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The Continuum Hypothesis : A Philosopher's View

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ABSTRACT

The intended scope of this article is to review the *Continuum Hypothesis*, **CH**, in foundational mathematics from the viewpoint of a phenomenologically influenced philosopher. Given the capital importance of Cantor's 19th century conjecture about the cardinality of continuum and the relevance it has acquired over the years in matters of mathematical ontology it is natural to motivate a discussion well beyond its place in foundational mathematics proper. This means that except for the continuing research toward its resolution by new and powerful theories following Cohen's pioneering forcing method in the 60s, one has come to inquire into the metatheoretical or even extra-theoretical nature of the *Continuum Hypothesis*, something that implies a philosophical inquiry into the ontological status of the question as such and into the ways it may be shaped by our natural intuitions of the continuous and the discrete. On the latter prompt, among others, and discarding platonistic tendencies I set out to provide an interpretation along subjective-constitutional tracks of the ontological status of the **CH** and ultimately of the ways it may be reducible to a question of the constitution of continuous unity in subjective-temporal terms. In the more formal part, I have tried to argue against a certain operationally motivated attempt at refut-

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ing **CH** as well as against attempts to resolve the issue of its undecidability relying on the more advanced large cardinals plus the inner models theory.

1. INTRODUCTION

As known the *Continuum Hypothesis*, **CH**, has been and remains one of the most intriguing questions of the mathematics of our time since its inception by Cantor in 1878. It is also known to be the first in the list of the 23 unresolved mathematical questions that D. Hilbert presented in the international congress of mathematicians in 1900 in Paris and, in spite of the research work by scores of competent or even renowned set-theorists spanning the decades that followed Cantor's conjecture, it is still regarded as virtually a question in suspense. For someone, outside the community of set-theorists or logicians, who might rightfully wonder why it is that **CH** is considered as perhaps the foremost in significance foundational question in mathematics the answer could be like this: the **CH** is a conjecture that states something simple, intuitively plausible and absolutely fundamental for the ontology of mathematics, namely that there is no set whose cardinality (i.e., its number of elements) is strictly between that of the integers and the real numbers. In formal language the cardinality of the real numbers c, ($c = 2^{\aleph_0}$), equals the first cardinality $\aleph_1$ in the canonical scale after the cardinality of the natural numbers $\aleph_0$ i.e., $c = 2^{\aleph_0} = \aleph_1$.

Simple as it may seem, this mathematical statement bridges in effect the ontological realms of the discrete and the continuous and it is for that matter that it acquires a fundamental significance in mathematics. Both in formal-theoretical terms, to the extent that everything dealing with the continuum in mathematics has to be treated essentially in terms of the first-order axioms of the Zermelo-Fraenkel (plus the *Axiom of Choice*) Theory, **ZFC**, and the predicative first order universe of set theory, and in metatheoretical, in fact, ontological terms to the extent that it may involve at once the intuitions of a multitude (of well-meant objects) and the intuition of the whole as a continuous unity. Given this background and the fact that it has been proved by K. Gödel in 1938 and by P. Cohen in 1963 that **CH** and its negation are both consistent with the predominant **ZFC** theory of our times, it has led some prominent set-theorists to express doubts about its decidability on purely formal-mathematical grounds insinuating on a possible extra-theoretical dimension of **CH**. Indeed this kind of ambiguities had led S. Feferman to point out in [5] that:

> I shall argue that for all intents and purposes, CH has ceased to exist as a definite problem in the ordinary sense and that even its status in the logical sense is seriously in question. (ibid., p. 2).

Yet for all the controversy concerning its nature and the philosophical relevance it has acquired all these years, it is undoubtedly true that **CH** has also spawned a surge of research activity either enriching existing or ushering in entirely new orientations and research fields in mathematical logic and set theory, referring primarily to Cohen's forcing method, and more recently to the theory of large cardinals in connection with the inner model theory, [1] the multiverse theories, etc. Moreover, the established undecidability of **CH** within the **ZFC** theory notwithstanding, there exists an ongoing research to recalibrate the issue in an advanced mathematical environment in terms of an extension of Gödel's constructible universe L, one that can accommodate a spate of new, ever larger cardinals. [2]

My own intention in this article is mostly focused on the philosophical-epistemological aspects of the **CH** question in view of the possibility of an extra-theoretical content and also in view of the influence it bears on the ontology of mathematics in their entirety taken either in a platonic sense or in a non-platonic, subjectively based and further phenomenologically influenced one. As my arguments are primarily of the latter category I set out to defend the position that there is a sense of immanent [3] continuous unity, in contrast with the apprehension we have of objects in general as well-meant, discrete re-presentations in the mind of real or imaginary objects of our environment, a kind of unity that pre-determines and conditions the conception we have in formal terms of any level of infinity and in this regard implicitly shapes our conception of **CH**. At the same time such immanent unity and the associated subjective constitutive modes may condition to a significant extent,

1. An inner model of an axiomatical theory, a concept widely applied in modern set-theoretical research, can be roughly said to be a transitive $\in$-model of the theory that contains all ordinals. See [18], p. 33.

2. See W. H. Woodin's work in [30], [31], [32].

3. The term immanent or immanence, widely used in Husserlian and generally phenomenological texts, can be roughly explained as referring to what is or has become correlative (or 'co-substantial') to the being of one's consciousness in contrast to what is 'external' or transcendent to it.

on the level of extra-theoretical evidence, the current state of undecidability of **CH** and the possibility of its prospective formal decidability. Given the philosophical inclination of the paper, on the one hand, and the advanced mathematical stuff the mathematical research on **CH** has generated over the years, on the other, I have tried to strike a balance between the philosophical discussion and the subtlety and, often, arduousness of the mathematical ideas involved.

It is indeed in philosophical terms that I am going to argue against Hoek's 'operationally' based interpretation of the falsity of **CH** in sec. 2. Further, in sections 3 and 4, I try to ground my position on an existing ontological dimension of **CH** and the way to address it by a subjectively founded, phenomenologically influenced approach. On the same grounds I argue in sec. 5 for the impossibility of resolving the issue of **CH**, at least in an ontological sense, by means of the progressing large cardinals theory.

2. WHY THE **CH** MAY NOT BE ADDRESSED ON EPISTEMIC-OPERATIONAL GROUNDS

In *Chance and the Continuum Hypothesis*, [10], D. Hoek has argued for the falsity of **CH**, based on certain results of Banach-Kuratowski (1929), further elaborated by Ulam (1930) and Solovay (1971), by essentially reducing the **CH** question to probabilistic procedures performed on continuum-sized sets for which however certain 'ontological' questions pertaining to the implicit presence of non-Lebesgue sets within them, are virtually left unanswered.

As it has happened with similar attempts to recalibrate formal principles in epistemic-operational terms, e.g., in [6], Hoek's is essentially a conceptual overlapping of the notion and the formal structure of the mathematical continuum with the epistemic content implied by the performance of concrete mathematical acts within the real world continuum of events. For example, what will be argued to be a vague, impredicative 'residuum' of continuum size mathematical objects is brought to bear an influence on well-defined processes such as the assignment of a chance value to any event in a sample space in contrast with the suppression of the notion of a chance free event. In other words one is faced with the question of coming to terms with the notion of continuum in strictly set-theoretical terms and then with the way it may be possibly founded in ontological sense through the epistemic relevance it may acquire in situations in which mathematical objects/state-of-affairs are generated by conscious hence discrete mathematical acts. This way Hoek proceeds to a refutation of **CH**, based on the Banach-Kuratowski's, Ulam's and Solovay's results mentioned above, essentially by treading on the 'blurry' lines between the following three levels of talking about the continuum. Talking of the continuum in purely formal-mathematical terms, talking of the continuum in ontological terms, and talking of the continuum in epistemic terms the latter implying its conceptual association with real world phenomena and the modes they may be mentally re-presented.

More concretely, events may be thought of as subsets of the outcome space $\Omega = [0°, 360°)$ of a rotating spinner so that, for instance, $(10°, 15°)$ represents the event that the pointer lands at an angle between $10°$ and $15°$. As normally expected the chance of an event is a real number between 0 and 1 expressing the objective likelihood that the event takes place. The following axiom **M** is a key axiom, adjoined to **ZFC** by Banach and Kuratowski, to prove the negation of **CH**, that is to prove $2^{\aleph_0} \neq \aleph_1$:

M: (Chance Measurability of the Continuum): Continuum-sized sets Ω admit of a total, countably additive measure *Ch* such that $Ch(\mathrm{x}) = 0$ for every $x \in \Omega$. [4]

Challenging Hoek's result on epistemological grounds, I turn my attention first on a key premise of this axiom, the totality premise **C**, which states that every event as a subset of Ω, $E \subseteq \Omega$, has a chance $Ch(E)$. While this sounds obvious for finite outcome cases it is not intuitively obvious and in fact may lead to faulty assumptions in an uncountably infinite outcome space. The reason is that in defining probability distributions on a continuum-sized outcome space, the assignment of chances is associated with the class of measurable sets which means that non-measurable (i.e., non-Lebesgue) sets of points correspond to chance-free events. In the following and in spite of Hoek's position that

> there is no good reason to think there is a real distinction in the world between chance-bearing and chance-free events corresponding to the mathematical distinction between measurable and non-measurable sets of points, [5]

there is a case to be made for the distorting influence epistemic-operational concerns may have on a purely formal concept of continuum.

4. Solovay has showed in 1971 that the theory **ZFC** + **M** is consistent. See [10], pp. 643-644.

5. See: ibid., p. 644.

The pedagogical effect of Hoek's reference to the spinner experiment is to show how on account of countable additivity the outcome space can never be divided into $\aleph_0$ minuscule events and on this fact come up with a conjecture on the size of the continuum. Yet one must prior have in mind that a) a minuscule event essentially corresponds to an infinitesimal (nonstandard) number, that is, one that has a chance smaller than any positive integer, and b) infinitesimal chances are considered non-zero chances smaller than any positive number so as to avoid paradoxical situations in which the uncountably infinite outcomes of a whirling spinner space would sum up to more than unit value in case they have standard real values. Of course there is no notion of minuscule events, in other words of infinitesimal numbers and in this sense of non-Lebesgue sets, but relative to a continuum-sized outcome space and as I'll show next this is exactly where the argumentation for or against the key premise **C** essentially reduces to. For a main argumentation of Hoek is epistemically grounded in the fact that the concept of chance has a version in reality or at least in a certain version of it and thus "is very fruitful in science, where probabilistic methods of prediction have met with great empirical success", so that he can dispense with the problematic case of single chances (i.e. of infinitesimal numbers) corresponding to single values of the angle the spinner could land on and thus validate the premise **B** [6] of axiom **M**. Therefore standard probabilistic inductive reasoning is in a certain sense subverted by the presumptive existence of chance-free events taken account of the fact that the predictability of standard statistical reasoning in science is pretty much dependent on the assumption that every event may have a chance.

As a matter of fact standard mathematical-probabilistic practice assigns a uniform measure to the class of Lebesgue subsets of a real interval, ignoring the non-Lebesgue sets, on grounds associated with the Vitali's theorem [7] negation of the translation (or rotation) invariance of countably additive, total chance functions on the continuum. If therefore non-Lebesgue sets of real numbers corresponding to non-Lebesgue propositions about the future are methodologically eliminated to secure translation invariance for total chance

6. Premise **B** is: No individual outcome has a positive chance of being realised, i.e., Ch(x) = 0 for any outcome $x \in \Omega$; ibid., p. 643.

7. Vitali's theorem states roughly that there is no total measure on the real line that is: (1) translation invariant (2) finitely or countably additive and (3) assigns measure 1 to the interval $(0, 1)$; see ([29], p. 233). Hoek's slightly different variant which nevertheless changes nothing to the discussion is that there is no total measure of the points on the circle that is $(1')$ rotation invariant $(2')$ countably additive and $(3')$ assigns measure 1 to the entire circle.

functions does that mean they must be also considered ontologically non-existent? And further in what sense could a non-Lebesgue set, taken to be a zero-measure scattered collection of isolated points in a purely set-theoretical sense, be linked to a lack of causal connection in physical terms for which Hoek admits that

> The hypothesis that scientists only ever reason about observable events stands in clear need of clarification and justification. The same is true for the view that there is a substantial class of unobservable events that exist in perfect causal isolation from those observable events. (ibid., p. 656).

In spite of all these metatheoretical ambivalences including the acceptance that empirical evidence can never be conclusive, Hoek seems eager to advocate that empirical evidence is a major attestor to the fact that all events, observable or not, have a chance and on such grounds put the claim that the **CH** is false.[8] Consequently in Hoek's view the truthfulness of the totality assumption **C** is founded, by abductive reasoning, on its implicitness in 'our ordinary scientific reasoning about chance' in spite of the fact that the ontological status of non-Lebesgue sets corresponding to non-Lebesgue propositions is left unaccounted for, e.g., in the case of the diffuse spinners of Hoek's example in which moreover, unlike other random procedures like infinite coin flips, events are real open intervals. In fact both Vitali's theorem entailing that the measure of all groups of points of a straight line and a total ordering of the continuum cannot co-exist, and Ulam's result that there is no nontrivial σ-additive measure on ω_1 [9] are essentially based on the 'irrelevance' of countable sets, and consequently of non-Lebesgue sets, in terms of measure-theoretical proofs.

As known in a general perspective Lebesgue measurability is inherently associated with the property of subsets of reals (or equivalently of the subsets of the Baire space ω^ω) to be everywhere dense in a way that by the topologization of the set of real numbers every other set of numbers, e.g. finite or countable subsets, lacking this topological property are measure-theoretically

8. The statement is formally associated: (a) with the Banach-Kuratowski theorem, i.e., a continuum-sized set Ω admits of a total, countably additive measure $\mathrm{Ch} : \mathcal{P}(\Omega) \to [0,1]$ with the property that $\mathrm{Ch}(x) = 0$ for every $x \in \Omega$, and (b) the fact that, in relation to Ulam's statement within **ZFC**, this is not true of the first uncountable cardinal $\aleph_1$; that is, $\aleph_1$-sized sets do not admit a measure of this kind. See [17], lemma 10.13, p. 132.

9. This is the symbol for the ordinal number corresponding to the first uncountably infinite cardinal $\aleph_1$ in the canonical scale.

'non-existent', a fact that may lead to ontological questions relative to the set-theoretical treatment of the Cantorian continuum itself.

In this sense it seems doubtful, at first, whether probabilistic reasoning based on Lebesgue measurability can decide the cardinality of continuum by virtue of ontological 'annihilation' of single points in measure-theoretical proofs and, second, one is left with the ontological dimension of the **CH** question virtually untouched. The latter relates to the incompatibility of an ontology of points as re-identifying individuals to the ontology of agglomerations of such points-individuals in the sense of real intervals that acquire an ontology of their own essentially different from that of their constituents. Even though I am going to discuss the philosophical aspects of the *Continuum Hypothesis* in a special section in the next, I point out as indicative of the philosophical depth this discussion may acquire Heidegger's view in *The Concept of Time*:

> Measuring amounts to determining something that is present [*Gegenwärtiges*] by means of something that is present [*Gegenwärtiges*]. The measurement [*Masszahl*] brings out how many times a given line segment fits into the whole, measured line, that is to say, it puts at one's disposal the measured line in its full presence [*Anwesenheit*]. Hence the key point in measuring time [*Zeitmessung*] is recourse to something that is available as present [*Anwesendes*] in every Now and which as such allows us to determine every Now. ([9], pp. 61-62).

Of course Heidegger was not trained as a mathematician like his one time mentor Husserl, yet his position above says quite a bit about the possibility of resolving the question of continuum by measurement methods. For if measuring means determining something that is present by something which is present too, then measuring a line segment by a given line segment gives no discomfort at all. Things start to get more complicated when one turns to the question of measuring a time interval by means of something available as present in every temporal now and which as such allows to determine every other now. For, if for Heidegger "measuring amounts to determining something that is present by means of something that is present" and therefore the measuring of a time interval can be carried out by means of something present each temporal now, then naturally the next question would concern the genus of what must be present each temporal now and by which we are about to determine every other now. Will it be a time interval itself in which case we would run into an interminable sequence of recurring intervals or a temporal point which, being the unique object of the intentional perception [10] of a human subject, yields its ontological foundation to the intuitive evidence of a subject's intending a vacuous 'general something' in the present now? But then, on the latter supposition, how could we run through an (indenumerable) infinity of such present nows to measure or *a fortiori* enumerate a time interval schematized in the form, e.g., of a line segment whose points may precisely thought to re-present the present nows?

Apparently this is the kind of queries one may be faced to if he is disposed to detach the continuum question out of the convenient realm of set-theoretical formalism and start viewing it as not a purely mathematical question going as far as reducing its ontological grounding to the exegetic capacities of the phenomenology of inner time. For if, in a purely natural attitude, enumeration by natural numbers is a fundamental mathematical act extendible in time and implying a conscious act for each temporal now of conscious apprehension, independently of the nature (real or abstract) of enumerated objects, then it is hardly possible, in the first place, to dispense oneself of temporal concerns in talking about mathematical objects involving indefinite extensionality properties, e.g. sets, classes, etc. One has then to inquire into the nature of talking about time as a progressing, discrete sequence of nows at variance with talking about time as the possibility of a subject's reflection on the totality of his already performed acts in the actual now. I leave the rest of this discussion for the next two more philosophical sections of the paper.

3. THE POSSIBLE PHENOMENOLOGICAL UNDERPINNINGS OF THE CONTINUUM QUESTION

Even as the working set-theorist can dispense himself, in his everyday research, with foundational questions going down a deep ontological level, it is still important, if formal axiomatical theories are not just a consistent game of abstract symbols according to some plausibly postulated axioms, to inquire into the nature of certain concepts that seem to be irreducible by the analytical means of the language of a formal axiomatical universe. A showcase of

10. The concept of intentional perception (*Wahrnehmung*) was widely applied by E. Husserl and can be roughly communicated to a non-phenomenologist as the 'enacted' a priori directedness of consciousness toward a specific 'something-here' (*Was-hier*), independently of content, in original presentation prior to explication. As the author would not like to enter into phenomenological 'technical' details more than what is absolutely necessary for certain arguments in this paper the interested reader may look, among others, into Husserl's *Ideas I*, [13], for further clarifications.

such concepts may be taken to be, on the one hand, the concept of formal individuals of set-theory as irreducible components of absolute formulas inside transitive classes under $\in$ inclusion and, on the other, the impredicative continuum generated by the application of the Power Set Axiom over the set of natural numbers (and potentially in iterative fashion *ad infinitum* over each induced power set) in contrast with a build-up of the sequence of cardinals $\aleph_n, \aleph_\omega, \aleph_\alpha, ...$ (where $\alpha = \aleph_\omega$), by the application of the Replacement Axiom. [11]

My intention in what follows is to show that these concepts have a non-analytical dimension, in fact their ontological status may be susceptible to a subjectively founded interpretation that highlights the way they implicitly impact the non-decidability of the **CH** question on the formal-theoretical level. I draw attention *a propos* to the debate, among logicians and scholars across a range of disciplines, that rages some decades now on the nature of the *Continuum Hypothesis*, in particular, S. Feferman's view in [4], that the *Continuum Hypothesis* is an inherently vague statement which cannot be settled by any new axiom added to the standard **ZF** theory. It happens that the same question, namely the paradoxes associated with the concept of continuum, is raised presumably in a broader context in E. Husserl's critique of the sciences lacking the

> type of full rationality that is constitutive for the idea of science [whose] naively enacted evidence leads to basic concepts and basic propositions that lead, in their consistent evaluation, to contradictions (continuum, paradoxes, and so on) ([16], p. 468).

Before discussing the specifics of the CH question from a philosophical, and in particular, phenomenological perspective it would be helpful to the non-

11. This situation was eventually a main argument of P. Cohen in [3] against the truth of **CH**. More precisely Cohen claimed that it is unreasonable to expect that any description of a cardinal, e.g. of $\aleph_1$ as corresponding to the set of countable ordinals, and of any larger cardinal "which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C", C being the power of the continuum, $C = \mathcal{P}(\mathbf{N})$, "generated by a totally new and more powerful principle, namely the Power Set Axiom", applied over the set of natural numbers $\mathbf{N}$. In this sense the cardinality of C as an 'incredibly rich set' generated by the power set axiom exceeds every cardinality $\aleph_n, \aleph_\omega, \aleph_\alpha, ...$ (where $\alpha = \aleph_\omega$), generated by any 'piecemeal process of construction' represented by the Replacement Axiom of **ZF** theory. Yet Cohen had the perspicacity to note at once that "Perhaps later generations will see the problem more clearly and express themselves more eloquently". See ibid., p. 151.

phenomenologically versed reader to point to some prompts that make the phenomenological discourse relevant with the issue at hand.

To start with, the thematic sphere of pure mathematical thought is not real nature as perceived by the senses but above all possible nature in the sense of nature as phenomenon that can be mentally re-presented. As Husserl put it in *Aufsätze und Vorträge* "The freedom of mathematics is the freedom of pure phantasy and of pure phantasizing thought" (*auth. trans.*, [14], p. 14). [12] Of course this is not meant as an arbitrary and random phantasizing but a kind of ascension, on the grounds of natural and practical mathematical experience, to an intuition of essences (*Wesensintuition*) and essential laws which is not reducible, however, to simple mental variation on perceived objects. In this sense the fundamental mathematical concepts are founded in intuition as generalities 'applicable' to individualities, the latter as identical objects of free variation in imagination which are bound as such to certain a priori modes of subjective constitution that have very little or even nothing to do with a sense of random, real world contingency. The relation to the facts of nature is thus based on the each time a priori possibilities to which refers each concrete, contingent fact as re-presented. On this recursive reference to a priori essential necessities can, for Husserl, be founded the exactness and rationality of the exact natural sciences based on pure mathematics.

Except for the mathematics of nature (mathematics as applied to nature, e.g., pure geometry, formal kinematics, etc) Husserl attributed a special importance to the formal mathematics of analysis in the sense of a formal ontology in which the fundamental object is the syntactical individual as the abstraction of a content-free 'something-in-general' in all its possible transformations. [13] This concept, even though it is a purely formal one in abstraction, it can yet be given a semantic content and further a noematic one as a, devoid of any material or other content, pure intentional object of consciousness in undisputed evidence. To the extent that such content may be ascribed to the objects as syntactical individuals of formal mathematical theory and in parallel terms to a notion of infinity as an indefinitely 'extending' and physically unconstrained immanent wholeness in consciousness, phenomenology in the deeper sense of transcendental phenomenology becomes relevant with the interpretation of mathematical questions, in particular questions involving the mathematical continuum. Husserl went as

12. "Die Freiheit der Mathematik ist die Freiheit der reinen Phantasie und des reinen Phantasiedenkens", [14], p. 14.

13. See further, footnote 18.

far as claiming that knowledge that does not draw its ultimate vindication from a fulfilled intuition reaching to the phenomenological pure subjectivity cannot achieve consummate rigorousness and scientificity, a situation underlying, in his view, the discord on the last foundations of mathematics and the generated paradoxes (ibid., pp. 19-20).

Of course the general concept of continuum in the phenomenological discourse is not restricted to the set-theoretical version as it involves a notion of intuitive continuum in the sense of a constituted temporal unity just as the concept of protean individuals is not restricted to the formal individuals mentioned above. Yet the kind of perspective provided may help elucidate an understanding of the continuum question by 'forcing' our way into the theory by extra-theoretical, that is, subjectively based and physically non-reductionistic means. This is the way for Heidegger to view, for example, the ordinary function of counting

> as the uncovering and making available of that which is present [*Anwesendes*] in its presentness [*Anwesenheit*]. To count is to render present. [...] Whether the Nows are 'counted' in terms of physical objects or psychological processes and 'data', it is always the 'time' we encounter in our expectant concern [*gewärtigenden Besorgen*] that we take account of. To count is to render present. ([9], p. 67).

In view of these considerations I draw attention to the following:

First, if the 'nows' are associated with the counting of physical objects or of psychological processes and 'data' and they are essentially the manifestations of the 'time' we encounter either in terms of original impressions (in Husserlian phenomenology) or of our expectant concern toward the future (in the Heideggerian narrative), then the 'nows' may apply both to the apprehension of an individual, let's say a formal one in a mathematical context, and an aggregation of such individuals in the form of an indefinite collection of them. This means that, independently of the way one may view the subjective enactment in the present now of apprehension, that is, independently of whether it is of a psychological or of an a priori transcendental character (in phenomenological concerns), one can have in the same terms the instant apprehension both of a formal individual as irreducible member of a (possibly indefinite) collection and the apprehension of the collection itself.

Second, one may recall any time in the present now of memory both a formal individual as such and a (possibly indefinite) collection of individuals in essentially identical acts of rememoration while preserving at the same time the intuition of individuality of each element in the collection ideally *ad infinitum*.[14] Therefore one may have the intuition of an indefinite collection as a noematic[15] whole in the present now something that implies a sense of 'inner' colligation of its elements necessarily presupposed to render a meaning of the collection as an objective whole in contrast with a supposedly random aggregation of elements deprived of some sense of meaning-giving connectedness. Furthermore, the intuition of an indefinite collection as a noematic whole is made possible both in the present now and as an instantaneous recall in memory, either at will or not, in a way that would be impossible without some sort of underlying immanent unity existing in advance of the reflection on the collection itself.

On such grounds this kind of unity, except for the conception of any set or class of abstract objects as a complete whole, must be also a prerequisite to establish the meaning-content of such abstract categorial forms as the relations of order, $\in$-inclusion, set-inclusion, absoluteness, etc. Moreover this continuous unity must necessarily involve a sense of inner temporality by being brought to reflection any instant in the present now and would be reasonably expected to belong to the subjective sphere rather than to the nature of the objectivities themselves for otherwise there would be no way to intuit the objective sphere the way it appears. It would further need to be founded still deeper, in a recursive mode, on a level of subjectivity that would dispense with all objectivity concerns for otherwise it would recur in an interminable cycle of reproducing itself as a 'subjectivity-conditioned' objectivity.

In a transcendental phenomenology perspective Husserl attributed the possibility to talk about finite or infinite collections of (formal) elements not

14. This may be seen as related with the intuition behind the *Axiom of Choice* insofar as this axiom can be extra-theoretically, indeed phenomenologically, reducible to intentional directedness toward an indefinite collection of objects-individuals potentially *ad infinitum*. I note that Heidegger had associated individuation, as a fundamental attribute of Dasein, with time in the sense that in achieving the authenticity of its being by projecting itself into the future, Dasein "enters the unique thisness and one-time-ness [*Diesmaligkeit*] of its thereness" ([9], p. 70). Naturally the intuition of individuation 'with respect to' in each temporal now presupposes the individuation of oneself in the deepest and most authentic sense.

15. A noematic object, which is a phenomenological term, is an object as meant, more specifically as constituted by certain modes as a well-defined object immanent to a subject's unity of temporal consciousness. Therefore it can be said to be given apodictically in experience inasmuch as: (1) it can be recognized by a perceiver directly as a manifested essence in any perceptual judgement (2) it can be predicated as existing according to the descriptive norms of a language and (3) it can be verified as such (as a reidentifying object) in multiple acts more or less at will. More in Husserl's *Ideas I*: [13], pp. 213-217.

only as colligations of object-elements in the actual now but as re-identifying objects-substrates of any sort of categorial formulas, to an act of a higher level, in fact not one of passive receptivity but one of productive spontaneity. This was termed a retrospective apprehension (*rückgreifendes Erfassen*), an act whose content is the thematization by the absolute constituting subjectivity of a collectivity of elements into an identifiable and re-identifiable meaning-object possibly posited as a substrate of judgments ([11], pp. 246-247). In this view Husserl characterized absolute time-consciousness as the original source of the constitution of the unity of identity in general, making clear that the outcome of temporal constitution is a universal form of order, of succession and coexistence of all immanent objects fed by our perception or imagination. Indeed except for the specific intentional modes of consciousness (i.e., the transversal and longitudinal intentionalities of consciousness), the passive unity of the pregivenness of a plurality of perceived or even imagined objects was ultimately attributed to a transcendental origin, the absolute ego thought to be the original source of self-constituting temporality. Husserl put it this way:

> [..] the unity of the intuition of time is the condition of the possibility of all unity of the intuition of a plurality of objects connected in any way, for all are temporal objects; accordingly, every other connection of such objects presupposes the unity of time. (ibid., p. 182).

Husserl was not the only prominent philosopher of the continental tradition to tie, to the one or the other extent, logical-mathematical and generally linguistic forms with constitutive subjective concerns. Heidegger treated mathematics as an ontological structure, a feature of the projection of Being underlying positive science, which is more basic than the science of mathematics or its elements and certainly overcomes mathematics meant as numerical determinations associated with counting and reckoning with time in the sense of quantifying time by means of encountered successive nows. Rather, as argued in *Being and Time*

> the essence of taking care of time does not lie in the application of numerical procedures in dating. What is existentially and ontologically decisive about reckoning with time must not be seen in the quantification of time but must be more primordially conceived in terms of the temporality of Dasein reckoning with time. ([8], p. 378).

And further in a bold enunciation of the primary role of subjectivity, in Heidegger's particular sense of Dasein, he pointed out

> That the 'I' comes to be defined as that which is already present for representation (the 'objective' in today's sense) is not because of any I-viewpoint or any subjectivistic doubt, but because of the essential predominance and the definitely directed radicalization of the mathematical and the axiomatic. ([7], p. 105).

It follows that insofar as the ontological foundation of mathematics may be at least partially reducible to the constitutive origin of human subjectivity there seems to be no way for the factor of inner temporality to be left unaccounted. In fact, for both Husserl and Heidegger and in spite of their otherwise diverging views on the nature of the absolute subjective factor this latter is essentially meant as origin of (inner) temporality.

One might plausibly ask at this point what relevance might have this talk in a context of discussion in which one has started to recalibrate the **CH** question by trying to reach beyond the strictly meant formal-theoretical universe.

Let's take the example of infinite sets and the way we may interpret the intuition of countably infinite sets in contradistinction with uncountably infinite ones. If we hold to the assumption above (§ 6), namely that the intuition of an indefinite collection as a noematic whole in the present now implies a sense of subjectively founded continuous unity which is more than a mere colligation of the elements of a psychological character, then one may reasonably wonder about the essential difference in the intuition of countably and uncountably infinite sets or classes if both are presented as completed wholes in the present now. My proposed interpretation may be articulated as follows:

Concerning denumerable infinity or potential infinity [16] as generated by adjoining a new element at a time to a pre-existing finite set, one talks about

16. In a phenomenological sense infinity in subjective terms may be viewed as reducible to a kind of immanent infinity totally unrestricted by real world, spatiotemporal constraints; see: [15], p. 45. Levinas, in his own narrative, sensing the transcendental character of immanent infinity stated that the idea of infinity has the exceptionality that its *ideatum* (i.e., infinity as presence) surpasses its idea, in fact the distance that separates the *ideatum* from its idea constitutes the content of the *ideatum* itself. The infinite is the attribute of a transcendental being as transcendental, the infinite is the absolutely other ([21], p. 49).

an idealization that plays a universal role in analytical logic, namely the fundamental form of 'this way ad infinitum' that produces infinity by iteration or more generally founds recursively enumerable processes and has as a subjective correlate the 'one can always do likewise'. Consequently we may have an infinity in the present now as a completed whole, yet this does not really stands as an ontological 'vacuity' since it is clear how, e.g., by some machine algorithm one can always go over to a next step and generate a new element and so on *ad infinitum*. If we are indeed ready to admit to some kind of ontological 'vacuum' this would rather be a situation in which there is no possibility of applying the subjective correlate 'one can always do likewise' meant as a simple intentional act of consciousness even in the absence of any real content of the intended object. One has in both cases (countable and uncountable) the intuition of infinity as an objective whole in the now of reflection, yet in the case of countability one has in principle the possibility of the reproduction of a process of 'nows' at will, whereas in the process of forming the class of subsets of a countably infinite set by the power-set axiom and *a fortiori* in the indefinite re-iteration of the same process of forming subsets, the intuition of a process of momentary 'nows' as applied to syntactical individuals of the theory is in perspective completely lost.

For what seems to be essentially the core matter of the continuum paradox, that is, having at once an intuition of a (uncountable) mathematical infinity as an objective whole in original presentation and yet a limited access to its 'inner' content, certainly not in terms of the sequential 'this way ad infinitum', the evident way to go through is to consistently project one's finistic categorial intuitions over the continuous horizon of infinity and at the same time take advantage of the 'breadth' of impredicative vagueness to prove or disprove higher infinity or continuum associated conjectures, first and foremost **CH** itself. In fact as I will try to show in the last section the attempt to resolve the **CH** question by appealing to higher order cardinals and the theory of inner models winds down to the non-eliminability of the formal or rather the ontological constraints of **CH**.

Perhaps the best intuitive picture we can have of the mathematical continuum are the open intervals of the real numbers, if not for the set (?) of real numbers itself, and yet continuum may not be conceivable at all as consisting of points (as zero-level elements) or of sets of points or of subsets of sets of points and so on, if it is to be an objective form of inner temporal unity. In other words, no formal language can establish a predicative universe to address the continuous unity, purportedly generated in subjective-constitutive terms by each one's absolute consciousness, [17] that founds *eo ipso* and in the present now of reflection all abstract categorial objects in virtue of objects of a formal theory. On such grounds even a mathematical sequence, independently of being prescribed by a recurrence formula or even being a lawless one, would not be conceivable as such, that is, as a complete mathematical object without a pre-existing immanent (temporal) unity. *A fortiori*, to have the intuition of applying indefinitely a unique choice, namely the possibility to apply the *Axiom of Choice* within the formal continuum, one has to take into account the continuum in subjective-constitutive and consequently temporal terms, implying by necessity the presence of a subject for whom the principle of the individuality of choice *ad infinitum* would be founded on his very constituting self. In simpler words the individuality of choice would be founded on a subject's being oriented intentionally, and therefore uniquely, each moment of reflection to a specific 'something-in-general', [18] independently of a content not even of a real existence of this 'something-in-general', and being conscious of such uniqueness by being conscious of his own unique self in reflection. In fact, talking on this level, it seems reasonable to deduce that it is thanks to such features of constitutive consciousness in the sense described that may be founded the possibility to project indefinitely such finitistic natural intuitions, abstracted as concepts of set theory, namely well-foundedness, well-ordering, transitivity, absoluteness, etc., to the indeterminateness of continuum. As a matter of fact these are principles that condition in an essential way the proof of the undecidability of **CH** in the respective proofs of Gödel and Cohen and underlie in one way or another more recent approaches toward resolving the *Continuum Hypothesis* as will be discussed in Sec. 5.

In the next I discuss the standard proof of the consistency of the negation $\neg$ **CH** with the predominant axiomatic theory **ZFC** in order to highlight its

17. Without wishing to enter into the 'esoterics' of the phenomenological philosophy I just note that absolute consciousness (essentially the absolute ego) in the Husserlian sense can be roughly communicated to be objectivity constituting consciousness as time constituting and not constituted and, far from a psychologistic description, as possessing certain a priori features, e.g., intentional directedness toward 'something-in-general'.

18. In *Formal and Transcendental Logic* Husserl thought of 'something-in-general' (or 'anything-whatsoever'; *Etwas-überhaupt*) as the unifying concept of both apophantic mathematics (derived from the aristotelian apophantics) and non-apophantic mathematics, i.e. mathematical analysis, set theory, the theory of ordinals, etc., in a sense that leaves out of question any relation whatsoever to a material content. In these terms formal mathematical disciplines are formal in the sense that they have as 'ontological' domain certain derived forms of the formal 'something-in-general' ([12], p. 77).

extra-theoretical constraints in view of my philosophical approach. Given the decisive role of the forcing method in guaranteeing the reliability of large cardinals theory toward resolving (among others) the **CH** question in recent research, I discuss almost exclusively the consistency of $\neg$ **CH** via the forcing method and I'll leave only a marginal place in the end of the section to refer to the implicit extra-theoretical assumptions involved in Gödel's proof of the consistency of **CH** with **ZFC**.

4. THE ONTOLOGICAL CONSTRAINTS IN THE UNDECIDABILITY OF CH

K. Gödel in 1938 and P. Cohen in 1963 established the consistency respectively of **CH** and $\neg$ **CH** with the **ZFC** axiomatical theory by constructing proper models of **ZFC**, the former by means of the constructible universe L generated by a recursive construction of logically definable sets along the ordinals, the latter by means of the extension $M[G]$ of a countable base model M without adding in effect new ordinals.

The constructible universe L is, by construction, a minimal inner model and precisely in virtue of its minimality has been disputed by many set-theorists as being too restrictive in comparison to the set-theoretic universe V, whereas the extended forcing model $M[G]$ offers the possibility of establishing various contradictory results by its 'flexibility' according to *ad hoc* manipulations in the base model M. Yet both are structured in ways that formally reflect similar constraints owing, on the subjective-constitutive level, to particular modes of constitution (discussed in the previous section) that essentially determine their 'ontology'.

Starting from Cohen's forcing method the alleged constraints may be summed up in the following:

As mentioned above, Gödel constructed L as the minimum inner model of **ZFC** + **CH** by recursively defining classes of logically definable sets. Now if we think of inner models in relation to outer models of a theory like **ZF** so that: (i) the binary relation on the inner model is the restriction of the binary relation of the outer model (ii) the transitivity property is preserved in passing from the outer to the inner model and (iii) ordinals are preserved between inner and outer models, then given Gödel's proof of $(L, \in) \models ZFC + CH$ one would make the plausible conjecture that if there is no nontrivial outer model of L then thanks to Gödel's 1938 results both **CH** and the *Axiom of Choice* (**AC**) would have been proved once and for all. Yet while generally for a model $(M, \mathcal{E})$ isomorphic to the segment $(V_\alpha, \in)$ of the set-theoretical universe V for some ordinal α, and for any model $(M', \mathcal{E}')$ such that $(M, \mathcal{E})$ is an inner model of $(M', \mathcal{E}')$ necessarily must be that $M = M'$, the same is not true when $(M, \mathcal{E})$ is countable. In short, the countability of the base inner model M permits in principle the construction of the non-trivial outer model $M[G]$ of forcing theory. Why this should happen in the first place? The answer in formal terms may be found in following the construction in the Cohen sense of a generic set G which does not belong to the countable base model M, curiously enough without adding any new ordinals to the base model. A standard way to achieve this is to primarily rely on the generic filter properties of G and the density of sets $D = \mathcal{P} \backslash G$ within M guaranteed by absoluteness of set-theoretical difference between the partially ordered set of forcing conditions $\mathcal{P}$ (within M) and the generic set G.[19]

In spite of the strict formalism employed to establish the existence of a non-trivial outer (forcing) model non-identifiable with an inner countable one, it is important to point to some key extra-theoretical prompts:

(**i**) the application of the *Axiom of Choice*, which is a principle not deducible from any other axiom or definition except for the need to establish a well-ordering principle for the real numbers. In section 3, I claimed that a subjectively founded interpretation of **AC** may be articulated in terms of a subject's intentional directedness toward each element of an indefinitely large collection (set or class) in preserving at the same time a sense of individuality of elements in an ideally indefinite extension.

(**ii**) the absoluteness of ordinals as the concrete form of the concept of absoluteness that runs through many proof-theoretic constructions of set-theory and generally of foundational mathematics, has been thought of by Cohen as having possibly a philosophical content running deeper than its formal significance within an axiomatical theory. One can possibly associate the absoluteness concept on the subjective phenomenological level with the invariable ontological character of empty-of-content individuals as lowest level 'substrates' of any analytical form and ultimately as evidences of intentional apprehension.

Ordinals as completely indeterminate from the point of view of analytical logic in terms of 'inner' content may become noematically determinate only in relation-to in the sense Husserl attributed to absolute substrates in *Experience and Judgment*:

19. More details and the proof of this assertion is found in Kunen's *Set Theory. An Introduction to Independence Proofs*; [20], p. 187.

> a 'finite' substrate can be experienced simply for itself and thus has its being-for-itself. But necessarily, is at the same time a determination, that is, it is experienceable as a determination as soon as we consider a more comprehensive substrate in which it is found. Every finite substrate has determinability as being-in-something, and this is true *in infinitum*. ([11], p. 137).

Knowing the invariance of ordinals across models and the possibility of reduction of absolute relations and functions to atomic formulas of individuals bound by logical connectives together with fundamental categorial properties (e.g., those of of inclusion, order, permutability, etc.), one can can get a subjectively founded notion of absoluteness underlying a strictly logical description.

(**iii**) the proof of the existence of a set G, the generic set, by virtue of which is generated the extended model $M[G]$ in the vast domain of which the original model M acts as a blueprint to produce various results contradicting hitherto proven ones, e.g., contradicting the validity of **CH**. By a simple straightforward proof, fundamentally supported by assumptions (**i**) and (**ii**), we obtain in case the generic G 'meets' forcing conditions in the original model M that contradict its filter and compatibility properties, that G cannot belong to M. Then $M \subset M[G]$, and consequently in general $M \neq M[G]$.[20] The existence of a generic set G not belonging to the countable base model M has been instrumental in Cohen's refutation of **CH** by the forcing method. Forcing method itself is known to have persistently obstructed the decidability of large infinity assumptions on the basis of designed axioms.

A. Badiou has attempted in *Being and Event* a kind of subjectively oriented, though not phenomenological properly meant, interpretation of the Cohen forcing theory in which the generic set G is termed the indiscernible for an 'inhabitant' of the countable base model M.[21] Largely in this sense we may eventually prove the existence of a set, the generic set G, with global properties over its domain by applying in its filter construction the countable terms of the 'and so on likewise' and yet make this set 'indiscernible' in the ontological domain by assigning a universal-existential formula of an indefinite scope contradicting the compatibility property of its genericity. In these terms the existence of the generic set is an ontological quasi certitude

20. See ibid., pp. 186-187, for the construction of G and the formal proof of $G \notin M$.

21. As a matter of fact Badiou has employed a virtually equivalent notion to that of a countable base model, namely that of a fundamental quasi complete situation based on four key hypotheses. See [2], pp. 392, 396. Engl. transl. pp. 356, 359-360.

since it is proved by means of the countability of the base model M, yet its existence is a 'matter of theological faith' for the inhabitants of M insofar as the generic G does not belong to M and is moreover proved indiscernible to M.

In fact the only way an inhabitant of M can have an intuition of the extended forcing model $M[G]$ is essentially due to the absoluteness property of the names defined by transfinite recursion on their rank, in the understanding that the stratification in the formation of names in M is guaranteed by the natural order and the absoluteness of ordinals. Therefore by the absoluteness of all terms and operations employed in the definition of names $\tau = < \sigma, p >$, where σ is a name and p a condition on M, 'being a name' in M and 'being a name' in the extension $M[G]$ in a general ontological sense coincide. It could be even said that the definition of the referential values of the names in $M[G]$,

$$\text{val}(\tau, G) = \{\text{val}(\sigma, G);\ \exists p \in G,\ < \sigma, p > \in \tau\}$$

are also comprehensible for inhabitants of M insofar as the generic set is viewed by them as simply a symbol 'designating an unknown transcendence' for which it makes perfectly sense to ask for each name in M whether the accompanying condition $p \in M$ belongs or not to the generic set G ([2], pp. 376-380). Therefore even though the generic set G does not belong and further may be an 'indiscernible' part of the model M, it can yet be made to exist in $M[G]$ and accomplish its primary *raison d' être*, namely to make manipulable (by absoluteness properties) the extended field $M[G]$ by any chosen combinatorics in the intuitively accessible countable base model M.

Essentially one has taken advantage of thinking in the intuitively clear terms of the 'so on likewise' and present an objective completed whole in the actual now (i.e., the generic set G) and at the same time have the intuition of another objective whole in actual presence in the indefinite domain of which he may assign an incompatibility formula by which it is proved impossible for the generic set G to belong to M. Just give a moment's thought and ponder how we could have the intuition of a set generated by a countable process performable *ad infinitum*,[22] and at the same time contradict this intuition by an incompatibility formula of an indefinite ontological scope, if not for a sort of continuous unity presumably belonging to the subjective and

22. I mean in this case the construction of the generic set by the Rasiowa-Sikorski lemma, i.e., by MA(ω). See [20], p. 187.

not to the objective sphere to which belong the formal elements themselves as objects of rational (mathematical) imagination.

Generally mathematical experience has shown at this level that it is hardly possible and largely unwanted to dispense with such principles, as a unique choice *ad infinitum*, well-foundedness, absoluteness, etc., that project the intuitions implied (but not founded) by the finiteness of the perceptual world to an indefinite domain presented in immanence as a complete whole independently of cardinality. This is a state-of-affairs which by all accounts cannot be founded on the objectivity of the world for then it would not be possible to have the intuition of a background unity free of spatiotemporal constraints upon which to perform meaningful mathematics and *a fortiori* the mathematics of infinity. Building foundational mathematics in formal-axiomatical terms and, in particular, approaching the question of continuum in these terms seems to be like 'tinkering' with the expressional tools of a formal theory upon the pre-existing, non-eliminable ground of a subjectively constituted immanent unity.

To a certain extent this also concerns Gödel's proof of the consistency of **CH** with **ZFC** which in spite of the restrictive character of the axiom $V = L$, due to the generation of the sets of the constructible universe L by logical definability, is partly conditioned either in Gödel's original version or in subsequent ones on the validity of the *Axiom of Choice*, the concept of absoluteness and certain actual infinity conditioned principles, implying by this fact alone the implicit acceptance of their alleged extra-theoretical content. [23]

5. WHY LARGE CARDINALS THEORY CANNOT RESOLVE THE CONTINUUM QUESTION?

Gödel's hopes to resolve the **CH** question on the assumption of some *ad hoc* large cardinal axiom/s were virtually shattered by the power of the forcing method to undermine established results through proper modifications at the 'ground level'. For instance, except for the well-known proof of $\neg$ **CH**, one may produce outer models of a given countable model in which **CH** holds. Moreover it is known that Cohen's outer model construction preserves all known large cardinals. It is largely telling on the limitations of the large cardinal approach that Levy and Solovay had proved in 1967, in [22], that the addition of a large (i.e., a measurable) cardinal to the **ZF** theory leaves the

23. More details in this approach the reader may find in [23].

CH question still undecidable. On these grounds and concerning Gödel's speculation that the key to resolve **CH** might be the development of the large cardinals theory, Woodin has claimed that

> by Cohen's method and by its adaptation to also produce outer models of a given countable model, in which (*auth add.*: except for $\neg$ **CH**) **CH** holds, this cannot happen. ([32], p. 201).

Except for these apparent inconveniences, the project of resolving the *Continuum Hypothesis* by means of large cardinals and equiconsistency results seems to be compromised by the following metatheoretical and presumably extra-theoretical grounds:

(a) By Gödel's Second Incompleteness Theorem, **ZFC** cannot prove by its own means that there exists a model for **ZFC**+φ, where φ is a large cardinal axiom, unless **ZFC**+φ is inconsistent.

(b) The undertaking of better comprehending the nature of **CH** in mathematical terms by narrowing the scope of investigation to the special case of projective sets has implicated the *ad hoc* assumption of the existence of an infinite class of special large cardinals (i.e., the Woodin cardinals). A fundamental axiom in the case at hand, the *Axiom of Determinacy* (*AD*), is refuted by the *Axiom of Choice* in an outstanding example of the possibility of manipulation of categorial-logical constructs in a set-theoretical environment across non-denumerable infinities. [24]

As mentioned earlier Gödel's constructibility axiom $V = L$ due to its guiding principle of logical definability is considered as being quite 'restrictive' with regard to the supposed amplitude of the set-theoretical universe V. Yet, due to its advantages in resolving at once except for the **CH** and the **AC** as such their projective versions as well and the fact that the constructible universe L is immune to Cohen's 'tinkerings' by the forcing method it was later seen that L could be so extended that large cardinal axioms might hold in it. However in the process of constructing new enlargements of L to accommodate ever larger cardinals in the context of the inner model program, it was proved that every specific enlargement so constructed was subject to a generalized form of Scott's theorem (see [18], p. 49). For example, in any enlargement of L constructed using the Mitchell-Steel methodology

24. See [17], pp. 627-628. The author wishes to state that for the highly technical notions of this section which the general reader might skip, the interested one is referred mainly to Kanamori's book *The Higher Infinite*, [18].

the enlargement would refute the existence of a very strong species of large cardinals, the supercompact cardinals ([27], p. 143). This given W. H. Woodin has proposed in [31] and [32] a universal extension of $V = L$, the $V =$ Ultimate $- L$ axiom, in terms of which one may deduce, contrary to his earlier views, that **CH** is true. This is however associated with a generalization of projective sets that touches upon the topological structure of the set of real numbers R in the sense that under $V = L$ every set $A \subseteq R$ is the image of a universally Baire set [25] by a continuous function $F : R \longrightarrow R$ which may further satisfy certain closure properties ([32], p. 210).

Ultimately in Woodin's approach the $V =$ Ultimate$-L$ axiom would be logically consequent to his so-called Ultimate $- L$ Conjecture which, except for being just a conjecture at the time being, hinges on the existence of very large cardinals under special conditions in an inner model by the Hamkins universality theorem. At the same time one must be able to prove in an almost circular turn that there is no generalization of the restrictive Scott's theorem for the axiom $V =$ Ultimate $- L$ in an inner model. Eventually in case $V =$ Ultimate $- L$ proves to be true it would likely be, in Woodin's view, the key missing axiom which would be consistent with all known large cardinal axioms and at the same time would resolve all questions, including of course the **CH** question, that have been made undecidable by Cohen's forcing method (ibid., pp. 217-218). [26] Very recently Woodin has proposed a scenario (the so-called N-scenario) that refutes the HOD conjecture, a fact that would presumably open the way to prove the validity of $V =$ Ultimate$-$ L in HOD. However the N-scenario, on the one hand, it is still an unproved hypothesis with too many technical prerequisites and, on the other, does not weaken in the least my ontological arguments on account of the *Continuum Hypothesis* as exposed in this article.

(c) As Woodin's undertaking to formulate the $V =$ Ultimate $- L$ axiom, on a generalization of projective sets to universally Baire subsets of R,

25. This is a technical topological definition involving the notions of continuity and openness within R. The interested reader may look for these details into [31], p. 95.

26. It happens that the validity of the $V =$ Ultimate $- L$ axiom as associated with the proof of the Ultimate $- L$ Conjecture proves to be a quite complex matter. In recent research Bagaria, Koellner and Woodin have shown that it might well be that the Ultimate $- L$ Conjecture fails (due to indications of consistency of a hierarchy of super large cardinals without the *Axiom of Choice*) in which case all welcome results of the validity of the Ultimate $- L$ Conjecture, among them the truth of **CH** and the identification of the set-theoretical universe V with the highly definable class HOD, collapse and then the 'chaos prevails'. More details on the matter in [1].

implies topological concepts inherently associated with the specific topological structure of R, a key objection on the metatheoretical level that can be raised might go like this: How may one *ad hoc* employ topological properties of the reals, given that the set of reals acquires its topological relevance essentially by defining topological openness in terms of the openness of real intervals, without implicitly admitting that there is no intermediate cardinality between R itself (or any of its open intervals) and any of its countable subsets? In other words without indirectly conceding to the validity of **CH**?

Moreover if by the application of the power set axiom on the set of natural numbers we reach the next level of infinity, the level of real mathematical continuum, how can we in ontological consistence involve objects belonging to this level of infinity, e.g. universally Baire sets, in the process of proving assumptions touching among others on the cardinality of this infinity, namely the **CH** question? Further, is there any ontological foundation in pursuing the application of the power set axiom beyond $\mathcal{P}(\mathbf{N})$ and reaching for successive infinities $\mathcal{P}(\mathcal{P}(\mathbf{N}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbf{N})))$, ..., except for reasons of mathematical 'recreation' as W. V. Quine once put it? [27]

If it can be indeed vindicated, as it was done in sections 3 and 4, the possibility of an extra-theoretical reduction of the mathematical continuum in subjective-constitutive terms by means of which we can have in abstraction an indefinite logical-mathematical field for formal axiomatization and predication, then by force of evidence this level of infinity is bound to re-emerge in explicit or implicit form any time we make use of principles or axioms, e.g., the *axiom of choice* or the *well-foundedness principle*, etc., that establish a kind of normativity on account of this indefinite formal field.

On these grounds it is extremely doubtful whether one can have a settlement of the question of continuum, at least in ontological terms, by explicitly or implicitly re-introducing the continuum level infinity in reaching to ever greater large cardinals with the hope to resolve the question on a lower-level infinity. And it is ultimately the implicit presupposition of the continuum level infinity in almost every meaningful mathematical activity and at any infinity level that the phenomenological narrative may acquire a measure of relevance on the matter to the extent that it seems to re-orient the ontological

27. In [26] W. V. Quine wrote: "I recognize indenumerable infinities only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in excess of such demands, e.g., $\beth_\omega$ [the cardinal number of $V_\omega(N)$ and of $V_{\omega+\omega}$] or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights" (ibid., p. 400).

discussion on the continuum far from any variant of platonism toward the ways a subject may constitute formal-mathematical objects including infinite ones as 'finitistic' noematic objects in the modes of his absolute constitutive capacities. To the extent that this may formally translate to the mathematical continuity, insofar as mathematics may be embedded in set theory,[28] and as the continuum question either implicitly or explicitly and certainly non-negligeably, as argued above, constrains the structure of higher than $\aleph_o$ infinity theories, it is questionable whether one may resolve the continuum question solely within the formal set-theoretical environment. Consequently this may pose even more clearly, even to the non-philosophically minded mathematician the need to approach the **CH** question as a multi-dimensional one that also includes certain non-eliminable parameters reducible to extra-theoretical, even phenomenological concerns.

To the extent that

> The connection between philosophy and mathematics is not made via the discipline of formal logic, but rests chiefly on the mathematics-ontology nexus[29],

both Husserl and Heidegger would have a say on the connection of mathematics with the question of being,[30] in consequence with the question of the constitution of continuous unity as a mode of constitution of being, in the particular exegetic approach of each one. Furthermore, such connection may also bring to the fore the Kantian project of transcendental philosophy with its own truth conditions, its own objects "including model structures, transcendental trees, and intentional objects" which "allows us to say, in the mathematical case, that the source of evidence is consciousness itself" ([25], p. 129).

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28. See Maddy's thoughts on this matter in [24].
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