}^\xi; \qquad (4.8)$$

(iv) as a $\binom{0}{2}$-tensor, the Ricci curvature is conditioned by the trace of a linear operator, so

$$\text{Ric}(\vec{X}, \vec{Y}) = \text{tr}\left(\vec{Z} \mapsto R_{\vec{X},\vec{Z}}\vec{Y}\right), \qquad (4.9)$$

which is why Ric is understandable in terms of a trace of the Riemann curvature tensor.

(3) $g_{\mu\nu}$ is the metric tensor.

5. With the addition of the cosmological constant, denoted by Λ, or by λ in [37, p. 151], $G_{\mu\nu} - \lambda g_{\mu\nu} = -\varkappa\left(\mathrm{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{T}\right)$, one gets $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\text{s}} + \Lambda g_{\mu\nu} = \varkappa \mathrm{T}_{\mu\nu}$.

6. See [25, p. 157].

(4) R_s is the scalar curvature, aka Ricci scalar [115] [116]; it is the trace of the Ricci curvature tensor, with regard to the Riemannian metric g (R_s is a local invariants of g),

$$R_\mathrm{s} \in \mathscr{C}^\infty(\mathcal{M}) \overset{\text{ιδ}}{=} \begin{cases} \mathrm{tr}(\mathrm{Ric}), & (4.10a) \\ R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} = R_\nu{}^\nu, & (4.10b) \\ g^{\mu\xi} g^{\nu\varrho} R_{\mu\nu\xi\varrho}. & (4.10c) \end{cases}$$

(5) $\varkappa$ is the Einstein gravitational constant [35],

$$\varkappa = \frac{8\pi G}{c^4}, \tag{4.11}$$

the strength of coupling between matter, or physical dimension, and geometric space.

(6) $\mathrm{T}_{\mu\nu}$ is the energy-momentum tensor,

$$\mathrm{T}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\Big[\sqrt{-g}\mathscr{L} = \Big(\sqrt{-g}\,\big(\frac{1}{2} g^{\mu\nu} \partial_\mu \digamma \partial_\nu \digamma - U\big)\Big)\Big]}{\delta g^{\mu\nu}}, \tag{4.12}$$

designating a quantity for the density of matter, which represents a disturbance in space-time, where

$$\mathscr{L} \overset{\text{viz}}{=} \mathscr{L}_\mathrm{m} = \frac{1}{2} \partial_\mu \digamma \partial^\mu \digamma - U(\digamma), \ \ \partial^\mu \digamma = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \digamma)}, \tag{4.13}$$

is the Lagrangian density of the matter-energy, with a scalar field $\digamma$ and a potential $U(\digamma)$.

(7) G_N is the Newtonian constant of gravitation, from $F = G_\mathrm{N} \frac{m_1 m_2}{æ^2 (m_1 - m_2)}$.

(8) $\left\{ {\mu\nu \atop \xi} \right\}$ etc. are the Christoffel symbols of the second kind [22],

$$\Gamma^\xi{}_{\mu\nu} \overset{\text{viz}}{=} \left\{ {\xi \atop \mu\nu} \right\} \tag{4.14a}$$

$$= g^{\xi\varrho} \Gamma^\varsigma{}_{\mu\nu} \left\langle \frac{\partial}{\partial x^\varsigma}, \frac{\partial}{\partial x^\varrho} \right\rangle \tag{4.14b}$$

$$= g^{\xi\varrho} \left\langle \nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\varrho} \right\rangle \tag{4.14c}$$

$$= \frac{1}{2} g^{\xi\varrho} \left\{ \frac{\partial g_{\nu\varrho}}{\partial x^\mu} + \frac{\partial g_{\mu\varrho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\varrho} \right\} \tag{4.14d}$$

$$= \frac{1}{2} g^{\xi\varrho} \big(g_{\nu\varrho,\mu} + g_{\mu\varrho,\nu} - g_{\mu\nu,\varrho} \big), \tag{4.14e}$$

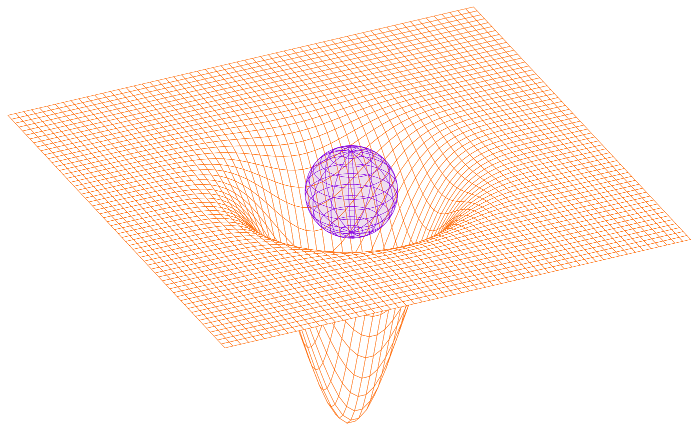


FIGURE 4.1 – Curved space-time in a #FF7518-grid within the framework of a (pseudo-)-Riemannian geometry, along the z-axis : we are looking at something like a non-3D space-time curvature, or non-$(1+3)$D space-time curvature, to be picky. The massive central body is #8803E3-colored

B. Riemann–Cartan Geometry

Let us look at two of the main features of Cartanian geometry,[7] or, for the sake of exactness, of Riemannian–Cartanian geometry, denoted by $\mathfrak{C}$.

(1) The notion of space is homogeneous or non-homogeneous. The Cartan homogeneous-space is modeled, at least locally, on the Klein geometry/on the coset space G/H. The Cartan non-homogeneous-space manifests deformations to the Klein structure (it is equivalent to a Klein geometry deformed by some curvature) ; the criterion by which to visualize a $\mathfrak{C}$-deformation is analogous to that used for introducing the Riemannian curvatures in Euclidean space.

(2) The Cartan space can be a flat or non-flat. In the first case, the Riemannian–Cartanian geometry is a generalization of the standard Euclidean one, and the curvature in the Cartanian structure is zero (the curvature vanishes at all points) ; in the second case, it is a more general conception of Riemannian geometry (see Section B.), and the fabric of Cartan space has blob-like, or hill-like, values—to recover the vivid language of W.K. Clifford [26, p. 158] = [28, p. 21]—in $\mathfrak{C}$-shape.

Apostil : Cartanian Structurality and Torsion Forms

We are going to underline some more technical aspects.

7. See e.g. [93, secc. 1.4, 1.5, 1.7].

(1) The Riemann curvature tensor

$$R_{\vec{X}\vec{Y}}\sigma_\mu$$

is succinctly stated by the 2-forms $\Omega^\nu{}_\mu$,

$$R_{\vec{X}\vec{Y}}\sigma_\mu = \Omega^\nu{}_\mu(\vec{X},\vec{Y})\sigma_\nu, \quad \mu,\nu = 1,\dots,n, \tag{4.15}$$

for all vectors $\vec{X},\vec{Y} \in \mathfrak{X}(\Upsilon)$, where $\Upsilon \subset \mathcal{M}$ is an open neighborhood, and $\mathfrak{X}(\Upsilon)$ is a vector space of vector fields on $\mathcal{M}$. The 2-forms $\Omega^\nu{}_\mu$ can alternatively be expressed as

$$\begin{aligned} \Omega^\nu{}_\mu &= d\omega^\nu{}_\mu - \omega^\xi{}_\mu \wedge \omega^\nu{}_\xi \\ &= \frac{1}{2} R^\nu{}_{\mu\xi\varrho}\omega^\xi \wedge \omega^\varrho, \end{aligned} \tag{4.16}$$

which is the second structural equation of É. Cartan [17, p. 133]. From here it is evident that

$$d\omega^\nu{}_\mu = \omega^\xi{}_\mu \wedge \omega^\nu{}_\xi + \Omega^\nu{}_\mu. \tag{4.17}$$

(2) Let

$$\tau\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

indicate a $\binom{1}{2}$-tensor field, or a tensor of type $(1,2)$. It will be the torsion tensor of the connection ∇ on a (pseudo-)Riemannian manifold. The $\binom{1}{2}$-tensor can be symbolized as $\tau \in \mathfrak{X}^1_2(\mathcal{M})$, and it is fixed by

$$\tau(\vec{X},\vec{Y}) = \nabla_{\vec{X}}\vec{Y} - \nabla_{\vec{Y}}\vec{X} - [\vec{X},\vec{Y}]. \tag{4.18}$$

For a torsion τ of ∇, it is easy to define a map

$$\tau^\nu\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M}), \quad \nu = 1,\dots,n,$$

by

$$\tau(\vec{X},\vec{Y}) = \tau^\nu(\vec{X},\vec{Y})\sigma_\nu. \tag{4.19}$$

The 2-forms $\{\tau^1,\dots,\tau^n\}$ are called *torsion forms*, useful for demonstrating the first structural equation of É. Cartan,

$$d\varphi^\nu = \varphi^\mu \wedge \omega^\nu{}_\mu + \tau^\nu. \tag{4.20}$$

C. Affine Connections by Cartan

What is an affine connection? Take a smooth manifold connecting nearby tangent spaces; given a vector space, an affine connection is a method of differentiating sections of vector bundles, and therefore tangent vector fields, which are treated as functions on the manifold under consideration. Cartan's interpretation of it is as follows.

We can symbolically write $H \subset G$ for a Lie subgroup in a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathring{\mathcal{P}}$ a fiber bundle in which there is a G-fiber (see Marginalia 3..1). From here one can define a Cartanian geometry $\mathfrak{C} = (\mathring{\mathcal{P}}, \omega_{\mathfrak{g}})$ of type (G, H) over a smooth manifold $\mathcal{M}$ as a principal fiber H-bundle

$$\pi\colon \mathring{\mathcal{P}} \xrightarrow{H} \mathcal{M},$$

equipped with a $\mathfrak{g}$-valued 1-form $\omega_{\mathfrak{g}} \in \Omega^1(\mathring{\mathcal{P}}, \mathfrak{g})$. Now,

$$\omega_{\mathfrak{g}} = \omega_{\mathfrak{h}} \oplus \omega_{\mathfrak{p}} \tag{4.21}$$

is a differential form on $\mathring{\mathcal{P}}$ corresponding to the Cartan connection, where $\mathfrak{p}$ is a Lie-like structure on a vector $\mathfrak{p}$-pace, in association with H- or $\mathfrak{h}$-module decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}. \tag{4.22}$$

In this respect it is noted that $\omega_{\mathfrak{g}}$ is a generalization of the Maurer–Cartan form ω_G [8], which is a $\mathfrak{g}$-valued 1-form on the group manifold G, with left invariant form,

$$\omega_G \in \Omega^1(G, \mathfrak{g}) \stackrel{\text{viz}}{=} \omega_G^{\text{L}} \in \Omega^1(G, \mathfrak{g}), \ \text{or}\ \omega_G^{\text{L}} \in \Omega^1(G) \otimes \mathfrak{g},$$

and right invariant form,

$$\omega_G^{\text{R}} \in \Omega^1(G, \mathfrak{g}), \ \text{or}\ \omega_G^{\text{R}} \in \Omega^1(G) \otimes \mathfrak{g}.$$

Here is a list of the three main peculiarities of the $\omega_{\mathfrak{g}}$-connection.

(1) Once introduced the symbol of the adjoint action adj, the first is

$$(\text{R}_h)^* \omega_{\mathfrak{g}} = \text{adj}(h^{-1}) \circ \omega_{\mathfrak{g}}, \quad h \in H. \tag{4.23}$$

(2) The second peculiarity is dictated by a linear isomorphism,

$$\omega_{\mathfrak{g}}(x)\colon \mathcal{T}_x \mathring{\mathcal{P}} \xrightarrow{\mathfrak{h}\oplus\mathfrak{p}} \mathfrak{g}, \tag{4.24}$$

for each point $x \in \mathring{\mathcal{P}}$.

(3) The third one by

$$\digamma_{\vec{X}} \in \mathfrak{X}(\mathcal{M}) = \omega_{\mathfrak{g}}^{-1}(\vec{X}), \tag{4.25}$$

for all vector fields $\vec{X} \in \mathfrak{h}$. It assumes that a fundamental vector field

$$\digamma_{\vec{X}}(x) = \frac{dt}{d}\Big|_{t=0} x\Big(\exp(t\digamma)\Big) \tag{4.26}$$

is on $\mathcal{M}$.

Marginalia 3..1 (A hint about the principal G-bundle). When the principal bundle is over $\mathcal{M}$, $\mathring{\mathcal{P}}$ is a surjective smooth map $\pi\colon \mathring{\mathcal{P}} \to \mathcal{M}$, or a $\mathscr{C}^\infty$ projection π of $\mathring{\mathcal{P}}$ onto $\mathcal{M}$, if there is a smooth right R-action of G on $\mathring{\mathcal{P}}$, that is,

$$\mathrm{R}_G\colon \mathring{\mathcal{P}} \times G \to \mathring{\mathcal{P}},\ \ \mathrm{R}_G(x,g) = x \cdot g,$$
$$\mathrm{R}_g\colon x \mapsto x \cdot g,$$

for each $x \in \mathring{\mathcal{P}}$, with $g \in G$. Since the fiber bundle of a principal G-bundle is isomorphic to G-space, we have $\mathring{\mathcal{P}}/G = \mathcal{M}$. This causes a principal G-bundle to be regarded as a smooth manifolds, or a $\mathscr{C}^\infty$ (smooth) G-bundle.

The left G-action on G corresponds to :

$$\mathrm{L}_G\colon G \times G \to G,$$
$$\mathrm{L}_g\colon G \to G\colon h \mapsto g \cdot h.$$

Curvature of Cartan space

Let

$$\kappa_\Omega \overset{\text{viz}}{=} \Omega^2_{\mathfrak{g}} = d\omega_{\mathfrak{g}} + \frac{1}{2}[\omega_{\mathfrak{g}}, \omega_{\mathfrak{g}}]^\wedge = d\omega_{\mathfrak{g}} + \omega_{\mathfrak{g}} \wedge \omega_{\mathfrak{g}}, \tag{4.29}$$

designate the curvature form of the connection $\omega_{\mathfrak{g}}$, knowing that $\Omega^2_{\mathfrak{g}}$ is a $\mathfrak{g}$-valued 2-form on the principal bundle $\mathring{\mathcal{P}}$.

By assigning to the symbol $\mathfrak{h}$ the value of a subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$, it should be noted that

(1) if $\Omega^2_{\mathfrak{g}}$ take values in the subalgebra $\mathfrak{h}$, the $\mathfrak{C}$-geometry is torsion free, and the κ-value lying in $\mathfrak{h}$ is zero;

(2) if $\Omega^2_{\mathfrak{g}}$ is decomposable, it comes out a reductive geometry with an H-module decomposition,

$$\kappa_\Omega = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{p}}. \tag{4.30}$$

Cartan $\mathfrak{h}$-subalgebra $\subset \mathfrak{g}$

Let us spend a few words about the subalgebra $\mathfrak{h}$. We write schematically,

$$\mathfrak{h} = \mathfrak{i}_{\mathfrak{g}}(\mathfrak{h}), \text{ and } \mathfrak{h} \subset \mathfrak{g}^0(\mathfrak{h}), \tag{4.31}$$

$$\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h}), \tag{4.32}$$

with a finite collection $\mathfrak{i}_1, \dots, \mathfrak{i}_k \trianglelefteq \mathfrak{g}$ (and sum $\mathfrak{i}_1 + \cdots + \mathfrak{i}_k$) of nilpotent ideals, $\mathfrak{i} \subset \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$. This means that $\mathfrak{h}$ is nilpotent, or better, a nilpotent Cartan subalgebra, with such an endomorphisms

$$\pi\colon \mathfrak{h} \to \mathfrak{gl}\big[\mathfrak{g}^0(\mathfrak{h})/\mathfrak{h}\big],$$

as long as

$$\pi(h)[x + \mathfrak{h}] = \big[h \in \mathfrak{h}, x \in \mathfrak{g}^0(\mathfrak{h})\backslash\mathfrak{h}\big] + \mathfrak{h}, \tag{4.33}$$

provided that $\mathrm{adj}(h)x \in \mathfrak{h}$. Ad in the end,

$$\mathfrak{h} \subset \mathfrak{g}^0\big[\mathrm{adj}(\mathfrak{h})\big] \subset \mathfrak{g}^0\big[\mathrm{adj}(x) \mid x \in \mathfrak{h}\big] = \mathfrak{h}. \tag{4.34}$$

The splitting Cartan $\mathfrak{h}$-subalgebra is simply expressible with

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\varphi_\lambda \neq 0} \mathfrak{h}^{\varphi_\lambda}, \tag{4.35}$$

by choosing $\varphi_\lambda\colon \mathfrak{h} \to \mathbb{K}$ as a linear functional on $\mathfrak{h}$ (see Marginalia 3..2), being aware that $\mathbb{K}$ is an $\mathbb{R}$-field or a $\mathbb{C}$-field.

Marginalia 3..2 (Weight of a 1-dimensional representation of a subspace). Incidentally, it has to be stressed that the linear functional φ_λ has to do with the so-called *weight of a 1-dimensional representation of a subspace* (pompous expression for a simple concept). Let $\mathfrak{W}$ be this subspace, with weight φ_λ. The $\mathfrak{W}$-subspace is the direct sum of its weight spaces,

$$\mathfrak{W} = \bigoplus_{\varphi_\lambda} \mathfrak{W}^{\varphi_\lambda}(\mathfrak{h}), \tag{4.36}$$

with the result that

$$\mathfrak{W}_{\varphi_\lambda}(\mathfrak{h}) = \bigcap_{x \in \mathfrak{h}} \mathfrak{W}_{\varphi_\lambda(x)}\big[\pi(x)\big], \tag{4.37a}$$

$$\mathfrak{W}^{\varphi_\lambda}(\mathfrak{h}) = \bigcap_{x \in \mathfrak{h}} \mathfrak{W}^{\varphi_\lambda(x)}\big[\pi(x)\big]. \tag{4.37b}$$

Gauge Frame in Riemannian–Cartanian Geometry

Let $\varUpsilon$ be an open set of a manifold $\mathcal{M}$, and $\vartheta_\varUpsilon$ a $\mathfrak{g}$-valued 1-form on $\varUpsilon$ in $(\mathfrak{g}, \mathfrak{h})$. The 1-form

$$\tilde{\vartheta}_\varUpsilon \colon \mathcal{T}_\upsilon \xrightarrow{(\vartheta_\varUpsilon)} \mathfrak{g},$$

accompanied by a canonical projection,

$$\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h},$$

is a linear isomorphism, for any $\upsilon \in \varUpsilon$. The pair $(\varUpsilon, \vartheta_\varUpsilon)$ is an illustration of a Cartan gauge.

Metric $\mathfrak{C}$-connection and 1-form of Type adj *for a Torsion*

(1) The metric $\mathfrak{C}$-connection is $Dg_{\mu\nu} = 0$, where D is the covariant derivative, cf. e.g. [125].

(2) Let

$$\alpha^\mu{}_\nu = (\omega^\mu_\nu)_1 - (\omega^\mu_\nu)_0 \tag{4.38}$$

be a $\mathfrak{g}$-valued tensorial 1-form of type adj on the principal G-bundle, such that

$$\mathrm{adj} \colon G \to GL(\mathfrak{g})$$

is the adjoint action of G on $\mathfrak{g}$, where G is a Lie group. Then

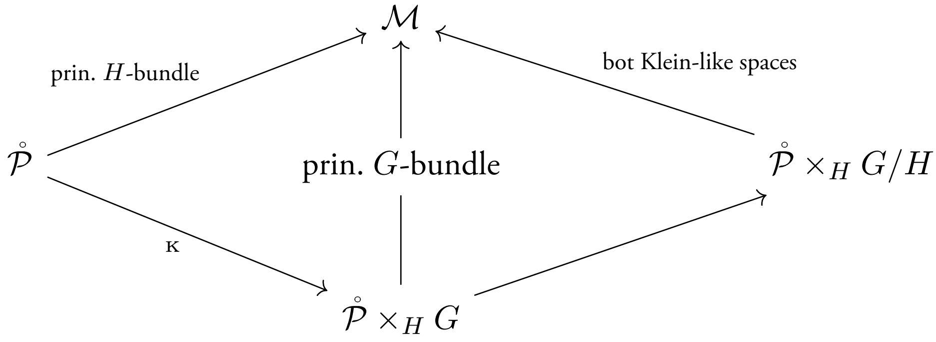
$$\alpha_{\mu\nu} + \alpha_{\nu\mu} = 0, \tag{4.39}$$
$$\varOmega_{\nu\mu} + \varOmega_{\mu\nu} = 0, \tag{4.40}$$

and, once it is setted that $\tau_{\xi\mu\nu} = g_{\xi\varrho}\tau^\varrho{}_{\mu\nu}$,

$$\alpha_{\mu\nu} = \frac{1}{2}\left(\tau_{\mu\varrho\nu} + \tau_{\nu\mu\varrho} + \tau_{\varrho\mu\nu}\right)\vartheta^\varrho. \tag{4.41}$$

D. $\mathfrak{C}$-Geometro-diagram

Here we summarize the Riemannian–Cartanian geometry with a diagram,



in which the maps

$$\kappa : \mathring{\mathcal{P}} \to \mathring{\mathcal{P}} \times_H G,$$
$$\varphi[\text{bot}] : \mathring{\mathcal{P}} \times_H G/H \to \mathcal{M},$$

hold—bot is an abbreviation standing for *bundle of tangent* Klein-like spaces, for this succession of H-spaces,

$$H \to G \to G/H.$$

The bundle

$$\mathring{\mathcal{P}} \times_H G \to \mathcal{M}$$

matches the principal right G-bundle, and it is promptly combined with the principal H-bundle $\mathring{\mathcal{P}} \to H \to \mathcal{M}$ and the action of H on G by left multiplication.

If we apply the diagram in the 4-dimensional $(1 + 3)$ geometry, the Einstein–Cartan space-time is a manifold usually symbolized with $\mathcal{M} \overset{\text{viz}}{=} \mathcal{U}^4$.

4. CARTANIAN CONNECTION AND GRAVITY

The adoption of Riemannian–Cartanian geometry, and its affine connection, consists in inserting two elements into the general theory of relativity.

(1) The first is the torsion

$$\tau_{\mu\nu\xi} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\xi}}\right) \tag{4.42}$$

associated with a spin structure, which is absent in the classical general relativity; e.g. in the affine connection ∇, or more properly $\nabla^{\mathring{\mathcal{T}}}$, by Levi-Civita [78], the torsion is zero, i.e. the torsion of ∇ vanishes identically ($\nabla g = 0$), for Einstein's theory of gravity onto a (pseudo-)Riemann geometry. [8]

Be advised that the Cartanian torsion coefficient is but the parts of the Christoffel symbols of the second kind (4.14), when in these parts there is a change of sign with indices reversed (antisymmetry).

(2) The second element is the local Lorentz invariance, that is a gauge field, [9] from Lorentz symmetry [82] [83] [84].

A. A Few Formulæ : Curved Space-Time + Spin & Torsion + Gauge Field

Let us try to fix some crucial items, so as to clarify the notions we have seen in the two previous Sections in order to mix them, in a single theory.

(1) Imagine a curved space-time; if it is endowed with torsion, it provides a new version of the Poincaré group (see Marginalia 4..3). The latter is a fundamental group (it is a topological & homotopy invariant), the first homotopy group, of a topological space.

(2) An Einstein–Cartan space-time offers a determination of the potential of the electromagnetic field as

(i) a scalar-valued 1-form $\omega_{\mathfrak{g}}[\digamma]$, with an invariance of $d\omega_{\mathfrak{g}}[\digamma]$ under the gauge transformation

$$\omega_{\mathfrak{g}}[\digamma] \mapsto \omega_{\mathfrak{g}}[\digamma] + d\zeta, \tag{4.43}$$

(ii) a covector-valued 0-form $\omega_{\mathfrak{g}}[\digamma_\mu]$, having a gauge field of this type :

$$\frac{1}{2}\left(\nabla_\mu \omega_{\mathfrak{g}}[\digamma_\nu] - \nabla_\nu \omega_{\mathfrak{g}}[\digamma_\mu]\right) \vartheta^\mu \wedge \vartheta^\nu = \\ \left(D\omega_{\mathfrak{g}}[\digamma_\mu]\right) \wedge \vartheta^\mu = d\omega_{\mathfrak{g}}[\digamma] - \omega_{\mathfrak{g}}[\digamma_\mu]\Theta^\mu_\tau, \tag{4.44}$$

8. This letter from Cartan [18, p. 7, originally in Fr., 8 May 1929] to Einstein is very explanatory : «[S]paces with a Euclidean connection allow of a *curvature* and *torsion* : in the spaces where parallelism is defined in the Levi-Civita way, the torsion is zero; in the spaces where parallelism is absolute (*Fernparallelismus*) [128, chap. XIII] [127] [19, 20] the curvature is zero [flat metric], thus these are spaces without curvature and with torsion».

The notion of *absolute parallelism*, which is a system preserving the metric but with non-zero torsion, is carefully considered by G. Vitali [127] already in 1924; but it was not until 1929 that he communicates it to German physicist, by a letter dated 11 February 1929. Einstein's first and independently use of this notion is in [38] [39] during an attempt to unify gravity with electromagnetism.

9. See R.J. Petti [100].

where

∇ is the differential operator, ϑ^μ, ϑ^ν are 1-forms, or dual coframes of n-tuple of vector fields $\vec{E}_1, \ldots, \vec{E}_n$, constructing an orthonormal basis of the tangent space—the metric tensor field here is $g = g_{\mu\nu}\vartheta^\mu\vartheta^\nu$, putting $g_{\mu\nu} = g(e_\mu, e_\nu)$, see Eq. (4.53),

Θ_τ is the torsion form, or the vector-valued 2-form, of the connection form $\omega \stackrel{\text{viz}}{=} \omega_{\mathfrak{g}}$, from which

$$\Theta_\tau \stackrel{\text{viz}}{=} (\Theta_\tau)^\mu = d(\Theta_\tau)^\mu + \omega^\mu{}_\nu \wedge \vartheta^\nu, \tag{4.45}$$

letting ϑ^ν be the basis, or a $\binom{1}{0}$-tensor valued 1-form.

(3) The curvature of Einstein–Cartan space-time is the surface density of the Lorentz transformations, whilst its torsion is the surface density of the Lorentz translations.

(4) In the Einstein–Cartan space-time there is no need for the Ricci curvature tensor (4.5) (4.6) (4.7) (4.8) to be symmetric; in fact, the insertion of the spin structure, in Riemann–Cartan geometry, ensures that the energy-momentum tensor $\mathrm{T}_{\mu\nu}$, on the right side of poly-Eq. (4.4), can be asymmetric :

$$\mathrm{T}_{\mu\nu} - \mathrm{T}_{\nu\mu} = \hat{S}^{\mu\nu\xi}{}_{,\xi}, \tag{4.46}$$

where $\hat{S}^{\mu\nu\xi} = -\hat{S}^{\nu\mu\xi}$ is the spin, or, to be exact, the spin angular momentum.

(5) At this stage, let us represent the Lagrangian density of the gravitational field, according to its variation :

$$\delta\mathscr{L} = \mathscr{L}_r \wedge \delta\omega_{\mathfrak{g}}[\mathrm{F}]^r + \frac{1}{2}\tau^{\mu\nu}\delta g_{\mu\nu} + \delta\vartheta^\mu \wedge \mathrm{T}_\mu - \frac{1}{2}\delta\omega^\mu{}_\nu \wedge \hat{S}^\nu{}_\mu, \tag{4.47}$$

by imposing $\omega_{\mathfrak{g}}[\mathrm{F}] = \omega_{\mathfrak{g}}[\mathrm{F}]^r e_r$, and $\mathscr{L}_r = 0$, because it is the Euler–Lagrange equation for $\omega_{\mathfrak{g}}[\mathrm{F}]$. Consider that if $\omega_{\mathfrak{g}}[\mathrm{F}] = \omega_{\mathfrak{g}}[\mathrm{F}]^r e_r$ is a k-form [7], [10] ergo

$$D\omega_{\mathfrak{g}}[\mathrm{F}]^r = d\omega_{\mathfrak{g}}[\mathrm{F}]^r + ь^{r\mu}_{s\nu}\omega^\nu{}_\mu \wedge \omega_{\mathfrak{g}}[\mathrm{F}]^s. \tag{4.48}$$

On condition that $\delta\mathscr{L} = 0$, then

$$g_{\mu\xi}\tau^{\xi\nu} - \vartheta^\nu \wedge \mathrm{T}_\mu + \frac{1}{2}D\hat{S}^\nu{}_\mu - ь^{s\nu}_{r\mu}\mathscr{L}_r \wedge \omega_{\mathfrak{g}}[\mathrm{F}]^s = 0. \tag{4.49}$$

10. Recall that a differential form ω of degree $k = \mathbb{Z}$, or just k-form, is a section of an algebra over an $\mathbb{R}$-field, by the exterior product of the cotangent bundle $\mathring{\mathcal{T}}^*\mathcal{M}$ of a manifold $\mathcal{M}$, so that $\omega_{\mathbb{R}} : \mathcal{M} \to \bigwedge^k \mathring{\mathcal{T}}^*\mathcal{M}$.

Marginalia 4..1 (Flat metric : Minkowski space-time, and its inner product). Minkowski space-time [87] [88] [89] is a a pseudo-Euclidean real vector 4-space $\mathfrak{M}^4 \overset{\text{viz}}{=} \mathbb{M}^4 = \mathbb{R}^4_{1,3}$: [11] its topology, unlike Euclidean 4-space, is not locally homogeneous; indeed, in $\mathbb{R}^4_{1,3}$ space vectors and time vectors are separated, see E.C. Zeeman [134]. In a sufficiently small neighborhood of a point, the Minkowski space-time is an excellent approximation of the 4-manifold in Einstein's theory of special relativity [34]; consequently, it is a flat 4-space, without matter-energy : $R_{\mu\nu} = 0$, since if $\mathrm{T}_{\mu\nu} = 0$, then $g_{\mu\nu}\left(R_{\mu\nu} - g^{\mu\nu}\frac{R_{\mathrm{s}}}{2}\right) = 0$. In the presence of a massive body, with the emergence of gravity, the Minkowski space(-time) transmutes into a non-pseudo-Euclidean 4-space.

Let $\{e_0, e_1, e_2, e_3\}$ be a basis, with $v = v^\mu e_\mu$ and $w = w^\nu e_\nu$, $\mu, \nu = 0, 1, 2, 3$. The 2-fold signature of the Minkowski metric tensor can be marked with $^{(1,3)^+}$, for $(+, -, -, -)$, and with $^{(1,3)^-}$, for $(-, +, +, +)$; conversely, the Euclidean signature has all positive signs. Once fixed the metric tensor of Minkowski space-time, η, $\eta^{\mu\nu}$, $\eta_{\mu\nu}$, the Minkowski inner product will be

$$g(v,w)^{(1,3)^+} = v^0w^0 - v^1w^1 - v^2w^2 - v^3w^3 = \eta_{\mu\nu}v^\mu w^\nu, \qquad (4.51)$$

$$g(v,w)^{(1,3)^-} = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3 = \eta_{\mu\nu}v^\mu w^\nu, \qquad (4.52)$$

11. A distinction is needed here. Minkowski space is a real vector space $\mathfrak{M}^n$ or $\mathbb{M}^n = \mathbb{R}^{1,n-1}$ or $\mathbb{R}^{n-1,1}$ of dimension $n \geqslant 2$, with a bilinear form g viz. $g_{\mathrm{M}} \colon \mathfrak{M}^n \times \mathfrak{M}^n \to \mathbb{R}$ on the tangent space at any point of $\mathfrak{M}$, claiming that g is symmetric $g(v,w) = g(w,v)$ and non-degenerate $g(v,w) = 0$, for each $v, w \in \mathfrak{M}^n$. When a generic map $g \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is prepared, it is determined that $g(v,w) = v^0w^0 + v^1w^1 + v^2w^2 + \cdots + v^{n-1}w^{n-1} - v^nw^n$ or $g(v,w) = -v^0w^0 + v^1w^1 + v^2w^2 + \cdots + v^nw^n$. The bilinear form $g(v,w)$ is usually said *Minkowski (or Lorentzian) inner product*, or *Minkowski (or Lorentzian) metric tensor*. If $\{e_0, \ldots, e_{n-q}, e_{n-q+1}, \ldots, e_n\}$ is a basis, setting $n = \dim(\mathfrak{M})$, with $v = v^0e_0 + \cdots + v^ne_n$ and $w = w^0e_0 + \cdots + w^ne_n$, one has

$$g(v,w) = v^0w^0 + v^1w^1 + v^2w^2 + \cdots + v^{n-q}w^{n-q} - v^{n-q+1}w^{n-q+1} - \cdots - v^nw^n, \qquad (4.50)$$

for a non-negative integer $q \in \mathbb{Z}_* = \{0\} \cup \mathbb{Z}_+$.

stating that g has index 1, and

$$\eta_{\mu\nu} = g(e_\mu, e_\nu) = \begin{cases} 1 \text{ if} \begin{cases} \mu = \nu = 0, \text{ with } \eta^{(1,3)^+}, \\ \mu = \nu = 1, 2, 3, \text{ with } \eta^{(1,3)^-}, \end{cases} \\ -1 \text{ if} \begin{cases} \mu = \nu = 1, 2, 3, \text{ with } \eta^{(1,3)^+}, \\ \mu = \nu = 0, \text{ with } \eta^{(1,3)^-}, \end{cases} \\ 0 \text{ if} \begin{cases} \mu \neq \nu, \text{ with } \eta^{(1,3)^+}, \\ \mu \neq \nu, \text{ with } \eta^{(1,3)^-}. \end{cases} \end{cases} \tag{4.53}$$

We pick out $x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3$, with the time (x^0) and the spatial (x^1, x^2, x^3) coordinates. It is customary to put $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, in such a way that, in standard coordinates (ct, x, y, z), the metric tensor of Minkowski space-stime is, doubly,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}^{(1,3)^+} dx^\mu dx^\nu, \tag{4.54}$$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}^{(1,3)^-} dx^\mu dx^\nu. \tag{4.55}$$

The letter c displays the speed of light in vacuum, while t the time, with $(t, x, y, z) \in \{-\infty, +\infty\}$. Eqq. (4.54) (4.55) are called line elements, in the course of an infinitesimal displacement vector at any point in $\mathbb{R}^4_{1,3}$.[12]

Marginalia 4..2 (Lorentz space-time). Lorentz space-time $\mathfrak{L}^4 \overset{\text{viz}}{=} \mathbb{L}^4 = \mathbb{R}^4_{1,3} = \mathcal{M}^4$ is a generalization of the Minkowskian one; it is a connected $\mathscr{C}^\infty$ (smooth) 4-manifold, having its correspondence in a $\mathscr{T}_2$ Hausdorff 4-space,

12. Eqq. (4.54) (4.55) are rewriteable in spherical coordinates (ct, ρ, θ, ϕ),

$$ds^2_{(1,3)^+} = c^2 dt^2 - d\rho^2 - \rho^2 (d\theta^2 - \sin^2\theta d\phi^2), \tag{4.56}$$

$$ds^2_{(1,3)^-} = -c^2 dt^2 + d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{4.57}$$

setting $x^0 = t$, $x^1 = x = \rho \sin\theta \cos\phi$, $x^2 = y = \rho \sin\theta \sin\phi$, $x^3 = z = \rho\cos\theta$. The set membership : $\rho \in [0, \infty), \theta \in [0, \pi], \phi \in [0, 2\pi)$. Things to know : ρ is the radius corresponding to the line segment moving from a point to the origin, θ is the colatitude, or the polar/zenith angle measured from the z-axis, and ϕ is the longitude, i.e. the azimuthal angle measured from the x-axis within the xy plane.

with a Lorentzian metric $\binom{0}{2}$-tensor $g^{(1,3)}$. [13] The alternative form is $\mathbb{R}^4_{3,1}$, $g^{(3,1)}$. The amphibolic nature of $\mathbb{L}^4 = \mathbb{R}^4_{1,3}$ coincides

(1) with the Minkowski's description $\mathfrak{M}^4 \overset{\text{viz}}{=} \mathbb{M}^4$, when the Riemann curvature tensor of the Levi-Civita connection is zero; its denomination is *flat Lorentzian space-time*, or *Lorentz–Minkowski space-time* : the infinitesimal neighborhood around any point in a curved 4-space is still a Minkowski-like flat 4-manifold, although there is a variation of g_{L} from point to point;

(2) with an incorporation into the general relativity; it provides a model for geometry of Einsteinian gravity (in which, to be more specific, the gravitational force is a manifestation of the curvature of space-time by the action of matter-energy), when it represents a hyperbolic 4-manifold; its denomination is *non-flat (curved) Lorentzian space-time*—here Lorentzian space-time and Einstein's gravitationally curved space-time are the same object, of mathematical design, patently.

Synoptically, one has

$$\mathbb{L}^4 = \mathbb{R}^4_{1,3} \begin{cases} \text{flat pseudo-Euclidean (Minkowskian-like) Lorentzian space-time,} \\ \text{non-flat (curved) non-pseudo-Euclidean Lorentzian space-time.} \end{cases}$$

Marginalia 4..3 (Poincaré group). The Poincaré group [101, § 12] [102], occasionally renamed *inhomogeneous Lorentz group*, is a 10-dimensional non-abelian Lie group

$$P_{\mathbb{M}} = P_{\mathbb{M}(1,3)} \cong O_{(1,3)} \rtimes \mathbb{R}^4_{1,3},$$

and it consists of the semi-direct product of the 4-parameter group of the translations [83] and of the 6-dimensional Lorentz group, [14]

$$\text{Л} = \text{Л}^{\uparrow}_{+} \cup \text{Л}^{\uparrow}_{-} \cup \text{Л}^{\downarrow}_{+} \cup \text{Л}^{\downarrow}_{-} = O_{1,3}(\mathbb{R}).^{15} \tag{4.60}$$

13. Lorentz space is an n-dimensional vector $\mathbb{R}$-space $\mathfrak{L}^n$ viz. $\mathbb{L}^n = \mathbb{R}^{1,n-1}$ or $\mathbb{R}^{n-1,1}$, characterized by a Lorentzian inner product g viz. g_{L},

$$g(v,w)^+ = v^0w^0 + v^1w^1 + v^2w^2 + \cdots + v^{n-1}w^{n-1} - v^nw^n, \tag{4.58}$$

$$g(v,w)^- = -v^0w^0 + v^1w^1 + v^2w^2 + \cdots + v^nw^n, \tag{4.59}$$

on the tangent space at any point of $\mathfrak{L}$, for each $v, w \in \mathfrak{L}^n$, with the signatures $(+,-,-,\ldots,-)$ and $(-,+,+,\ldots,+)$.

The Lorentzian metric tensor $g = g_{\mu\nu}dx^\mu dx^\nu$ is a pseudo-Riemannian metric, and the signature of the quadratic form is that of Eqq. (4.58) and (4.59).

14. Cf. H. Poincaré [102, p. 130] : «Lorentz's idea [83] can be summed up as follows : if we are able to bring a translation upon a whole system, without modification of any observable

We can be more accurate. Any transformation of—the Lie algebra of—the Poincaré group is an isometry of the Minkowski space-time, in conformity with

the 4-vector generators of the global translations, labeled by Γ^{T}, i.e. 4 components of the translation generators, for the Minkowski metric, and

the 6-vector generators of the Lorentz rotations, denoted by Γ^{R} (vector representation of the Lorentz group by 4×4 matrices).

The commutation relations of Γ^{T} plus Γ^{R} are

$$\left[\Gamma^{\mathrm{T}}_{1}, \Gamma^{\mathrm{T}}_{2}\right] = 0, \tag{4.61a}$$
$$\left[\Gamma^{\mathrm{T}}_{1}, \Gamma^{\mathrm{R}}_{23}\right] = i\left(\eta_{12}\Gamma^{\mathrm{T}}_{3} - \eta_{13}\Gamma^{\mathrm{T}}_{2}\right), \tag{4.61b}$$
$$\left[\Gamma^{\mathrm{R}}_{12}, \Gamma^{\mathrm{R}}_{34}\right] = i\left(\eta_{14}\Gamma^{\mathrm{R}}_{23} - \eta_{13}\Gamma^{\mathrm{R}}_{24} - \eta_{24}\Gamma^{\mathrm{R}}_{13} + \eta_{23}\Gamma^{\mathrm{R}}_{14}\right). \tag{4.61c}$$

About the group of the translations, the final reference is offered by the Lorentz–Minkowski space-time, $\mathfrak{L}^4 \overset{\text{viz}}{=} \mathbb{L}^4 = \mathbb{R}^4_{1,3}$ or $\mathbb{R}^4_{3,1}$, that is, $\mathcal{M}^4$ or $\mathfrak{M}^4 \overset{\text{viz}}{=} \mathbb{M}^4$, establishing that $\mathfrak{M}^4 \overset{\text{viz}}{=} \mathbb{M}^4 = \mathbb{R}^4_{1,3}$ is the typing for the Minkowski space-time. As a quick reminder, the Minkowski space-time is a 4D real vector space, or, more elaborately, a pseudo-Euclidean vector 4-space with strongly asymptotic flatness. For more details on all this, see e.g. [93, chap. 3].

phenomena, it is because the equations of an electromagnetic medium are not altered by certain transformations, which we will call *Lorentz transformations*; two systems, one of which is motionless, the other in translation, thus become exact images of each other».

15. For the benefit of completeness :

$$Л = \begin{cases} Л_{+} \text{ (proper L. g.)}, \\ Л_{-} \text{ (improper L. g.)}, \\ Л^{\uparrow} \text{ (orthochronous L. g.)}, \\ Л^{\downarrow} \text{ (non-orthochronous, or heterochronous, L. g.)}, \\ Л^{\uparrow}_{+} \text{ (proper orthochronous, or restricted, L. g.)}, \\ Л^{\uparrow}_{-} \text{ (improper orthochronous L. g.)}, \\ Л^{\downarrow}_{+} \text{ (proper non-orthochronous, or heterochronous, L. g.)}, \\ Л^{\downarrow}_{-} \text{ (improper non-orthochronous, or heterochronous, L. g.)}. \end{cases}$$

The $P_{\mathbb{M}}$-group has four disjoint components, $P_{\mathbb{M}}{}^{\uparrow}_{\pm}$ e $P_{\mathbb{M}}{}^{\downarrow}_{\pm}$, each of which contains the Lorentzian components ($Л\mathrm{I}^{\uparrow}_{\pm}$ and $Л\mathrm{I}^{\downarrow}_{\pm}$), so that

$$P_{\mathbb{M}} = \begin{cases} P_{\mathbb{M}}{}^{\uparrow}_{+} = P_{\mathbb{M}+} \cap P_{\mathbb{M}}{}^{\uparrow} \\ P_{\mathbb{M}}{}^{\uparrow}_{-} = P_{\mathbb{M}-} \cap P_{\mathbb{M}}{}^{\uparrow} \\ P_{\mathbb{M}}{}^{\downarrow}_{+} = P_{\mathbb{M}+} \cap P_{\mathbb{M}}{}^{\downarrow} \\ P_{\mathbb{M}}{}^{\downarrow}_{-} = P_{\mathbb{M}-} \cap P_{\mathbb{M}}{}^{\downarrow}, \end{cases} \tag{4.62}$$

where $\uparrow$ and $\downarrow$ are the inequalities. It is worth noting that $P_{\mathbb{M}}{}^{\uparrow}_{+}$ is

non-compact,
doubly-connected,
not (semi)-simple.

Einstein–Cartan Equations : a Couple of Pocket-Formulæ for a Torsioning Force in a $(1+3)$ *Signature*

Let $\kappa_{\Omega} \overset{\text{viz}}{=} \Omega^2_{\mathfrak{g}} \overset{\text{viz}}{=} \Omega^{12}$ be the curvature of the Cartan connection, or rather, a $\mathfrak{g}$-valued 2-form on $\mathring{\mathcal{P}}$, better laid down as

$$\kappa_{\Omega} \overset{\text{viz}}{=} \Omega^2_{\mathfrak{g}} \overset{\text{viz}}{=} \Omega^{12} = \frac{1}{2} R_{\mu\nu}{}^{12} dx^{\mu} \wedge dx^{\nu}$$
$$= \left(\partial_{[\mu}\omega_{\nu]} + \omega_{[\mu|}{}^{1}{}_{3}\omega_{|\nu]}{}^{32} \right) dx^{\mu} \wedge dx^{\nu} = d\omega^{12} + \omega^{1}{}_{3} \wedge \omega^{32}. \tag{4.63}$$

The equations for the Einstein–Cartan theory will be

$$\frac{1}{2}\Omega^{12} \wedge ш^{3} \varepsilon_{1234} = -\varkappa \overset{3}{\vartheta}_{4}, \tag{4.64a}$$
$$\frac{1}{2}\Omega^{3} \wedge ш^{4} \varepsilon_{1234} = -\varkappa \left(\overset{3}{\chi}_{12} \right), \tag{4.64b}$$

where $\overset{3}{\vartheta}$ is a 3-form field (to which the energy-momentum of the matter field is linked), $ш$ is the basis 1-form, $\varkappa = \frac{8\pi G}{c^4} = 1$ is the Einstein gravitational constant (4.11), and $\overset{3}{\chi}$ is an antisymmetric tensor-valued 3-form.

Einstein–Cartan Structure via Palatini Identity (Variational Principle of the Gravitational Action)

Another way to obtain the Einstein–Cartan space-time is to use Palatini identity [97] onto a Riemann–Cartan geometry, introducing the constraint constituted by the combo spin + torsion. It is a question of deriving the gravitational equations from a variational principle, as, in ordinary mechanics, the Euler–Lagrange equations [43] [44] [45] [46] [47] [72] [73] [74, 75] are derived from the principle of Hamilton [51, pp. 10-11] [52] [53]. The equations must maintain a totally invariant condition.

(1) The first thing to do is go back to the energy-momentum tensor (4.12) and the Lagrangian density of the matter-energy (4.13). Which can be defined as a functional in the calculus of variations.

Let

$\mathscr{S}_{\mathrm{m}}$ be the action functional, referred to as a matter action (to outline the dynamics of gravitational fields within the interactive scheme of matter-geometry),

δ be the metric g-variations, and

$\mathfrak{D} \subset \mathbb{M}^4$ be a 4-volume—domain of integration of dimension 4—corresponding to a region of Minkowski space-time (Marginalia 4..1).

The variation of the action integral can be written with this symbolism,

$$\begin{aligned}\delta\mathscr{S}_{\mathrm{m}} &= \int_{\mathfrak{D}\subset\mathbb{M}^4}\left\{\frac{\partial\cdot\sqrt{-g}\mathscr{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\cdot\sqrt{-g}\mathscr{L}}{\partial(\partial_\xi g^{\mu\nu})}\partial_\xi\delta g^{\mu\nu} + \cdots\right\} d^4x \\ &= \int_{\mathfrak{D}\subset\mathbb{M}^4}\left(\delta\cdot\sqrt{-g}\mathscr{L}\right) d^4x = \left(\frac{1}{2}\int_{\mathfrak{D}\subset\mathbb{M}^4}\sqrt{-g}\mathscr{L}\mathrm{T}_{\mu\nu}\delta g^{\mu\nu}\right) d^4x.\end{aligned} \tag{4.65}$$

The variation of $\mathscr{S}_{\mathrm{m}}$ (4.65) conducts us to the Einstein–Hilbert gravitational action [62] [63], the action from which it is possible to reconstruct the poly-Eq. (4.4),

$$\mathscr{S}_{\mathrm{EH}} = -\frac{c^4}{16\pi G_{\mathrm{N}}}\int R_{\mathrm{s}}\sqrt{-g}d^4x = -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4} R_{\mathrm{s}}\sqrt{-g}d^4x, \tag{4.66}$$

whilst the variation of $\mathscr{S}_{\mathrm{EH}}$ is

$$\begin{aligned}\delta\mathscr{S}_{\mathrm{EH}} &= -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4}\delta\left(R_{\mathrm{s}}\sqrt{-g}\right) d^4x = -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4}\delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right) d^4x \\ &= -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4}\left(R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} + g^{\mu\nu}R_{\mu\nu}\delta\sqrt{-g} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\right) d^4x \\ &= -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4}\sqrt{-g}\left\{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\mathrm{s}}\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right\} d^4x \\ &= -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4}\sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu}\delta R_{\mu\nu}\right) d^4x.\end{aligned} \tag{4.67}$$

(2) The second thing to do is set up the variation $\delta R_{\mu\nu}$ of the Ricci curvature tensor (4.8), and the variation $\delta\Gamma$ of the Christoffel symbols

$$\delta R_{\mu\nu} = \partial_\xi \delta\Gamma_{\mu\nu}{}^{\xi} - \partial_\nu \delta\Gamma_{\mu\xi}{}^{\xi} + \delta\Gamma_{\mu\nu}{}^{\xi}\Gamma_{\xi\varrho}{}^{\varrho} +$$

$$+ \quad \Gamma_{\mu\nu}{}^{\xi}\delta\Gamma_{\xi\varrho}{}^{\varrho} \quad - \quad \delta\Gamma_{\mu\xi}{}^{\varrho}\Gamma_{\nu\varrho}{}^{\xi} \quad - \quad \Gamma_{\mu\xi}{}^{\varrho}\delta\Gamma_{\nu\varrho}{}^{\xi}. \tag{4.68}$$

Afterwards, take the covariant derivative ∇, with the aim of achieving the Palatini identity,

$$\mathbb{I}_{\mathrm{P}} = \left\{ \delta R_{\mu\nu} = \nabla_\xi \left(\delta\Gamma_{\mu\nu}{}^{\xi} \right) - \nabla_\nu \left(\delta\Gamma_{\mu\xi}{}^{\xi} \right) \right\}. \tag{4.69}$$

The variation formula for the Einstein–Hilbert (4.67) will look like :

$$\delta\mathscr{S}_{\mathrm{P}} =$$

$$-\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4} \left\{ \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\mathrm{s}} \right) \delta g^{\mu\nu}\sqrt{-g} + \sqrt{-g} \right. \tag{4.70a}$$

$$\left. \times \left[\nabla_\xi \left(g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^{\xi} \right) - \nabla^\mu \left(\delta\Gamma_{\mu\xi}{}^{\xi} \right) \right] \right\} d^4x,$$

$$= -\frac{1}{2\varkappa}\int_{\mathfrak{D}\subset\mathbb{M}^4} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\mathrm{s}} \right) \delta g^{\mu\nu}\sqrt{-g}\, d^4x, \tag{4.70b}$$ [16]

It is the Palatini formalism of $f(R)$ gravity. Finally, the above-mentioned combo must be added, see D.E. Neville [92].

The Kibble–Sciama Model

The Kibble–Sciama (KB) model [65] [123] produces his own version of the Einstein–Cartan theory. In the words of T.W.B. Kibble [65, p. 212], his argument leads

> from the Lorentz invariance of the Lagrangian to the introduction of the gravitational field [. . .], by considering the parameter group of inhomogeneous Lorentz transformations, involving variation of the coordinates as well as the field variables. It is then unnecessary to introduce *a priori* curvilinear coordinates or a

16. In Eqq. (4.65) (4.66) (4.70) d^4x stands, concisely, for a 4-dimensionality, $d^4x = d(t, x, y, z)$; alternatively, $dx^0dx^1dx^2dx^3$, or $dx^1dx^2dx^3dx^4$.

Riemannian metric, and the new field variables introduced as a consequence of the argument include the vierbein [tetrad] components [17] [. . .] as well as the "local affine connection".

Let us get to the brass tacks, with the two princely equations of the KB model :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\mathrm{s}} = 8\pi \mathrm{T}_{\mu\nu}, \tag{4.72}$$

and

$$\tau^{\xi}{}_{\mu\nu} + \delta^{\xi}_{\mu} \tau^{\varrho}{}_{\nu\varrho} - \delta^{\xi}_{\nu} \tau^{\varrho}{}_{\mu\varrho} = 8\pi \hat{S}^{\xi}{}_{\mu\nu}, \tag{4.73}$$

from which one has

$$\tau^{\xi}{}_{\mu\nu} = 8\pi \left\{ \hat{S}^{\xi}{}_{\mu\nu} + \frac{1}{2} \delta^{\xi}_{\mu} \hat{S}^{\varrho}{}_{\nu\varrho} + \frac{1}{2} \delta^{\xi}_{\nu} \hat{S}^{\varrho}{}_{\varrho\mu} \right\}. \tag{4.74}$$

5. EINSTEIN–CARTAN SPACE-TIME OWING TO $\mathcal{J}$-BUNDLE, PLUS CLIFFORD BUNDLES $\mathcal{C}\ell(\cdot\mathcal{M}^4_{ß,\tau}, \cdot)$

A. 4-Dimensional Structure by (Riemann–)Einstein–Cartan

Eqq. (4.64), for the Einstein–Cartan theory, are specified by the spin-torsion interaction. The algebro-topological identity—which is none other than a Lorentz–Minkowski's space-oriented and time-oriented manifold (Marginalia 4..1 and 4..2), whose provenance is (pseudo-)Riemannian and Cartanian—can be called *Riemann–Einstein–Cartan space-time*, or *Einstein–Cartan space-time*, and thus symbolized :

$$\text{space-time in a 4Dspin-torsion balancing} \begin{cases} \left(\mathbb{R}^4_{ß,\tau(1,3)}, g_{\mathrm{EC}}, \nabla^{\omega_{\mathring{\mathcal{P}}}} \right), & (4.75\text{a}) \\ \mathbb{R}^4_{ß,\tau(1,3)} = \mathbb{M}^4_{ß,\tau} \overset{\text{viz}}{=} \mathcal{M}^4_{ß,\tau} \overset{\iota\delta}{=} \mathfrak{M}^4_{ß,\tau}, & (4.75\text{b}) \end{cases}$$

17. Set of four linearly independent vector fields. The tetrad formalism of the vector fields, for the tetrad bases, can be expounded in this manner, $\{\varepsilon_{(\alpha)}\}$ or $\{\varepsilon_{\hat{\alpha}}\}$, with $\alpha = 1, 2, 3, 4$, and the Ricci rotation coefficients $\gamma_{\alpha\beta\lambda}$ undergo the following math-symbolization,

$$\gamma_{(\alpha)(\beta)(\lambda)} = \varepsilon_{(\alpha)}{}^{\mu} \varepsilon_{(\lambda)}{}^{\nu} \nabla_{\nu} \varepsilon_{(\beta)\mu}, \tag{4.71a}$$

$$\gamma_{\hat{\alpha}\hat{\beta}\hat{\lambda}} = \varepsilon_{\hat{\alpha}}{}^{\mu} \varepsilon_{\hat{\lambda}}{}^{\nu} \nabla_{\nu} \varepsilon_{\hat{\beta}\mu}. \tag{4.71b}$$

where $\nabla^{\omega_{\mathring{\mathcal{P}}}}$ is a covariant derivative (operator) on $\mathbb{R}^4_{ß,\tau(1,3)}$, cf. Eq. (4.98).

B. Lorentz and Spinor Bundles

Returning to what has already been mathematized [93, sec. 3.5.1], let us go to define the Lorentz bundle and the spinor bundle.

(1) The Lorentz bundle is the principal $Л^{\uparrow}_{+}$-bundle over $\mathbb{R}^4_{1,3}$ (spacetime), with $\mathring{\mathcal{P}}_{Л}$,

$$SO^{+}_{1,3}(\mathbb{R}) = Л^{\uparrow}_{+} \hookrightarrow Л(\mathbb{R}^4_{1,3}) \xrightarrow{\mathring{\mathcal{P}}_{Л}} \mathbb{R}^4_{1,3}. \tag{4.76}$$

(2) The spinor bundle is the principal $SL_2(\mathbb{C})$-bundle over $\mathbb{R}^4_{1,3}$ (spacetime), with $\mathring{\mathcal{P}}_{ß}$,

$$SL_2(\mathbb{C}) \cong Spin^{+}_{1,3}(\mathbb{R}) \hookrightarrow ß(\mathbb{R}^4_{1,3}) \xrightarrow{\mathring{\mathcal{P}}_{ß}} \mathbb{R}^4_{1,3}. \tag{4.77}$$

The spinor configuration is the spinor bundle (4.77) plus the map

$$\varphi \colon ß(\mathbb{R}^4_{1,3}) \to Л(\mathbb{R}^4_{1,3}),$$

under three conditions,

(i) $\mathring{\mathcal{P}}_{Л}\big(\varphi(x)\big) = \mathring{\mathcal{P}}_{ß}(x)$,

(ii) $\mathring{\mathcal{P}}_{Л} \circ \varphi = \mathring{\mathcal{P}}_{ß}(x)$,

(iii) $\varphi(x \cdot g) = \varphi(x) \cdot Spin(g)$, for all $x \in ß(\mathbb{R}^4_{1,3})$ and $g \in SL_2(\mathbb{C})$.

(3) Let us recall that

(i) $SL_2(\mathbb{C}) \cong Spin^{+}_{1,3}(\mathbb{R})$, the—3-dimensional complex a/o 6-dimensional real—special linear group of 2×2 matrices over the complex field, [18] is the 2-fold covering of $SO^{+}_{1,3}(\mathbb{R})$;

(ii) $SO^{+}_{1,3}(\mathbb{R})$ is the indefinite special orthogonal group of linear transformations of $\mathfrak{M}^4 = \mathbb{R}^4_{1,3}$, oka the restricted Lorentz group (cf. Marginalia 4..3), $SO^{+}_{1,3}(\mathbb{R}) = Л^{\uparrow}_{+}$. [19]

18. It has 4 complex numbers, a/o 8 real numbers : $\mathbb{C}^4$ equates $\mathbb{R}^8$; except that the unit determinant takes away 2 of its 8 degrees of freedom : $8 - 2 = 6$. Geometrically, $SL_2(\mathbb{C})$ is diffeomorphic to the 3-sphere. For an inclusion map, one has $\iota \colon \mathbb{S}^3 \hookrightarrow \mathbb{C}^2 = \mathbb{R}^4$.

19. Do not forget these congruences :

$$SO^{+}_{1,3}(\mathbb{R}) = Л^{\uparrow}_{+} \cong \mathfrak{M}\ddot{o}\mathfrak{b}(\hat{\mathbb{C}}) \cong PSL_2(\mathbb{C}) \cong \frac{SL_2(\mathbb{C})}{\{\pm \mathbb{I}\}},$$

With the aforementioned groups the famous spinor map can be built, in order to have the universal covering group $SL_2(\mathbb{C}) \to SO^+_{1,3}(\mathbb{R})$,

$$\begin{array}{ccc} SL_2(\mathbb{C}) \cong Spin^+_{1,3}(\mathbb{R}) & \xrightarrow{\ \varsigma\ } & SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+ \\ & \searrow^{\text{H}\circ\varsigma} & \Big\downarrow \text{H} \\ & & GL(\mathfrak{M}) \end{array}$$

thru the representation of $\text{Л}^\uparrow_+$ on a vector $\mathbb{R}$-space $\mathfrak{M}$ as a homomorphism H in 6D of $SO^+_{1,3}(\mathbb{R})$ into a general linear group $GL(\mathfrak{M})$, relying on the spinor map

$$\varsigma\colon \Big(SL_2(\mathbb{C}) \cong Spin^+_{1,3}(\mathbb{R})\Big) \longrightarrow \Big(SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+\Big).$$

(4) We are ultimately able to achieve the following diagram :

$$\begin{array}{ccccc} & & & \overset{\mathring{\mathcal{P}}_{\text{ß}}}{\curvearrowright} & \\ SL_2(\mathbb{C}) \cong Spin^+_{1,3}(\mathbb{R}) & \hookrightarrow & \text{ß}(\mathbb{R}^4_{1,3}) & & \mathbb{R}^4_{1,3} \stackrel{\text{viz}}{=} \mathfrak{M}^4 \\ & & \Big\downarrow \varphi & \nearrow_{\mathring{\mathcal{P}}_{\text{Л}}} & \\ SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+ & \hookrightarrow & \text{Л}(\mathbb{R}^4_{1,3}) & & \end{array}$$

under which

$$\varphi \times Spin\colon \text{ß}(\mathbb{R}^4_{1,3}) \times SL_2(\mathbb{C}) \to \text{Л}(\mathbb{R}^4_{1,3}) \times SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+.$$

(5) The Lorentz and spinor bundles are equivalent to the product bundle,

$$SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+ \hookrightarrow \mathbb{R}^4_{1,3} \times \Big(SO^+_{1,3}(\mathbb{R}) = \text{Л}^\uparrow_+\Big) \to \mathbb{R}^4_{1,3}, \qquad (4.78)$$

$$SL_2(\mathbb{C}) \cong Spin^+_{1,3}(\mathbb{R}) \hookrightarrow \mathbb{R}^4_{1,3} \times SL_2(\mathbb{C}) \to \mathbb{R}^4_{1,3}, \qquad (4.79)$$

respectively.

related to the Möbius group, and $SO^+_{1,3}(\mathbb{R}) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$, in close liaison with the Klein 4-group [67] = [68] :

$$\frac{O_{1,3}(\mathbb{R})}{SO^+_{1,3}(\mathbb{R})} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

C. Cartan $\boldsymbol{k}$-Forms & $\mathcal{J}$-Bundles; $\boldsymbol{C\ell}$-Bundles, $\boldsymbol{C\ell}$-$\boldsymbol{k}$-Form, and $\binom{4}{k}$-Space

(1) The bare Cartan bundle over the cotangent bundle of an n-dimensional (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ is the set of equalities

$$\bigwedge \mathring{\mathcal{T}}^* \mathcal{M} = \bigcup_{x \in \mathcal{M}} \bigwedge \mathcal{T}_x^* \mathcal{M} = \bigcup_{x \in \mathcal{M}} \bigoplus_{k=0}^{n} \bigwedge^{k} \mathcal{T}_x^* \mathcal{M}, \quad k = 0, \dots, n. \tag{4.80}$$

Once the set $\bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} = \bigcup_{x \in \mathcal{M}} \bigwedge^k \mathcal{T}_x^* \mathcal{M}$ is isolated, the k-forms bundle is the piece

$$\bigwedge^{k} \mathring{\mathcal{T}}^* \mathcal{M} \subset \bigwedge \mathcal{T}_x^* \mathcal{M}, \tag{4.81}$$

with the exterior algebra $\bigwedge \mathcal{T}_x^* \mathcal{M}$ of the cotangent (vector) space $\mathcal{T}_x^* \mathcal{M}$.

(2) Let $C\ell(\mathcal{M}^4_{ß,\tau}, g)$ be a Clifford bundle (of differential forms). The Cartan $\mathcal{J}^1$-bundle, or 1-jet bundle, over

$$\bigwedge \mathring{\mathcal{T}}^* \mathcal{M} \hookrightarrow C\ell(\mathcal{M}^4_{ß,\tau}, g), \text{ wh. Eq. (4.75b) holds,}$$

with an embedding in the $C\ell$-bundle, is explicated by

$$\mathcal{J}^1 \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{ß,\tau} \right) = \bigcup_{x \in \mathcal{M}^4_{ß,\tau}} \mathcal{J}^1_x \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{ß,\tau} \right), \text{ wh. Eq. (4.75b) holds.} \tag{4.82}$$

It is noteworthy that :

(i) the above-mentioned $C\ell$-bundle is a vector bundle involved with a principal G-bundle $\mathring{\mathcal{P}}_{SO^+_{1,3}(\mathbb{R})}$ of orthonormal frames, in the contingent case, of oriented Lorentzian tetrads (see Section B.); we explicitly add the spin-Clifford principal G-bundle, or covering spin-bundle, $\mathring{\mathcal{P}}_{Spin^+_{1,3}(\mathbb{R})}$, et voilà :

$$C\ell(\mathcal{M}^4_{ß,\tau}, g) = \mathring{\mathcal{P}}_{\{SO^+_{1,3}(\mathbb{R}),\, Spin^+_{1,3}(\mathbb{R})\}} \left(\mathbb{R}^4_{ß,\tau(1,3)} \right) \times_{\text{adj}} (\mathbb{R}_{1,3}); \tag{4.83}$$

(ii) the Clifford bundle $C\ell(\mathcal{T}_x^* \mathcal{M}^4_{ß,\tau}, g_x)$ is a vector $\mathbb{R}$-space isomorphic to the exterior algebra

$$\bigwedge \mathcal{T}_x^* \mathcal{M}^4_{ß,\tau} = \bigoplus_{k=0}^{4} \bigwedge^{k} \mathcal{T}_x^* \mathcal{M}^4_{ß,\tau}, \text{ wh. Eq. (4.75b) holds,} \tag{4.84}$$

of the cotangent space $\mathcal{T}_x^* \mathcal{M}^4_{ß,\tau}$. The bit $\bigwedge^k \mathcal{T}_x^* \mathcal{M}^4_{ß,\tau}$ is a $\binom{4}{k}$-space of k-forms, or a k-space of dimension $\binom{4}{k}$.

(3) Each space section, which we typify with $\mathit{\Gamma}_\varsigma$, of $C\ell(\mathcal{M}^4_{\text{ß},\tau}, g)$, is a Clifford k-form field

$$\omega^\alpha_{C\ell} \in \mathit{\Gamma}_\varsigma \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right) \hookrightarrow \mathit{\Gamma}_\varsigma \left\{ C\ell(\mathcal{M}^4_{\text{ß},\tau}, g) \right\}, \ \ \alpha = 1, \ldots, n,$$

for $k = 0, \ldots, 4$. Then the Lagrangian density, for the 4-dimensional Einstein–Cartan structure, in the prescription of $\omega^\alpha_{C\ell}$, gains this explicitness :

$$\mathscr{L}_{\text{EC}} \colon \mathit{\Gamma}_\varsigma \left\{ \mathcal{J}^1 \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right)^{n+2} \right\} \xrightarrow{\mathscr{L}_{\text{m}} \text{ for 4D space-time with spin-torsion } \mid \, \omega^\alpha_{C\ell}} \mathit{\Gamma}_\varsigma \left(\bigwedge^4 \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right). \quad (4.85)$$

NB. In Eq. (4.85), for convenience, this 1-jet bundle is chosen :

$$\mathcal{J}^1 \left\{ \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right)^{n+2} \right\} = \bigcup_{x \in \mathcal{M}^4_{\text{ß},\tau}} \mathcal{J}^1_x \left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \times \bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \times \cdots \times \bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right) \quad (4.86)$$

over $\left(\bigwedge \mathring{\mathcal{T}}^* \mathcal{M}^4_{\text{ß},\tau} \right)^{n+2} \hookrightarrow \left\{ C\ell(\mathcal{M}^4_{\text{ß},\tau}, g) \right\}^{n+2}$.

Marginalia 5..1 (Exterior covariant derivative). On that account, the symmetries of space-time (4.75) can be revised-rewritten by virtue of geometrophysical differential notions, the first of which is the exterior covariant derivative. Suppose $\mathrm{C}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \in \mathit{\Gamma}_\varsigma \left(\mathcal{T}^{k+r}_s \mathcal{M} \right)$ is a set of components, when there is a system of coordinates $x^1, \ldots, x^n$, such that

$$\mathrm{C}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}(x^1, \ldots, x^n) \in \mathit{\Gamma}_\varsigma \left(\bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} \right),$$

where the fields $\mathrm{C}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}$ are the $(r + s)$-indexed k-forms. The exterior covariant derivative, in such a backdrop, is a covariant differential

of vector-valued differential k-forms, or, with greater precision,
of $(r + s)$-indexed k-form fields of $\mathrm{C}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}$.

Labeled by d_∇, we state the exterior covariant derivative as

$$d_\nabla \colon \mathit{\Gamma}_\varsigma \left(\bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} \right)_{\mathscr{C}^\infty} \to \mathit{\Gamma}_\varsigma \left(\bigwedge^{k+1} \mathring{\mathcal{T}}^* \mathcal{M} \right)_{\mathscr{C}^\infty},$$

for $0 \leqslant k \leqslant 4$.

The spin-torsional symmetries of the 4-manifold (4.75), which rest on the 3-forms of energy-momentum of all matter fields, cf. Eq. (4.64a), are perfectly regulated by Eq. (4.99), exhibiting the Euler–Lagrange equations for the Yang–Mills theory (see Section A.).

6. Quantum Yang–Mills Gravity

The gauge status of Einstein's theory of gravitation is still inconsistent with the quantum gauge theory in comparison with the other three fundamental interactions. Nevertheless the Einstein–Cartan theory is the confirmation [1] that a geometry of Yang–Mills (YM) [133], when it comes to treating with a gravitational field, can take root on a Cartanian background, in which the Yang–Mills Lagrangian equation,

$$
\begin{aligned}
\mathscr{L}_{\mathrm{YM}} &= -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) && (4.87a)\\
&= -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha} && (4.87b)\\
&= -\frac{1}{4}\sum_{\alpha}\Big(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + \mathrm{B}_{(g)} f^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu\Big)^2, && (4.87c)
\end{aligned}
$$

stands out; remember that

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu \tag{4.88}$$

is the Yang–Mills strength tensor. In poly-Eq. (4.87)

$F_{\mu\nu}/F^{\mu\nu}$ is the electromagnetic (field) tensor, or Maxwell tensor, oka (field) strength tensor, and it is invariant under global and local Lorentz transformations,

A^α_μ is the vector potential, or vector A-field, [20] whose triple entity epitomizes three gauge bosons,

$f^{\alpha\beta\gamma}$ are the (gauge group) structure constants of SU_2, the special unitary group of degree 2,

$\mathrm{B}_{(g)}$ is the (gauge) coupling constant.

For those interested in learning more, it is advisable to read E.W. Mielke [86, chap. 7. *Yang's Theory of Gravity*, pp. 137-159], and J.-P. Hsu & L. Hsu [64, part II. *Quantum Yang–Mills Gravity*, pp. 91-213].

20. An arrow above, $\vec{A}$, is sometimes included, in some literature.

A. Geometro-topological Yang–Mills Schema

Yang–Mills geometry is a gauge theory with non-Abelian symmetry. Let us now try to explore some mathematical essential knots.

Yang–Mills Lagrangian via Electromagnetic 2-Form

It is possible to write a Lagrangian for the Yang–Mills schema, by embracing an alternative comprehension to that of poly-Eq. (4.87), namely, by identifying the electromagnetic (field) tensor $F_{\mu\nu}F^{\mu\nu}$ with a 2-form

$$F = \tfrac{1}{2}F_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \tag{4.89}$$

in Minkowski space-time (Marginalia 4..1) having a signature of the metric tensor $\eta^{(1,3)^-} \overset{\text{viz}}{=} (-,+,+,+)$. Let $\star$ be the Hodge dual (the reference is the Hodge star operator, e.g. in de Rham cohomology). By selecting—through a gauge fixing (gf) procedure—a vector potentials, or A-field, as a gauge, more neatly, as a gauge potential, the $\mathscr{L}$-invariant 4-form will bear this aspect :

$$\mathscr{L}_{\text{YM}}\left[A_{\text{gf}}\right] = -\frac{1}{2}F \wedge \star F. \tag{4.90}$$

Yang–Mills $\boldsymbol{O_n}(\mathbb{K})$-Action

Imagine some massive body m. For simplicity, we grab the orthogonal group $O_n(\mathbb{K})$, and a scalar field $\digamma_{\text{и}}$. The Yang–Mills action, dropped in a perspective of algebraic & Lie groups, is

$$\mathscr{S}_{\text{YM}} = \int_{\mathbb{K}} \left(\sum_{\text{и}=1}^{n} \frac{1}{2}\partial_{\mu}\digamma_{\text{и}}\partial^{\mu}\digamma_{\text{и}} - \frac{1}{2}m^2\digamma_{\text{и}}^2 \right) d^4x. \tag{4.91}$$

Yang–Mills Topological Action, with Second Chern Number

Eq. (4.91) also has its own topological variant. Take a connection $\omega_{\mathring{\mathcal{P}}}$ on the principal G-bundle over a 4-manifold $\mathcal{M}$,[21] and let $\varOmega^{\nabla}$ be the curvature 2-form of $\omega_{\mathring{\mathcal{P}}}$. And here is a 4D-map,

$$\star : \varOmega^{\nabla}(\mathcal{M})_{\mathbb{R}} \to \varOmega^{\nabla}(\mathcal{M})_{\mathbb{R}}, \text{ i.e. } \varOmega^{\nabla}(\mathbb{R}^4) \xrightarrow{\star} \varOmega^{\nabla}(\mathbb{R}^4),$$

making use of the Hodge operator, under the now-familiar $\star$ (`\star`) symbol, again.

21. $\mathcal{M}$ is for $\mathcal{M}^4$ or $\mathfrak{M}^4$ viz. $\mathbb{M}^4$, that is, $\mathfrak{M}^4$ viz. $\mathbb{M}^4 = \mathbb{R}^4$.

If we connote the self-dual and anti-self-dual parts of $\varOmega^{\nabla}$ with

$$\varOmega^{\nabla}_{\pm} = \frac{1}{2}\left(\varOmega^{\nabla} \pm \star\varOmega^{\nabla}\right), ^{22} \tag{4.92}$$

and the second Chern number with

$$\mathring{c}_2 = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \operatorname{tr}\left(\varOmega^{\nabla}\wedge\varOmega^{\nabla}\right) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \left|\varOmega^{\nabla}_{+}\right|^2 - \left|\varOmega^{\nabla}_{-}\right|^2, \tag{4.93}$$

the Yang–Mills action, in the first instance, becomes

$$\mathscr{S}_{\mathrm{YM}} = \frac{1}{2}\int_{\mathbb{R}^4} (F\wedge\star F) = \frac{1}{4}\int_{\mathbb{R}^4}\left(\det(g_{\mu\nu})^{\frac{1}{2}} F_{\mu\nu}F^{\mu\nu}\right) d^4x, \tag{4.94}$$

by placing F in the guise of a 2-form, just like in Eq. (4.89); and, subsequently, it comes to be

$$\mathscr{S}_{\mathrm{YM}}[\omega_{\mathring{\mathcal{P}}}] = \int_{\mathbb{R}^4} \left|\varOmega^{\nabla}\right|^2 d^{-} = \int_{\mathbb{R}^4} -\operatorname{tr}\left(\varOmega^{\nabla}\wedge\star\varOmega^{\nabla}\right) \tag{4.95a}$$

$$= \left\|\varOmega^{\nabla}\right\|^2_{L^2} = \int_{\mathbb{R}^4}\left|\varOmega^{\nabla}\right|^2 d^{-} = \int_{\mathbb{R}^4}\left|\varOmega^{\nabla}_{+}\right|^2 + \left|\varOmega^{\nabla}_{-}\right|^2 d^{-} \tag{4.95b}$$

$$= \frac{1}{2}\int_{\mathbb{R}^4}\left|\varOmega^{\nabla}_{[+]}\right|^2 d^{-} + 8\pi^2\mathring{c}_2, \tag{4.95c}$$

by requiring that

$d^{-} = \det(g_{\mu\nu})^{\frac{1}{2}} = \sqrt{\det(g_{\mu\nu})}$ is the (Riemannian) volume element, [23]

L^2 is a (pre-)Hilbert-space with an inner product in the norm $\|\cdot\|$,

$\mathring{c}_2$ is directly related to the second Chern class $\mathring{\mathscr{C}}_2(\mathcal{M})$, [24] see S.-S. Chern [21] and A. Grothendieck [50].

Yang–Mills–Higgs Action on $\mathbb{R}^n$ through the Principal G-bundle

We start with the usual n-dimensional (pseudo-)Riemannian manifold $(\mathcal{M}, g)$. Let us say that $\omega_{\mathring{\mathcal{P}}}$ is the value of a gauge connection form—trivially known as *gauge connection*—on $\mathring{\mathcal{P}}$. A gauge transformation of a principal G-bundle

$$\pi_{\omega_{\mathring{\mathcal{P}}}} : \mathring{\mathcal{P}} \to \mathcal{M}$$

22. If $\varOmega^{\nabla}_{+} = 0$ a connection is remarked as anti-self-dual.
23. The $^{-}$ (\bbmu) symbol is for a measure on a space/on a set.
24. The $\mathring{c}_2$-number of $\mathcal{M}$ is but its Euler class; and $\mathring{c}_2$ may coincide with the so-called *instanton number*.

is a diffeomorphism $\varPhi \colon \mathring{\mathcal{P}} \to \mathring{\mathcal{P}}$, so that

$$\pi_{\omega_{\mathring{\mathcal{P}}}} \circ \varPhi = \pi_{\omega_{\mathring{\mathcal{P}}}}, \tag{4.96}$$

$$\varPhi(x \cdot g) = \varPhi(x) \cdot g, \tag{4.97}$$

for each $x \in \mathring{\mathcal{P}}$, and $g \in G$.

Let $\digamma_{H^0}$ be the Higgs scalar field, alias a space-time scalar $\digamma_{H^0}$-field. The reference articles are those of F. Englert & R. Brout [42], P.W. Higgs [59] [60] [61].

The Yang–Mills–Higgs (YMH) bundle action, for the quadruple $(\mathring{\mathcal{P}}, \pi_{\omega_{\mathring{\mathcal{P}}}}, \mathcal{M}, G)$, can be reported in such terms :

$$\begin{aligned} &\mathscr{S}_{\mathrm{YMH}}\left[\omega_{\mathring{\mathcal{P}}}, \digamma_{H^0} \;\middle|\; \mathring{\mathcal{P}} \xrightarrow{\varPhi} \mathring{\mathcal{P}}\right] \\ &\quad = \int_{\mathbb{R}^n} \left(\frac{1}{2} \varOmega^{\nabla} \wedge \star \varOmega^{\nabla} + \frac{1}{2} \nabla^{\omega_{\mathring{\mathcal{P}}}} \digamma_{H^0} \wedge \star \nabla^{\omega_{\mathring{\mathcal{P}}}} \digamma_{H^0} - \left(U \circ \digamma_{H^0}\right) dV_g \right) \\ &\quad = \frac{1}{2} \left\| \varOmega^{\nabla} \right\|^2 + \frac{1}{2} \left\| \nabla^{\omega_{\mathring{\mathcal{P}}}} \digamma_{H^0} \right\|^2 - \int_{\mathbb{R}^n} \left(U \circ \digamma_{H^0}\right) dV_g, \end{aligned} \tag{4.98}$$

by adopting $\nabla^{\omega_{\mathring{\mathcal{P}}}} \digamma_{H^0}$ as a covariant derivative, and $U \circ \digamma_{H^0} = U\left(\digamma_{H^0}\right)$ as the Higgs potential; $\varOmega^{\nabla}$ is, again, the curvature form of $\omega_{\mathring{\mathcal{P}}}$, whilst dV_g is for the volume form of g on $\mathbb{R}^n$.

Yang–Mills–Euler–Lagrange Equation(s)

The Euler–Lagrange equations (cf. Section A.) for the Yang–Mills theory have, in consequence, a brachylogical equality :

$$d_{\omega_{\mathring{\mathcal{P}}}} \star \varOmega^{\nabla} = 0. \tag{4.99}$$

This is not to say that all Yang–Mills $\omega_{\mathring{\mathcal{P}}}$-connections are solution of (4.99).

7. A FORM OF SPIN-TORSION INTERACTION ? THE CONUNDRUM OF THE DISCRETENESS (DISCONTINUITY), NODULARITY, AND SINGULARITY IN $\mathbb{R}^4_{\boldsymbol{\beta},\boldsymbol{\tau}(\mathbf{1},\mathbf{3})}$

So do the sums of this short survey. The transition from Einsteinian to Cartanian theory of gravity is only a *partial* conversion from the notorious mollusc to a tensor networks. Something like that can be dug up clearly in a reflection of P.W. Bridgman [2, p. 199] :

> The events, in terms of which the world is to be described in general relativity theory, are thought of as intersection nodes of the coordinate "mollusc" [40, §§ 28-29]. [25] No matter what the [space-time] transformation of [the four] coordinates, the intersection nodes cannot be transformed away, but persist in all systems, and it is this invariant background of nodes of intersection that corresponds to the physical "reality". But there is no general relativity theory of what the nodes represent. The implication seems to be that they represent some sort of discreteness or singularity in the solution of the underlying equations, and that there is nothing more to be said about the situation than the mere fact of the existence of the discontinuities.

But this has its own complication : the geometry of space-time, to fit quantum theory, must be Euclidean. Already M.P. Bronštejn [3, p. 150] = [4, p. 276] had clear ideas about it :

> [I]t is possible [...] to construct a completely self-consistent quantum theory of gravity within the framework of special relativity (i.e. when the space-time continuum is "Euclidean" [*raumzeitliche Kontinuum ein „Euklidisches" ist*]). However, within the domain of General Relativity theory, where deviations from "Euclideanness" can be arbitrary large, the situation is quite different.

Could the Bridgman-like nodes of intersection, or more elegantly, the tensor networks, with the entanglement amalgamating them together, be a physico-geometric path to illuminate a type of quantum gravity, showing that a smooth and continuous space-time may emerge from discrete bits of quantum information ?

But wait, there is more. The Cartanian spin-torsion coupling, in a Poincaré (see Marginalia 4..3) gauge theory of gravity (thru which the Einstein–Cartan space-time possesses torsion in addition to curvature), is inside

25. Cf. Einstein [40, § 28, pp. 65-66] : «This non-rigid reference-body [*nichtstarre Bezugskörper*], which might rightly be called a "reference-mollusc" [*„Bezugsmolluske"*], is essentially equivalent to any Gaussian 4-dimensional coordinate system. That which gives the "mollusc" [*„Molluske"*] a certain comprehensibility, as compared to the Gaussian coordinate system, is the (really unjustified) formal preservation of the separate existence of the space coordinates as opposed to the time coordinate [*formale Wahrung der Sonderexistenz der räumlichen Koordinaten gegenüber der Zeitkoordinate*]. Every point on the mollusc is treated as a space-point [*Raumpunkt*], and every material point which is at rest relatively to it is at rest, so long as the mollusc is considered as reference-body. The general principle of relativity demands that all these molluscs can be used as reference-bodies with equal rights and equal success in formulating the general laws of nature ; the laws must be entirely independent of the choice of mollusc».

a smooth framework; e.g. the (jet) $\mathcal{J}$-bundle (see Section C.) is imbued with a smoothness of points, lines, and surfaces; it is not with a series of Cartan-like bundles that it is possible to generate some discrete bits, or nodes. Perhaps it is coercive to change the type of tensor networks; an alternative for e.g. is the AdS/MERA (Multi-scale Entanglement Renormalization Ansatz) correspondence, which nonetheless has several deficiencies and ideological niggles.

A. How Far Can We Get without the Micro-scale ?

The grafting of the algebraic relation between the torsion and spin tensors into general relativity pays off when the density of matter is high, without having to get down to the Planck scale,

$$\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G_{\mathrm{N}}}{c^3}},$$

or to a quantum foam-like space-time, in keeping with Wheeler's [131, p. 509] [132, pp. 1-2, 6] grandiose audacity. Briefly, the Einstein–Cartan theory is a classical limit of a quantum gravity theory, still in fieri, when its torsional and spinor conception is accepted. Cf. M. Reuter, J. Gutenberg [109, 9.1. *Quantum Einstein–Cartan Gravity*, pp. 220-223].

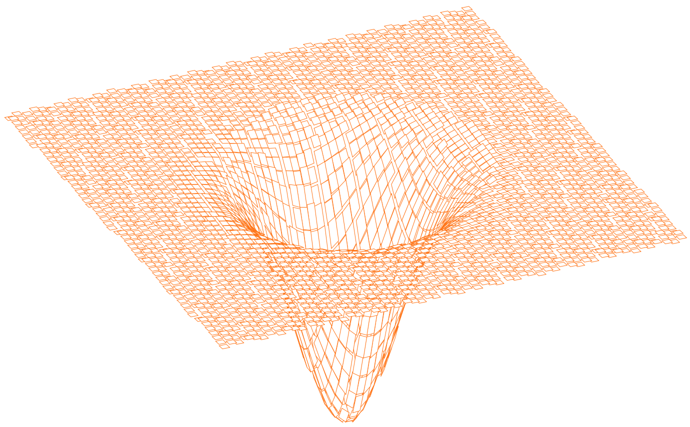


FIGURE 4.2 – Semi-destruction of a non-3D—or non-$(1+3)$D—spatio-temporal #FF7518-grid with increasing gravitational field strength. Exactly like in the Fig. 4.1, here is a crude representation along the z-axis. Actually, when the gravitational force becomes infinite (façon de parler), is mandatory to switch to an unknown tiling, whether periodic or aperiodic, as a covering of the plane; we must correspondingly return to a Flatland-like scenario

B. Infinitesimal Curvature and Torsion vs. Spongiform Topology or Fractal Geodesics

Do not forget that, in the Cartanian context, it is fair to talk of infinitesimal connection, and hence the same curvature & torsion are delineated in infinitesimal granules. But in the quantum world, where one fantasizes about a spongiform topology, this makes no sense, as far as we know.

That is why it is legitimate to conceive a gauge theories where space-time is a non-differentiable continuum but a fractal σχῆμα, as the scale relativity theory does, see L. Nottale and T. Lehner [95] [96].

The *focus argumenti* is that the continuum, both in the mathematico-physical and in the purely mathematical domain—see [93, sec. 9.2.1.1., Margo 9.2.1]—arises from the discontinuum, the discretum.

C. Multi-Dimensionality and Fractality of Fiber Structures

All of this brings up an old problem about dimensionality. There is a passage from B. Mandelbrot & R.L. Hudson [85, VII. *A Dimension to Measure Roughness*] that best sums up the *vexata quæstio* :

> Look at a ball of thread and think about it first from the idealized viewpoint of Euclid. Assume it is five inches in diameter, made of fiber a fraction of an inch thick. From a long distance away, you can barely see the ball ; it is, effectively, a point—of no dimension, according to classical geometry. Hold it in your hand, and it resolves to a normal, three-dimensional ball. Bring it up closer : You see it is a tangle of one-dimensional fibers. Closer still, and the fibers are clearly three-dimensional strands. Keep going until the atoms resolve in an electron microscope : Back to zero-dimensional points again. So what is this ball of thread, anyway ? Zero, one, or three dimensions ? It depends on your point of view. For a complex natural shape, dimension is relative. It varies with the observer. The same object can have more than one dimension, depending on how you measure it and what you want to do with it. And dimension need not be a whole number ; it can be fractional.

D. A Look Beyond the Hedge I : On the Quandary of the Curvature Tending toward Infinity

This is why the Einstein–Cartan theory

(1) is applied to black holes, where the (spin-)torsion of the geometry of the space-time fabric—in which the individual threads in weft and warp are

intertwined—is assumed to be relevant. Not without a certain complacency, any physico-mathematical speculation, that wants to bypass the narrow scope of empirical evidence (in the absence of experiments to verify some theoretical prediction), has the same pace as a novelist's imagination ; [26]

(2) is probed to generate models, for avoiding singularities in the presence of astronomical objects capable of generating an extreme deformation of space-time, whose curvature tends toward infinity. See (in order of appearance) A. Trautman [126], B. Kuchowicz [70], N.J. Poplawski [103] [104] [105] [106] [107], S. Desai and N.J. Poplawski [30].

E. A Look Beyond the Hedge II : Black Holes for a Non-singularity

See the #E30382-point in Fig. 4.3, where the spiraling #003153-torsion fatally kisses the spiraling #DAA707-torsion. The #E30382-entity is a *math-free spot*, or a *primitive non-space* in which the physico-mathematical dimension is without comprehension. The Einstein–Cartan construction helps to find a *narrativum artificium* for this bewitching quandary.

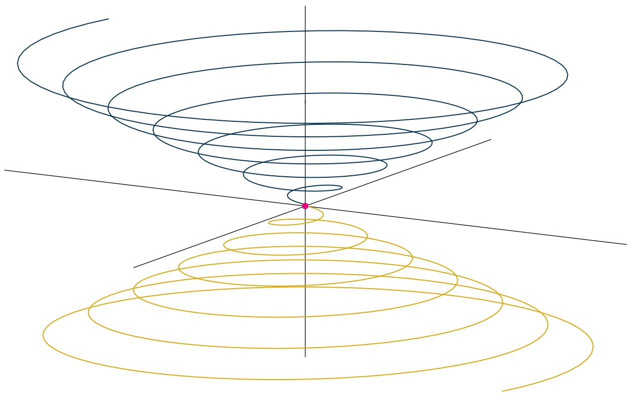


FIGURE 4.3 – Graphic simplification of a Cartanian non-3D—or non-$(1+3)$D—space-time with double spiraling torsion. Try to visualize, imaginatively, an action à la Petti–Poplawski [99] [107] of a #003153-black hole, on one side, and of a #DAA707-white hole, on the other.

26. Never hurts to remember these words of A. Einstein & L. Infeld [41, p. 33] : «Physical concepts are free creations of the human mind, and are not, however it may seem, uniquely determined by the external world».

F. In-depth Explorations

(1) Investigations on the Yang–Mills & gauge theories, are (abc order) in R. Cianci, S. Vignolo, and D. Bruno [23] [24], R.W.R. Darling [29, chap. 10. *Applications to Gauge Field Theory*, pp. 223-250], S.K. Donaldson [31], C. Doran, A. Lasenby, A. Challinor, and S. Gull [32], T. Eguchi, P.B. Gilkey and A.J. Hanson [33], M. Göckeler, T. Schücker [49, chap. 4. *Gauge theories*, pp. 43-60], C. King [66], J. Labastida and M. Marino [71, sec. 2.1. *Yang–Mills theory on a four-manifold*, pp. 12-14], F. Lenz [77], J.W. Morgan [90], G.L. Naber [91], G. Rudolph, M. Schmidt [120, chap. 6. *The Yang–Mills Equation*, pp. 461-543], L. Sadun and J. Segert [122].

(2) Insights on the Einstein–Cartan theory and surrounding areas, are (abc order) in S. Capozziello, R. Cianci, C. Stornaiolo, S. Vignolo [5] [6], R.T. Hammond [54] [55] [56], F.W. Hehl, P. von der Heyde, and G.D. Kerlick [57], F.W. Hehl and Y.N. Obukhov [58], H. Kleinert [69], A. Lasenby, C. Doran and S. Gull [76], E.W. Mielke [86, chap. 5. *Einstein–Cartan Theory*, pp. 95-107], R.J. Petti [98] [99] [100], N.J. Poplawski [108, sec. 2.5.1], W.A. Rodrigues, Jr. and E.C. de Oliveira [119], V. de Sabbata [121], I.L. Shapiro [124].

NOTE. The three figures 4.1, 4.2, and 4.3 are build by availing the `tikzpicture` code, and subsequently enhanced with `Sketch`, a vector graphics editor.

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