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On recent applications of the theory of scale relativity in hydrodynamical turbulence

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RÉSUMÉ

On recense ici des travaux communs en collaboration étroite avec Laurent Nottale qui a fondé et développé la théorie de la relativité d'échelle (notée RE en abrégé) depuis près de quarante ans (voir en référence deux livres techniques cités de Laurent Nottale). Nous revenons d'abord sur les principes de la RE et sur son application à la fondation de la mécanique quantique dans l'espace des positions avant de décrire ensuite comme application des nouveaux résultats récents qui ont été obtenus en turbulence des fluides cette fois dans l'espace des vitesses, en précisant d'abord quels sont les problèmes actuels qui restent encore ouverts en turbulence hydrodynamique, puis de montrer ce qu'apporte de nouveau la RE dans ce domaine dont on en donnera trois applications concrètes sur respectivement :
1) la turbulence homogène et isotrope à trois dimensions ;
2) la turbulence en rotation pertinente en géophysique et en astrophysique ;
3) l'étude des flots cisaillés comme le jet turbulent.

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ABSTRACT

We review here common works in straight collaboration with Laurent Nottale who has funded and developed the theory of scale relativity (SR in brief for over forty years (see in reference two technical quoted books of Laurent Nottale). We make first a recall on the underlying principles of SR and on its application to the foundation of quantum mechanics in the space of positions before to describe next as applications new recent results obtained on fluid turbulence now in the space of velocities. We shall prior list known open problems in hydrodynamical turbulence and show what are the novelties brought in by the SR theory in this field by giving three concrete applications on respectively:

1) The homogeneous and isotropic three dimensional turbulence
2) The rotating turbulence pertinent in geophysics and astrophysics
3) The study of shear flows such as the turbulent jet.

1. INTRODUCTION

Our plan of exposition will be the following:

2) On epistemology about scale relativity
3) On the principles and foundations ot the scale relativity theory (SR)
4) Implementation of SR in position space: foundation of quantum mechanics
5) On the status of quantum mechanics viewed by scale relativity
6) Some open problems in fluid turbulence
7) Implementation of SR in velocity space: application to turbulence
8) Application: choice of three selected papers and comments on their main results
9) Provisory conclusions

2. EPISTEMOLOGY

Very interesting novelties for epistemololgy are emerging within the scale relativity theory (SRT). First of all the general remark that physical systems in general are not scale invariant is pertinent here: since in SRT the underlying fractal structure space-time has no reason to be self-similar: the fractals here are not always self similar. One fundamental property of fractals is precisely their implication of scale dependence. This idea of scale dependence is also particularly useful when one is dealing with hydrodynamical flows in the current multi-fractal approach of turbulence. as opposed to the case of mono-fractal systems. Thus in general the observation of a given system will also depend at which scale it is considered. In SRT the scale dependence is made explicit rather than using internal measurements of dimension equal to the fractal dimension, which may be hindered. This is true for both the self similar ad non self similar cases. Another directly related property is that the resolution associated with any measurement is also scale dependent.

Another fundamental feature in SR is the possibility to describe physical systems which show quantum like properties but at a macroscopical level with a diffusion constant specific for a given system but which is distinct from the Planck constant ($\hbar$) suitable in the microscopic case. For a deep comprehensive epistemological approach induced by SRT one can refer to the work of the philosopher V. Bontems[[1]].

3. ON THE PRINCIPLES AND FOUNDATIONS OT THE SCALE RELATIVITY THEORY

In this introduction we recall first shortly some essential features of the scale relativity theory (SRT), while a much more detailed exposition can be found in L. Nottale books [[2],[3]]. The SR theory relies first mainly on the principle of relativity. We shortly recall the relativity principle which assumes that the law of physics are expressed identically in two different inertial frames : these laws are invariant by a change of inertial frame. In general relativity it is called the principle of covariance with the same content but valid now in any frame (inertial or not). The laws are then said to be covariant. It should be noticed that the general philosophical assumption is the universality of the laws of Nature whatever is the system considered. In physics we assume that these laws can be suitably represented by equations and the principle of covariance sets that the laws are invariant within a change of system of coordinates. The general purpose of the theory of scale relativity (SRT) aims at the description of a non-differentiable continuum by founding itself on the relative character of any scale (namely, one can never define an absolute scale, only scale ratios have physical meaning). It is based on a fundamental theorem according to which a continuous and nondifferentiable space is fractal, i.e. explicitly dependent on the resolution scale and divergent when this scale tends to zero. This concept can be extended to a space-time and not to space only in a relativistic version of the theory. The principle of relativity is also now extended in SR in order to include the spatial $\delta x_i, (i = 1,2,3)$ and temporal δt resolutions as well which should now transform as do their associated coordinates (x,t) in a change of inertial frames. Namely if the coordinates are (x,t) in a given frame R and becoming x',t' in another frame R' thus the resolutions should also follow the same law of transformations when going from $(\delta x, \delta t)$ to $(\delta x', \delta t')$. However coordinates can be put into vectors while resolutions might become tensor objects. In a fractal -space time a function $f(x,t)$ becomes explicitely scale dependent as $g(x,\delta x,t,\delta t)$ in a frame R and by a change of inertial frames we thus get a function $g'(x', \delta x',t', \delta t')$ in R'. The function g can exhibit also an intrinsic fractality character but anyway it becomes here fractal at least through its coordinates themselves being now fractal (scale dependent) variables. Besides it should be noticed that any measure in physics on a given variable is always performed with a finite resolution, which justifies the necessity to handle the resolutions of the variables as well as the variables themselves. It is possible to define the notion of derivative also for a fractal function but we shall not develop this point in detail here. Also the derivation can be defined and applied to the scale resolutions as well. The general idea of a non differentiable continuum space time was already put forward by J.A Wheeler starting in the 1960's with his consideration of a quantum foam at small scales for the four dimensional space- time. This idea has made progresses over time and for instance recently studies on two dimensional quantum gravity using the notion of causal local triangulation (by the help of simplexes in interaction in order to triangulate the space-time into cells) have shown that the space-time behaves indeed like a fractal at small scales with a fractal dimension $D_F = 2$. The SRT relies on this same hypothesis (see for example the reference [2]). However in quantum gravity it is assumed that the fractality of space time is present only at the very small Planck scale lp ($lp = \hbar(G/c^3)^{1/2}= 1.6 \cdot 10^{-35} m$), while in SRT the fractality exists at all scales and it manifests at small scales till the de Broglie length for a given particle (see below). In SRT at each point we have an infinite curvature due to the non differentiability.

A. Covariant derivative in SR, link with the covariant derivative in general relativity

The generalized covariant derivative in SR can be constructed geometrically at different levels in analogy with the one which is used in general relativity involving the curvature of space-time. But now the fractality (with $D_F = 2$) implies a local dedoubling of derivatives at each point and the emergence of stochastic underlying processes leading to a diffusion coefficient (or variance) and then to a term in $D\triangle$ in this covariant derivative , where $\triangle$

is the Laplacian operator in three space dimensions). This covariant derivative reads, $V*$ here being a complex velocity due to the local dedoubling, as follows:

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}*.\nabla - i\mathcal{D}\Delta. \tag{5.1}$$

One notices that the two first terms of this derivative in the right hand side are built on the same ground than the usual Eulerian derivative with a convective term in V*.grad , while the total structure of this derivative is close to an usual Schrodinger operator in $id/dt + D\triangle$ (more precisely it will appear of the kind $i(D)d/dt + D^2\triangle$). But the diffusion coefficient D is equal to $\hbar/2m$ (m particle mass) only in the case of usual quantum mechanics in the microscopic domain. The fact to have a possible different D opens the path to behaviors like in quantum dynamics but at a macroscopic level. It can be shown that this derivative given by (5.1) is covariant according to the principle of relativity as recalled above.

-Notion of fractal (virtual) geodesics

The equation $\frac{\widehat{d}}{dt}V* = 0$ defines the equation of the geodesics for a particle of velocity $V*$. This is the manifestation of the principle of equivalence resulting from the relativity for a peculiar object. At this level the particle will be formed through its network of (virtual) geodesics. While particles with zero mass like photons can be treated as usual particles (as bosons), in addition the origin of the electromagnetic field can be understood as fields of dilatation and contraction transformations of the space-time (for example emanating from electrons).

In the case of no particle the vacuum itself of the space -time can be considered as well as being fractal. The nature of the fractal(s) is unknown but it may be coming from any or many different non self similar structures.

The covariant derivative is the manifestation of the presence of underlying stochastic processes, like Brownian motion for example, as it will be seen below. We shall see later on that the explicit writing of the covariant derivative (with V*) applied on the same V* allows to transform a Newton equation of motion (or a Navier-Stokes equation for a fluid) into a Schrodinger equation. Indeed the SRT allows to obtain Schrodinger like equations in various situations (see [[2],[3]]).

Other remarkable properties are put forward by SRT, they are described in ref [[2],[3]], we just recall some of them in the following:

-the emergence of a scale of transition fractal-non fractal at the de Broglie wavelength ($\lambda_{dB} = \hbar/p$) of a given particle of momentum p. The fractal space being at the small scale below λ_{dB}. This wavelength may be generalized in the case of vacuum by introducing the vacuum energy.

-the emergence of a stochastic potential (noted as Q) as a potential energy of the fractal space-time.

-If we do not take the limit of zero resolutions for dt or dx we can keep the memory of the motion of given stochastic 'trajectories'.

-Chaos and hyper -chaos may appear beyond relevant integral scales, but SRT is acting beyond the usual chaos.

-SRT can appear in x (like in QM) but also in v space (like in turbulence) or even in both x space and v space with a transition inbetween the two behaviors according to the relevant scales (this can seen in the case of turbulence).

-Structure of the space-time can be explored for example as toy model according to the choice of an arbitrary selected fractal set (example: with Cantor sets).

-Conditions for possible application of SRT will be given later on.

4. IMPLEMENTATION OF SR IN POSITION SPACE: FOUNDATION OF QUANTUM MECHANICS

The initial purpose of SRT was to derive quantum mechanics (QM) from the principle of relativity, by going beyond the principle of correspondence which is usually used to go from classical mechanics to quantum one. We recall here the SRT in position space by a short technical reminder. From Newton to Schrödinger equation: The laws of motion are obtained in this theory by writing the fundamental equation of dynamics in a fractal space (more generally, in a fractal space-time, but only fractal space will be considered in the present work). In the absence of exterior field, it reduces to a geodesics equation, which takes (in terms of a covariant derivative that describes the effects of the fractal geometry), the form of the equation of free Galilean motion in empty space, $\Gamma = 0$ (Christoffels symbol describing non zero curvature effects).

The non-differentiability and the fractality of coordinates implies at least three consequences [[2, 3]]:

(1) The number of possible paths is infinite. Their description therefore naturally becomes non-deterministic and probabilistic. These virtual paths are identified to the geodesics of the fractal space. The ensemble of these paths constitutes a fluid of geodesics, which is characterized by its velocity field.

(2) Each of these paths is itself fractal. The velocity field is therefore a fractal function, i.e., a function that is explicitly dependent on resolutions and divergent when the scale interval tends to zero (this divergence is the manifestation of non-differentiability).

(3) The non-differentiability also implies a two-valuedness of the derivative of the coordinates, $(V_+,\ V_-)$, i.e. of the velocity when these coordinates are space coordinates (but it may be any fractal variable). Indeed, two definitions of the velocity field now exist, which are no longer invariant under a transformation $|dt| \to -|dt|$ in the nondifferentiable case.

These three properties of motion in a fractal space lead to describing the velocity field of geodesics in terms of a complex fractal function $\widetilde{\mathcal{V}} = (V_+ + V_-)/2 - i(V_+ - V_-)/2$. The $(+)$ and $(-)$ velocity fields can themselves be decomposed in terms of a differentiable part $v_\pm$ and of a fractal (divergent) fluctuation of zero mean $w_\pm$, i.e., $V_\pm = v_\pm + w_\pm$. Therefore the same is true for the full complex velocity field, $\widetilde{\mathcal{V}} = \mathcal{V}(x,y,z,t) + \mathcal{W}(x,y,z,t,dt)$.

Jumping to elementary displacements along these geodesics, this reads $dX_\pm = d_\pm x + d\xi_\pm$, with (in the case of a critical fractal dimension $D_F = 2$ for the geodesics)

$$d_\pm x = v_\pm\ dt, \quad d\xi_\pm = \zeta_\pm \sqrt{2\mathcal{D}}\,|dt|^{1/2}. \tag{5.2}$$

When it is applied to standard quantum mechanics, the constant $\mathcal{D} = \hbar/2m$ is just a re-expression of the Planck constant. In the scale-relativity approach it is generalized to a macroscopic constant of dimension $[L^2\mathrm{T}^{-1}]$ homogeneous to a viscosity and emerging from the properties of the fractal medium.

This case is particularly relevant since it corresponds to a Markov-like situation of loss of information from one point to the following, without correlation nor anti-correlation. Here $\zeta_\pm$ represents a dimensionless stochastic variable such that $<\zeta_\pm>= 0$ and $<\zeta_\pm^2>= 1$. The parameter $\mathcal{D}$ characterizes the amplitude of fractal fluctuations.

These various effects can be combined in terms of a total derivative operator [2, 3] which generalizes the Euler derivative to a fractal space with (5.1) above. Newton's fundamental equation of dynamics becomes, when it is written in terms of this operator

$$m\,\frac{\widehat{d}}{dt}\,\mathcal{V} = -\nabla\phi. \tag{5.3}$$

In the absence of an exterior field ϕ, this is a geodesic equation (i.e., a free inertial Galilean-type equation),

$$\frac{\widehat{d}}{dt}\mathcal{V} = \left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right)\mathcal{V} = 0. \tag{5.4}$$

The next step consists of making a change of variable in which one connects the velocity field $\mathcal{V} = V - iU$ to a complex function $\psi = e^{iS/S_0}$ (where S is the action, now complex because the velocity field is itself complex), according to the relation

$$m\mathcal{V} = -i\,S_0\,\nabla \ln \psi. \tag{5.5}$$

This equation is but the standard relation between momentum and action $P = \nabla S$, that provides a new expression (now exact) for the principle of correspondance. The parameter S_0 is a constant for the system considered (it identifies to the Planck constant $\hbar$ in standard quantum mechanics). Thanks to this change of variables, the equation of motion can be integrated under the form of a Schrödinger equation [2, 3], generalized to a constant which may be different from $\hbar$,

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\phi}{2m}\psi = 0, \tag{5.6}$$

where the two parameters introduced hereabove, S_0 and $\mathcal{D}$, are linked by the relation:

$$S_0 = 2m\mathcal{D}. \tag{5.7}$$

By setting finally $\psi = \sqrt{P} \times e^{i\theta}$, with $V = 2\mathcal{D}\nabla\theta$, one can show [11, 3] that $P = |\psi|^2$ gives the number density of virtual geodesics. This function becomes naturally a density of probability when the geodesics are manifested in terms of effective particles. The function ψ, being solution of the Schrödinger equation and subjected to the Born postulate and to the Compton relation, owns therefore all the properties of a wave function.

Further studies in SRT have established also the Pauli and the Dirac equations as well.

5. ON THE STATUS OF QUANTUM MECHANICS VIEWED BY SCALE RELATIVITY

We enumerate here some specific features of QM in SR to be compared with the usual view of QM. First of all the structure of the space time underlying the QM is defined since it is set to be fractal from the beginning, while its structure is assumed to be Minkowskian in usual QM. One important item connected to this, when the nature of the fractal is specified, is that one needs to generalize the Fourier transform when going from x space to p (momentum space) or conversely. Also the notion of indeterminism can be explained here: if a fractal structure is given/chosen (by its equation) as a toy model one can say that the space-time is hence deterministic, but not the geodesics since due to non differentiability they become fractal at each point and hence unpredictable. The wave function is no more postulated in SR like in usual quantum mechanics (QM) but it is now built as a manifestation of the fluid of virtual geodesics, quite in the spirit of the Feymann path integral approach. The fundamental relation which bridges the algebraic tool of QM to the geometric tool of SR is the following fundamental relation:

$$V = -2iD\nabla \ln \psi. \tag{5.8}$$

Where $\Psi(x,t)$ is the wavefunction and D is the diffusion coefficient already introduced above. This last relation comes from the usual definition of the velocity in mechanics by the relation $V = grad(S)/m$, where S is the action on a given particle. On the r.h.s of this equation Ψ is a primordial variable, without microscopic internal theory explaining its structure. On the l.h.s of the equation we have the field velocity of the geodesics. Thus this field contains only a part of the information about these geodesics which are themselves forming the internal structure of the wave function. The notion of internal to the wavefunction in SR means that we are set on one of these geodesics. A possible analogy could be made wih the Einstein general relativity equations: $Gij = \kappa Tij$. The matter distribution here is described by a rich complexity but only its impulsion-energy tensor Tijintervenes to create the Einstein tensor Gij and thus the curvature of space-time.

Other features worth to be noticed are the following: In SRT all axioms of the QM are demonstrated: - correspondence principle: it becomes strict equation instead of a mere correspondence and it is proved;
- Schrödinger equation (and Pauli, Klein-Gordon, Dirac): are demonstrated instead of set as postulated;
- Born axiom $P = |\Psi|^2$ is proved;
- the QM of identical particles is recovered.
- Entanglement is accounted for as property issued from the fact that even a single "particle" is constituted from an infinity of fractal geodesics.
- Young slit experiment / interferences: this is understood as a property of the infinite set of fractal geodesics;

- spin is pure quantum property without classical counterpart: it is explained as a property of fractal spiral geodesics.

6. SOME OPEN PROBLEMS IN FLUID TURBULENCE

Turbulence remains a complex problem still mis-understood in physics; its consists in a spatio-temporal chaos involving a large number of degrees of freedom. Pseudo- random processus are at work but there are correlated inbetween in a subtle and still hidden way, that remains to be discovered better. We list here some open problems in this field on which SRT is hoped to contribute to shed some light. The equation to be solved is the celebrated Navier Stokes (noted NS) equation which reads, here in the incompressible case for which the density $\rho = cte$ thus we get the relation $div(v) = 0$, where $f(x,t)$ is a possible external (stochastic or deterministic) force or forcing term, ν being the fluid viscosity and $v(x,t)$ the fluid velocity:

$$dv/dt + (v.grad)v = -grad(p)/\rho + f + \nu \triangle v. \qquad (5.9)$$

This equation has to be completed by initial and boundary conditions. One can define the Reynolds number Re as the ratio of the non linear term to the dissipative term: $Re = /(v.gradv)/\nu \triangle v/$ According to the value of Re the flow can be in different regimes: in the laminar regime (Re low), at the transition to turbulence (Re intermediate) or in full developed turbulence (for large Re). According to us three important open problems at least are remaining unsolved in turbulence, concerning the following related questions: the existence of energy cascades, the presence of anomalous dissipation and the phenomenum of intermittency.

A. Possible existence of energy cascades

One should be aware that even the hypothesis invoked of existence of cascade processes involving the rate of energy transfer between scales (noted ϵ) is only postulated in turbulence but it has not yet been demonstrated from the NS equations, but it seems to be well observed in both experiments and numerical simulations at least for the energy spectrum. However one should make an important distinction between what may occur in physical space and what is meaningful in a statistical approach when performing ensemble averages. Besides other mechanisms can yield the same energy spectrum than the one predicted by the Kolmogorov, with [[5]] and [[6]] theories but without appealing to any cascade process. For instance recently we have designed possible tests for detecting cascades and have applied them on real turbulent data to study the energy transfer among scales but in presence of possible singularities (for example as explosion of solutions in v in finite time starting from smooth initial data); it should be noticed that with singularities only energy transfer may exist but without even the need of invoking a specific cascade mechanism[[7]]. Also for example two different systems may exhibit a quite similar energy spectrum but with otherwise very different physical properties. It seems obvious that the knowledge of only the second order structure function (correlations of velocities) is not sufficient to characterize entirely a given system.

B. Anomalous dissipation

If one looks again at the rate of energy transfer which is just the time derivative of the kinetic energy density Ec ($Ec = \rho v^2/2$), one observes that this rate decreases as a function of the Reynolds number Re, but above a (universal) threshold value for Re, it reaches a saturated plateau constant value. This phenomenum is still poorly understood, but it could be due to 'inertial' damping by opposition to usual dissipation by viscosity, but now occurring in the inertial range , which is defined for scales l as follows ($l_K << l << L$), l_K being the Kolmogorov scale ($l_K = (\nu^3/\epsilon)^{1/4}$) and L the integral scale of the flow, often typically equal to the dimension of the device. Indeed mathematicians have shown that this inertial dissipation, they have noted as Dis(v), vanishes in the energy balance for smooth solutions of the hydrodynamical equations, but the term Dis(v) could become non zero for 'weak' solutions (like the self-similar Leray weak solutions) of the NS equation. If we set $\epsilon = cte$, for stationary states by equating/ balancing this rate to the energy dissipation rate defined by: $\epsilon_l = \nu grad(v)^2$ we see that grad(v) should diverge when the viscosity goes to zero, thus indicating possible singular behavior for the velocity.

C. Intermittency

The intermittency consists in the deviations to the Gaussian law for the distribution of random processes, the latter law being valid for perfectly uncorrelated random ones, for instance with the observation of long non vanishing tails of the probability distribution functions of the velocities and above all of the accelerations in the fluid. Physically intermittency refers to a spatio-temporal structure exhibiting alternating phases of small scales of intense activities with large resting ones. The smaller is the involved scale, the stronger will be the intermittency, but whose spatial structure is still remaining very complicated; It is thus useful to design proper methods for its visualization. Various mechanisms for explaining intermittency have been put forward. For example the Dis(v) term just quoted above could play here a crucial role. It can be shown that singularities, of the Leray self similar kind with Sedov-Taylor exponents for energy conservation, could bring a decisive contribution to this mechanism [[8]]. But the nature of the different kind of singularities and their eventual co-existence remain still an open problem. In the scale relativity approach, the intermittency in turbulence is explained by the new contribution of an acceleration Aq which diverges at points where the velocity pdf ($P = |\Psi)|^2$) goes to zero. It is also interpreted in terms of 'macro' quantum jumps in-between local harmonic oscillator like potential wells in velocity space, see below in section 8 ([9]).

7. IMPLEMENTATION OF SR IN VELOCITY SPACE: APPLICATION TO TURBULENCE

A. Stochastic 'seed' for turbulence

Contrary to the stochastic seed seen for QM ($\delta x \sim \delta t^{1/2}$) coming from the fractal character of space-time, here a fluid becomes non differentiable in velocity space v when it is turbulent and then it develops its own fractality mainly in this space. Now the fractality will be manifested by the matter itself and not by the space-time as it was considered initially. As in superconductivity the medium (matter) plays a similar role than space for the objects which are imbedded into it. In inertial domain range, fundamental scaling relations have been found by A. Kolmogorov for velocity increments, with $\delta v \sim \delta x^{1/3}$ (Eulerian case) and $\delta v \sim \delta t^{1/2}$ (Lagrangian case). In a similar way, fractality with fractal dimension$D_F = 2$ in velocity space is expressed by this last relation. This is just the universal Kolmogorov [[5]] scaling law in Lagrangian representation. It can be obtained by simple dimensional analysis based on the assumption that the various scale dependences be driven by the ϵ (it has units of density of energy/time) between scales in a turbulent postulated cascade, which is also the energy finally dissipated into heat at Kolmogorov viscous scales ($l < l_K$). One should notice that this relation $\delta v \sim \delta t^{1/2}$ is indeed more general than the scaling derived just above in the Lagrangian case. It behaves like the scaling $\delta x \sim \delta t^{1/2}$ for a Brownian like motion in the usual position space. Besides the use of stochastic differential equations (SDE's) relying on these scalings is now a standard method used in turbulence studies, when the position x or/and the velocity v are becoming random processes.

B. Motivations to apply SRT to turbulence

They can be given as the followings: On open question was to find such a suitable medium in natural systems or to build it in a laboratory experiment. However, such systems are expected to be fractal only on a limited range of scales. Two additional constraints should be added to the three above conditions for manifesting such a macroscopic Schrödinger regime: (iv) a large enough range of fractal scales (Ref. [[3]], Chap. 10) and (v) Newtonian dynamics.

Indeed, in the application of the theory to quantum mechanics, we considered that space-time was fractal and non-differentiable below the de Broglie scale, without any lower limit, and the dynamics is naturally Newtonian. However, when it is applied to a fractal medium instead of a fractal space, a lower scale is expected for its fractality, and the diffusive aspects of the medium may involve a Langevin-type dynamics instead of a Newtonian one. This reduces the number of systems where such a new physics should be implemented.

Another strong argument for the application of the scale relativity approach to a fully turbulent fluid is that it is directly adapted to the Lagrangian description of such a fluid (and therefore to a comparison with Lagrangian experiments). Indeed, one obtains the Schroödinger-type description by identifying the wave function with a manifestation of the velocity field of fractal geodesics.

Numerical simulations of fractal geodesics (Ref. [[3]], Chap. 10) have been performed in the context of standard quantum mechanics. They have allowed us to recover the probability densities which are solutions of the Schrödinger equation in a direct way, with- out writing it explicitly. These simulations anticipate the application of scale relativity to turbulence. Indeed, in Lagrangian turbulence experiments, one follows Lagrangian small particles which are considered as valid tracers of the fluid elements. In our framework, the trajectories of these tracers can then be considered as concrete manifestations of the virtual fractal geodesics introduced in the scale relativity approach.

C. SR description versus turbulence

The application of the SRT [2, 3] to turbulence, is supported by many elements. Indeed, let us recall the various ingredients of this theory and put them in correspondence with some recognized characteristics of turbulent fluids.

- **Scale dependence**. The SRT aims at describing systems which are explicitly dependent on scales. It is well known that this is just the case for a fully developed turbulent fluid.
- **Scale variables**. In SRT, one describes this scale dependence through the introduction of one or several scale variables. For example, a standard time-dependent function $f(t)$ is replaced by a two-variable function $f(t,\delta t)$ depending on time t and time-scale δt.

  In turbulent fluids, it is well-known that several physical quantities are explicitly scale dependent in the inertial range, i.e., from the Kolmogorov dissipative small scales (l_K) to the integral large scales (L) where the energy is injected. For example, in the Lagrangian description in terms of fluid particle trajectories, the accelerations measured for small test particles are explicitly dependent on the time interval $\tau = \delta t$.
- **Relativity of scales**. The theory of scale relativity relies on the fact that the various scales are not absolute, but only relative. Indeed, only ratios of scales do have a physical meaning, not a scale by itself. This new relativity is therefore expressed in terms of multiplicative groups instead of the usual additive groups of motion relativity. However, the relevant scale variables being actually given by logarithms of scale intervals ratios, e.g. $\ln(\tau/T)$, one recovers standard additive groups in terms of these logarithmic variables.

  The study of turbulent fluid just involves a scale description in terms of such variables, for example $\ln(\tau/T_L)$ or $\ln(\tau/\tau_\eta)$ in Lagrangian representation, where T_L is the Lagrangian integral time-scale and τ_η the Kolmogorov dissipative time-scale. The scale relativity aspect of turbulence is manifested by the need of such reference scales in the definition of the scale variables, since the dimensioned scale interval $\tau = \delta t$ has no meaning in itself, but only the ratio between this scale and the reference scale.
- **Chaotic trajectories**. The application of the scale relativity theory to the macroscopic realm is specific of chaotic systems, at time-scales larger

than their horizon of predictibility [2, Chap. 7.2]. On these timescales (larger than about 10 to 20 Lyapunov times), the strict determinism is lost and one is led to use a stochastic description. This ensures the first condition underlying the scale relativity description, according to which there is an infinity (or at least a very large number) of possible trajectories whatever the initial conditions.

It is well known that the fluid element trajectories in a turbulent fluid are chaotic. In effect, the predictibility of individual trajectories is lost after some Kolmogorov times.

— **Scale laws**. One of the three main conditions upon which the scale relativity description relies is the fractal dimension $D_F = 2$ of trajectories. In position space, it is expressed by the fact that the space increments and the time increments are no longer of the same order, since $\delta x \sim \delta t^{1/2}$.

In a similar way, fractality with dimension $D_f = 2$ in velocity space is expressed by the relation $\delta v \sim \delta t^{1/2}$. This is just the universal Kolmogorov scaling law in Lagrangian representation [[5]]. Recall that it can be obtained by simple dimensional analysis based on the assumption that the various scale dependences be driven by the mere energy ε transferred between scales in the turbulent cascade (which is also the energy finally dissipated into heat at Kolmogorov viscous scales). In Eulerian representation, one finds $\delta v \sim \delta x^{1/3}$.

— **Irreversibility**. The third condition on which the obtention of a Schrödinger-type equation relies in the theory of scale relativity is local irreversibility. In the new description, velocities are fractal functions, i.e. explicitly scale dependent functions $v(t,\delta t)$. Their derivative (the acceleration) must be defined from two points, the second point being taken after [i.e., from the velocity increment $v(t + \delta t, \delta t) - v(t,\delta t)$] or before the initial point [$v(t, \delta t) - v(t - \delta t,\delta t)$]. There is no *a priori* reason for the two increments to be the same, which leads to a fundamental two-valuedness of the acceleration (that we describe in terms of complex numbers).

It is widely known that the trajectories of fluid elements in a turbulent fluid are irreversible. Here this local irreversibility takes a new meaning, when it is accounted for by this doubling of the acceleration vector and combined with the fractality of trajectories in velocity space.

— **Newtonian regime**. As we have recalled, we need Newtonian dynamics (linking the force to the second derivative of the variable) to obtain a Schrödinger form of the motion equation from fractality and nondifferentiability. A Langevin regime (in which the action of a force is a velocity instead of an acceleration) does not yield this result [3, Chap. 10].

The basic equations of fluid mechanics are the Navier-Stokes equations which are clearly of Newtonian nature (i.e., they involve the second derivative of the variable), even if they contain a dissipative viscous term. The same is true after jumping to velocity space: the basic variable becomes the velocity vector and the equation of dynamics is just the time derivative of the Navier-Stokes equations.

— **Range of fractal scales**. The last condition is that the range of scales involving a $D_f = 2$ fractal-type behavior be large enough for the relation $\delta v \sim \delta t^{1/2}$ be fulfilled, at least in an effective way [3, Sec. 10.3.2].

In turbulent fluids, this means establishing the conditions under which the K41 scaling can be observed. The range of scales where it manifests itself is the inertial range, which is limited by the Kolmogorov dissipative scale τ_η and the integral scale T_L. As recalled in the introduction, their ratio is given by:

$$R_\lambda = 15(Re)^{\frac{1}{2}} \approx \frac{T_L}{\tau_\eta} = \frac{R_\lambda}{2\,C_0}, \tag{5.10}$$

i.e., according to the estimated values of $C_0 = 4$ to 7 , the range of scales is $\approx R_\lambda/10$. The transition to fully developed turbulence is estimated to be at $R_\lambda \approx 500$, yielding a scale ratio 50, while $R_\lambda \approx 1000$ yields $T_L/\tau_\eta \approx 100$, as experimentally observed. A well defined (effective) Kolmogorov regime is observed under these conditions.

D. Experimental support for the application of SRT to turbulence

Infinite number of virtual trajectories

In a turbulent fluid, the trajectories in v-space are no longer deterministic. Namely, under the same initial conditions of velocity and acceleration (v_0, a_0) [and more generally (x_0, v_0, a_0)], the subsequent evolution of a fluid particle is not determined on time-scales larger than the Kolmogorov dissipative time-scale τ_η. This is supported by all turbulence experiments such as Lagrangian-type experiments where one follows small particles considered to be valid tracers of the fluid particles. In von Karman contra-rotative experiments, its has been shown that particles of size $<\approx 100\ \mu$m achieved such valid tracers.

If we consider trajectories in v-space starting from nearby initial conditions in velocities and accelerations. It is clear that during the first instants (of some τ_η's), there is a memory of the initial conditions and a partial determinism, after which the trajectories diffuse in a Brownian-like chaotic way. This is in agreement with the observed correlation time of acceleration of $\approx 2.5\tau_\eta$.

This supports a description in terms of stochastic scale-dependent variables, $v = v(t,\delta t)$.

Scaling laws

The basic stochastic (and Lagrangian K41) scaling law in the inertial range $\delta v \sim \tau^{1/2}$ (where τ is the time increment, $\tau = \delta t$) or equivalently for accelerations $\sigma_a(\tau) \sim \tau^{-1/2}$, can be shown to be present in an effective way in Lagrangian experimental data of fully developed turbulence. This law is observable locally in individual segments, not only for the full data (3 millions velocity values). This is an important point since, as we shall see, the new structures pointed out here, concerning in particular the PDF of velocities, are purely local.

As a direct consequence, the expected scaling law for the acceleration a and for its increment da are $a \sim da \sim \delta t^{-1/2}$. This is also confirmed in the experimental data.

E. SR approach to turbulence in v-space

Here we derive the equation of dynamics in v-space in the form of a Schrodinger equation , by a computation analogous to the one performed previously in x space for obtaining QM.

The application of the scale relativity approach to fluid turbulence in the inertial range (where the K41 relation $\delta v \sim \delta t^{1/2}$ holds) amounts to just shift the variables (x, v, a) to $(v, a, \dot{a})$. Now, the velocity V is the primary variable, while the fundamental local irreversibility issued from non-differentiability leads to a two-valuedness of the acceleration field, (A_+, A_-), which is represented in terms of a complex acceleration,

$$\mathcal{A} = \frac{A_+ + A_-}{2} - i\,\frac{A_+ - A_-}{2} = A_R - iA_I. \qquad (5.11)$$

In the inertial range, and neglecting for the moment the Langevin term $-v/T_L$, the new Lagrangian description starts with two stochastic differential equations (SDE) in v-space:

$$dV_+ = A_+ \, dt + dV_{\xi+} \tag{5.12}$$

$$dV_- = A_- \, dt + dV_{\xi-} \tag{5.13}$$

where the scale dependence of the stochastic fluctuation reads:

$$dV_{\xi\pm} = \zeta_\pm \sqrt{2\mathcal{D}_v}\, |dt|^{1/2}, \tag{5.14}$$

where $\mathcal{D}_v$ is a fundamental parameter, specific of the turbulent flow under consideration, which is homogeneous to a diffusion coefficient in v-space. This relation is the expression of the [[5]] scaling in the ScR approach.

Such linear SDEs should yield Gaussian processes. However the experimentally observed stochastic behavior of velocity increments and of accelerations is far from Gaussian in turbulent flows (their probability distribution function (PDF) shows large tails which have been observed beyond 50 σ). This is fully explained by the scale-relativity process, which generates a new acceleration component

$$A_q = \pm \mathcal{D}_v \; \partial_v \ln P_v \tag{5.15}$$

(where $P_v(v)$ is the velocity PDF, see here below), that is at the very origin of this non-Gaussianity and of intermittency. Because of this new contribution, the coefficient $\mathcal{D}_v$ differs from its expected expression in the K41 [[5]] regime,

$$\mathcal{D}_v = \frac{1}{2} C_0 \varepsilon = \sigma_v^2 / T_L, \tag{5.16}$$

where C_0 is Kolmogorov numerical constant, ε is the energy transfer rate, σ_v is the velocity dispersion and T_L the integral time-scale.

In this model, there still exists a purely stochastic background which remains fully Gaussian and which serves as seed for the fractal / scale-relativity process. This has been demonstrated by an analysis of Lagrangian experimental data: it has been shown that, in the intermittent alternation of calm periods and bursts, the calm zones (for which the new acceleration component vanishes) remain fully Gaussian, while the burst zones (generated by A_q) exhibit a highly non-Gaussian acceleration PDF characterized by large tails, see below with Ref.[[8]].

Let us follow the successive steps of the derivation of these results in the v-space case. The reduced variable $\zeta_\pm$ is taken here to be a dimensionless

Gaussian stochastic variable such that $<\zeta_\pm>= 0$ and $<\zeta_\pm^2>= 1$. The parameter $\mathcal{D}_v$ characterizes the amplitude of the fluctuations. It is apparented to a diffusion coefficient, but its meaning here is more general.

Using the Ito calculus, the various effects of fractality and non-differentiability can be combined in terms of a total derivative operator acting in v-space:

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{A}.\nabla_v - i\,\mathcal{D}_v \Delta_v, \tag{5.17}$$

where the gradient and Laplacian operators are just the standard ones in terms of velocity coordinates, $\nabla_v = (\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$ and $\Delta_v = \partial^2/\partial v_x^2 + \partial^2/\partial v_y^2 + \partial^2/\partial v_z^2$.

The Navier-Stokes equation (reduced to the Euler equation in the inertial range by neglecting for the moment the viscous term) writes in Lagrangian form $dv/dt = F = -\nabla p/\rho$. In the incompressible case, its derivative with respect to time reads:

$$\frac{da}{dt} = -\frac{\nabla \dot{p}}{\rho}. \tag{5.18}$$

In order to account for the various effects described here, one replaces d/dt by the new total derivative operator $\widehat{d}/dt$. One therefore obtains the new equation of dynamics in v-space:

$$\frac{\widehat{d}}{dt}\mathcal{A} = \left(\frac{\partial}{\partial t} + \mathcal{A}.\nabla_v - i\,\mathcal{D}_v \Delta_v\right)\mathcal{A} = \dot{F}, \tag{5.19}$$

where F contains the pressure gradient term and possibly an applied external force. A final question to be discussed is the fact that the force expression from which we started, $F = -\overline{v}p/p$ is valid in Eulerian coordinates, while its expression in Lagrangian coordinates has a more complicated form involving a Jacobian transform. However, our final result in v-space can be expressed in terms of the velocity components Vi alone, which now play the role of primary coordinates. Such a force depending just on the coordinates therefore applies both to the Eulerian and Lagrangian form of the Navier-Stokes equations (and their derivative).

F. Macroscopic Schrödinger equation in v-space

Let us now consider the force in v-space, that we can formally write $\dot{F}(v)$. In the turbulent case considered here, it is well known that it manifests itself in terms of an energy cascade of eddies. The pulsating or oscillatory nature of the motion of fluid (or test) particles in eddies has led us to formalize them as a sum of oscillators. These oscillators are solutions of the well-known motion equation for harmonic (HO), damped (DHO) or anharmonic (AHO) oscillators. One of the advantages of this representation is that, in all cases, the force that generates these oscillators derives from a potential, in x-space as well as in v-space. As we shall see, this method becomes particularly useful with the Schrödinger form of the motion equation that we shall obtain, since quantum oscillator solutions, harmonic (QHO) and damped (QDHO) are well known and largely studied in standard quantum mechanics.

Force and potential in v-space The new formulation involving the derivative of the Navier-Stokes equations has introduced the time derivative of a force in x space. It is therefore important to study in more details the nature of such a derivated force. We can prove that when the force in x-space derives from an x-potential, the force in v-space also derives, now, from a v-potential. (see also in our paper 2 in section 8), this property is applied in particular to the pressure gradient term. One can show that an important contribution to the total force can be written in terms of a v-gradient of a potential ϕ_v, such that:

$$\frac{\widehat{d}}{dt}\mathcal{A} = -\nabla_v \phi_v. \tag{5.20}$$

Then we introduce a wave function ψ_v as a re-expression of the action $\mathcal{S}_v$ which is now complex (since the dynamical variables are complex):

$$\psi_v = e^{i\mathcal{S}_v/2\mathcal{D}_v}. \tag{5.21}$$

It can be decomposed in terms of a modulus and a phase:

$$\psi_v = \sqrt{P_v} \times e^{i\theta_v/2\mathcal{D}_v}. \tag{5.22}$$

The main point here is that the PDF of velocities is given by the square of the modulus of the wave function, $P_v(v) = |\psi_v|^2$, while its phase is linked to the real part of the complex acceleration through the relation $A_R = \nabla_v \theta_v$. The constant $\hbar_v = 2\mathcal{D}_v$ is therefore the macroscopic equivalent in v-space of the constant $\hbar$ of standard quantum mechanics (more generally, when $m \neq 1$, they are related by $\hbar_v = 2m\mathcal{D}_v$).

Schrödinger equation in v-space Finally, the time derivative of the Navier-Stokes fluid equations takes, after integration on v, the form of a macroscopic

Schrödinger equation in v-space:

$$\mathcal{D}_v^2 \, \Delta\psi_v + i\,\mathcal{D}_v \frac{\partial}{\partial t}\psi_v - \frac{1}{2}\phi_v\,\psi_v = 0. \tag{5.23}$$

This Schrödinger equation describes the effects of the part of the turbulent force ($F = -\nabla p/\rho$ + possible forcing terms) which can be written in terms of the gradient of a potential in v-space. We have seen that this can be done, at least formally, for any force which derives itself from a potential in x-space, among which the most important one, the pressure gradient $-\nabla p/\rho$.

We shall now see that there is more to that, since this v-potential takes in a universal way the form of an harmonic oscillator.

Link to the cascade of eddies In Ref. [9], our argument for the description of the v-space force in terms of HO potentials relied on the experimental observation of the cascade of eddies. The motion of a Lagrangian particle driven by an eddy is expected to be locally described by an HO or DHO (also forced), which has a priori no reason to be centered around $(v_x, v_y) = (0, 0)$). The new property obtained here, according to which any force in x-space deriving from a potential results in an off-center harmonic oscillator force in v-space, justifies and reinforces this choice. The experimental laboratory data fairly supports this conclusion [9].

Recall finally that the emergence of a v-Schrödinger equation is founded on the inertial range properties, in particular the scaling relation $\delta v^2 \sim \delta t$. In the framework of K41 dimensional analysis, this relation is the Lagrangian temporal equivalent of the Eulerian spatial scaling relation $\delta v^3 \sim \delta x$. However, the temporal relation on which the scale-relativity approach is founded (through its identification with a process of fractal dimension $D_F = 2$) is far more general and does not depend on the mere fluid mechanics, being the manifestation of a general Markov process.

This Schrödinger equation does not contain all the contributions to the dynamics, but it can be considered as a kind of kernel to which other effects (non potential terms, Langevin term, viscosity, etc.) can be added.

G. New acceleration / force component

The complex acceleration field writes, in terms of the wave function,

$$\mathcal{A} = -2i\,\mathcal{D}_v\,\nabla_v \ln\psi_v, \tag{5.24}$$

so that we are now able to establish the expression for the two new acceleration components A_+ and A_-:

$$A_+ = +\mathcal{D}_v \, \frac{\partial_v P_v}{P_v} + \partial_v \theta_v, \tag{5.25}$$

$$A_- = -\mathcal{D}_v \, \frac{\partial_v P_v}{P_v} + \partial_v \theta_v. \tag{5.26}$$

In many situations which may be relevant to the turbulence case, in particular for a harmonic oscillator potential expected to describe the motion of a particle driven in the largest eddies of the turbulent cascade , the solutions of the Schrödinger equation are real, i.e. $\theta_v \approx$cst and then $\partial_v \theta_v \approx 0$. Under this approximation (which is supported by the experimental data [9]), the new acceleration can then be written as:

$$A_q = \pm \mathcal{D}_v \; \partial_v \ln P_v(v). \tag{5.27}$$

In an equivalent way, it can be considered as a component of the force appearing in the right-hand side of the Navier-Stokes equations, i.e., of the pressure gradient $-\nabla p/\rho$.

8. APPLICATION: CHOICE OF SELECTED PAPERS AND SHORT COMMENTS ON THEIR MAIN RESULTS

A. Turbulence and scale relativity [[9]]

The paper 1 is on the topic of homogeneous and isotropic 3D Turbulence.

ABSTRACT

We develop a new formalism for the study of turbulence using the scale relativity framework (applied in v-space, following de Montera's proposal). We first review some of the various ingredients which are at the heart of the scale relativity approach (scale dependence and fractality, chaotic paths, irreversibility) and recall that they indeed characterize fully developed turbulent flows. Then we show that, in this framework, the time derivative of the Navier-Stokes equation can be transformed into a macroscopic Schrödinger-like equation. The local velocity Probability Distribution Function (PDF), $P_v(v)$, is given by the squared modulus of a solution of this equation. This implies the presence of null minima $P_v(v_i) \approx 0$ in this PDF. We also predict

a new acceleration component, $A_q(v) = \pm\mathcal{D}_v\, \partial_v \ln P_v$, which is divergent in these minima. Then we check these theoretical predictions by data analysis of available turbulence experiments: (1) Empty zones are in effect detected in observed Lagrangian velocity PDFs. (2) A direct proof of the existence of the new acceleration component is obtained by identifying it in the data of a laboratory turbulence experiment. (3) It precisely accounts for the intermittent bursts of the acceleration observed in experiments, separated by calm zones which correspond to $A_q \approx 0$ and are shown to remain perfectly Gaussian. (4) Moreover, the shape of the acceleration PDF can be analytically predicted from A_q, and this theoretical PDF precisely fits the experimental data, including the large tails. (5) Finally, numerical simulations of this new process allow us to recover the observed autocorrelation functions of acceleration magnitude and the exponents of structure functions.

In the paper, we first compare the characteristics of turbulent fluids to the various principles underlying the construction of the scale relativity theory (SRT) (Sec. II). We show in Sec. III how the various physical and mathematical tools of scale relativity are fully supported by experimental data of turbulent flows. We also briefly review the basic mathematical methods by which one constructs the wave function and the geodesics equation, showing that they are well-known and proven methods widely used in stochastic descriptions of turbulence: it is just their special combination which is specific of scale relativity. In Sec. IV, the Schrödinger form of the equation of motion is derived, first in position space and then in velocity space for application to turbulence. Section V describes the main implications and theoretical predictions that one can expect from the new approach, in particular, those which can be experimentally put to the test: the main one is the prediction of the existence of a new acceleration component $A_q(v) = \pm\mathcal{D}_v\, \partial_v \ln P_v$, where $P_v = |\Psi|^2$ is the local PDF of velocity given by the square of the modulus of a wave function Ψv, which is solution of a Schroödinger-like equation. We list in Sec.VI some experimental observations and results that already come in support of these theoretical expectations and we end by a discussion and conclusion in Sec. VII. To illustrate we select here only one major result of this paper which is the prediction of the observed PDF(a) of the accelerations a of the turbulent flow. In particular the high acceleration tail is connected to intermittency and is explained by the new acceleration Aq which diverges (as a singularity) at the minima of the PDV(v) where $P_v(v_i) \approx 0$. For example see Fig. 5.1 showing the prediction of this PDF(a) as compared to observations in a typical turbulent flow, where a more standard approach to turbulence remains unable to make such kind of prediction.

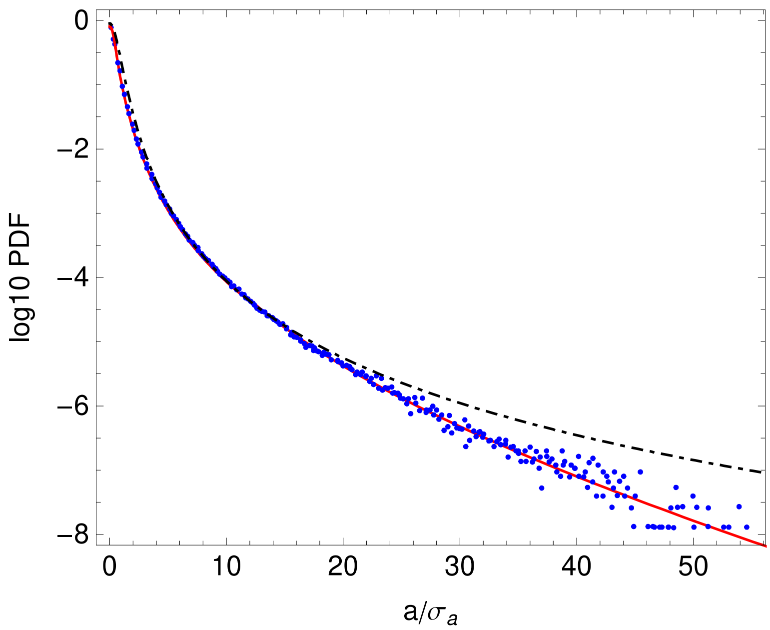

Figure 5.1 – Comparison between the observed acceleration PDF in Bodenschatz et al data ([9] ($R_\lambda = 690$, 10^8 values up to ~ 55 sigmas, blue points), the $(1 + (a/\sigma_a)^2)^{-2}$ law expected as a first approximation (dot-dashed black curve) and the PDF corrected for both small and large accelerations (red continuous curve). The corrected curve perfectly fits the data within experimental uncertainties.

One naturally expects an exponential cut-off to the a^{-4} purely inertial law from a relative damping mechanism as explained in the paper. This is supported by the experimentally observed PDF's.

Another remarkable feature in the SR approach, is the interpretation of the intermittency process as of 'macro' quantum jumps in-between local harmonic oscillator like potential wells in velocity space. Figure 2 illustrates this interesting phenomenon.

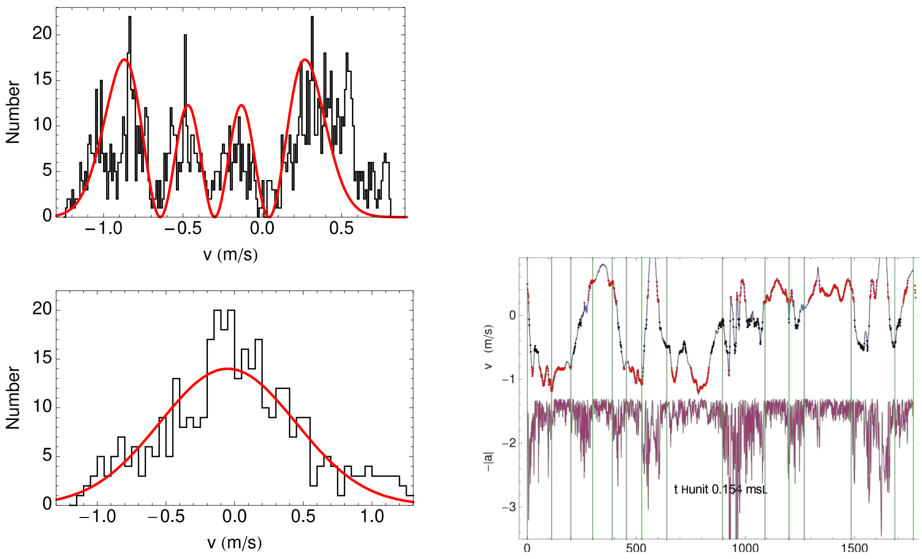

Figure 5.2 – Left figure: observed PDF of velocity for the sub-sample [1-1800] of Seg3398 of Mordant's experiment #3 (man290501), [10]. Top left figure: observed PDF of velocities for $|a| < \sigma_a$ (black histogram), compared with a solution of the macroscopic Schrödinger equation for a quantized harmonic oscillator with $n = 3$ (red curve). External peaks not accounted by this theoretical PDF just fit the corresponding classical oscillator. Down left figure: observed PDF of velocities for $|a| > \sigma_a$ (black histogram), compared with a Gaussian of standard deviation 0.7 m/s (red curve). Right figure: detailed analysis of the evolution of velocity and acceleration with time for the same sub-segment. The top curve shows the time evolution of velocity $v(t)$. The bottom curve shows the time evolution of the acceleration magnitude ($|a(t)|$ (reversed, varying here from 0 to 1000 m/s^2). We have underlined in red the points lying in the main external peaks of the local PDF of velocity and in black those lying in the secondary internal peaks (top left figure). The vertical green lines mark the limit between the calm periods of acceleration ($<\sigma_a> \approx 90$ m/s^2) and the intermittent bursts ($<\sigma_a> \approx 330$ m/s^2).

B. Scale-relativity approach to turbulence in the presence of a Coriolis force ([7])

This paper 2 is an extension of the previous one to account for the presence of a Coriolis force which is suitable for fluid in rotation with applications to geophysics and astrophysics.

ABSTRACT

In this paper we examine the predictions of the scale-relativity approach for a turbulent fluid in rotation. We first show that the time derivative of the governing Navier-Stokes equation in usual x-space can be transformed into a Schrödinger -like equation in velocity space with an external vectorial field to account for the rotation, together with a local harmonic oscillator potential in v-space (noted VHO). The coefficients of this VHO are given by second order x-derivatives of the pressure. We can then give formulae for the velocity and acceleration PDFs. Using a simple model of anisotropic harmonic oscillator, we compare our predictions with relevant data from both direct numerical simulations (DNS) and oceanic drifters velocity measurements. We find a good agreement of the predicted acceleration PDF with that observed from drifters, and some possible support in DNS of the existence of gaps in the local velocity PDF, expected in the presence of a Coriolis force.

An important new contribution in this work is the demonstration (which was only checked on data in the previous paper) that the force in v space due to the gradient of the pressure can be decomposed into a sum of gradients of harmonic oscillator potentials. The result is satisfactory since with this modelization of the force we are able for example to predict the PDF(a) in good agreement with observations in the case of data coming from oceanic drifters put at the surface of the ocean, relevant to describe the local geostrophic turbulence.

Figure 3 (Fig. 5.3 shows this feature, again here the PDF(a) scales again mainly like $1/a^4$. Of course for large a in both turbulence without and with rotation there is also an exponential cut-off of the PDF (as shown in figure 1 above, not here).

C. The Turbulent Jet in the Scale-Relativity Framework [11]

The paper 3 describes the properties of a typical shear flow: the free round jet

ABSTRACT

We apply the scale-relativity theory to the turbulent round jet. In this theory, the time derivative of the Navier-Stokes equations is integrated in terms of a macroscopic Schrödinger equation acting in velocity-space. This

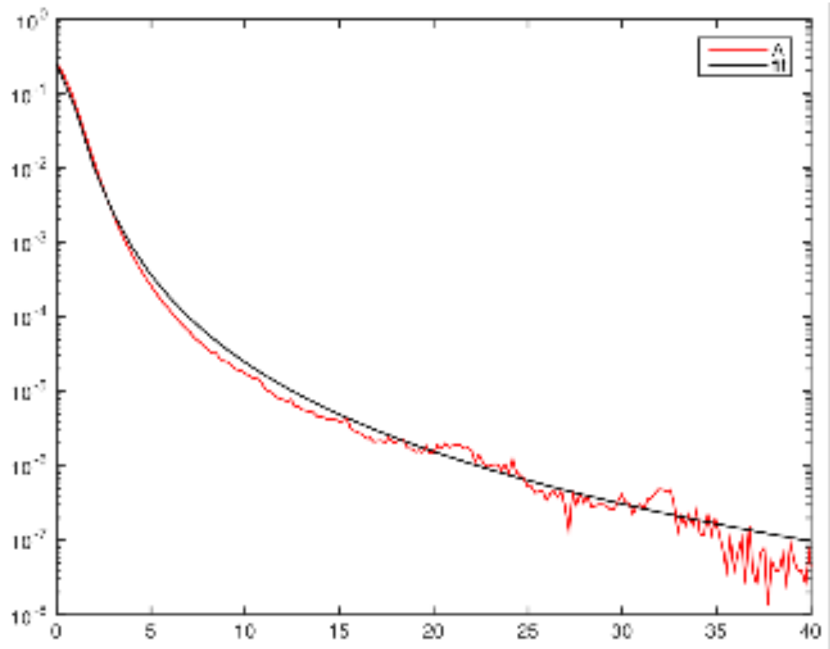

Figure 5.3 – Acceleration PDF obtained for a subsample of drifters (red curve) compared to the function $(1 + (a/\sigma_a)^2)^{-2}$ (black curve) theoretically predicted by the scale-relativity theory of turbulence [9].

equation involves a constant $\hbar_v$ which can be identified with the energy dissipation rate, while the pressure gradient manifests itself as a quantum harmonic oscillator (QHO) potential. The squared modulus of its solutions yields the probability density function (PDF) of velocities. The Reynolds stresses can then be derived from this PDF, so that the closure problem is solved in this case. This allows us to obtain a theoretical prediction for the turbulent intensity radial profile (and therefore for the pressure) which agrees with the experimental data. The ratio of axial over radial velocity fluctuations is found to be $R = 1.3 - 1.4$ from QHO properties, in good agreement with its experimental values; we theoretically predict a jet opening angle $\alpha = 1/2R^3$ accounting for its universal value $\approx 1/5$; the mean ratio of turbulent intensity amplitudes over jet centerline axial velocity is predicted to be $X = 19/80$, in good agreement with its universally measured value $\approx 1/4$; from these parameters we derive possible values of the radial turbulent intensity amplitude $\mu = 0.19 - 0.22$ consistent with experiments; finally, we find a correlation coefficient of velocities $\rho = 2\,\alpha$, suggesting an explanation for the universal value ≈ 0.4 observed for the turbulent jet and for all free shear flows.

We give just a glimpse about this paper here: The form of a new Schrodinger equation as derived by the SRT for the jet case has allowed to make new prediction and checking of their agreement with observations for the key parameters which are defined as follows: $R = \sigma_u/\sigma_v$ and the related quantities defined in the above abstract. The main feature is that R is now quantized due to the occurrence in the contributing v-potential of two coupled harmonics oscillators (in u and v in the axial and radial directions). The quantification of R allows to make further the right numerical predictions for the relevant parameters just quoted as compared to observations.

9. CONCLUSIONS

In this short review we have recalled the initial aim of SRT to found quantum mechanics from the principle relativity under the hypothesis of a fractal space-time. Then we have described the conditions for the application of the theory to turbulence based now on the fractal behavior of the matter itself now in v space. With our selected examples we have shown that SRT can bring new light on many aspects of turbulence which are uncovered in the standard approach. The powerful tool at hand is relying on the linearization of non linear equations in suitable spaces. This approach can be considered as a special aspect of the branch of idempotent analysis in mathematics.

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