Minicourse (4 +1 lectures)

The Chalker-Coddington Network Model of the Integer Quantum Hall Transition

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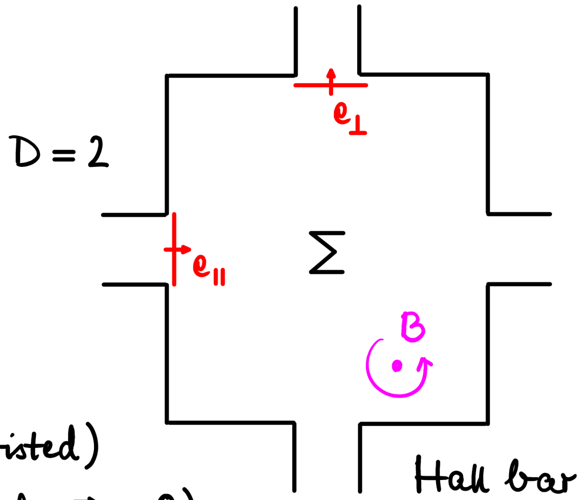
Workshop "Probability and Mathematical Physics: Walks, Loops, Spins, Fields"

Lecture 1: Quantum Hall Basics

1.1 Conductance

(Hall and dissipative)



[Assume d.c. limit]

a) The current-density $(D-1)$-form j (twisted) is closed: $dj=0$ ("Kirchhoff's rule", $\mathrm{div}\,\vec{j}=0$).
Current $I \equiv [j] \in H^{D-1}(\Sigma, \mathcal{L})$.

b) The electric-field 1-form E is exact: $E = -d\phi$.
Voltage $V \equiv [E] \in H^1_c(\Sigma)$.

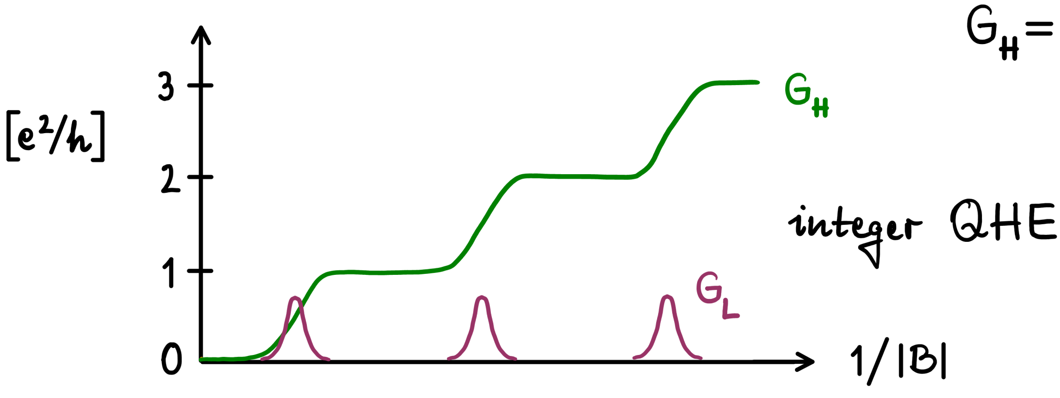
c) "Power" pairing: (non-degenerate)

$$H^{D-1}(\Sigma,\mathcal{L}) \otimes H^1_c(\Sigma) \longrightarrow \mathbb{R},$$
$$[j] \otimes [E] \longmapsto \int_\Sigma j \wedge E.$$

d) Conductance G: $H^1_c(\Sigma) \longrightarrow H^{D-1}(\Sigma,\mathcal{L}) \cong H^1_c(\Sigma)^*$,

$$V \longmapsto I = GV.$$

- Decompose conductance as $G = G^{sym} + G^{skew}$ (dissipative + Hall).
[Onsager relation: $G^{sym}(-B) = +G^{sym}(B)$, $G^{skew}(-B) = -G^{skew}(B)$.]
Quantum Hall Effect: $G^{sym}=0$, G^{skew} quantized (in some range of B).
[Warning/disclaimer: experiments measure the resistance, $R = G^{-1}$.]
Given cohomology generators $e_\parallel, e_\perp \in H^1_c(\Sigma)$, let $G_L = e_\parallel(Ge_\parallel)$,
$G_H = e_\parallel(Ge_\perp)$.



1.2 High-Field (Semiclassical) Limit

<u>Caveat</u> : Assume the single-electron approximation!

(justified by low-temperature limit, Kubo linear response theory, and disorder stronger than electron-electron interactions)

- Charged particle in a (static) field E, B as a Hamiltonian system (M, Ω, H):
  - — phase space $M = T^*\Sigma$,
  - — symplectic form $\Omega = dp_i \wedge dq^i + \frac{1}{2} e B_{ij}\, dq^i \wedge dq^j$,
  - — Hamiltonian $(D=2)$: $H = \frac{p_1^2 + p_2^2}{2m} + e\phi(q^1, q^2)$.

- High-field limit: cyclotron frequency $\omega_c = \frac{|eB|}{m}$ largest scale. Can take average over one cycle of the fast variables (p_1, p_2); cf. Arnol'd.

  $\longrightarrow$ Dimensional reduction (DR):

  $$M_{DR} = \Sigma, \quad \langle q^1 \rangle_{cycle} \equiv x, \quad \langle q^2 \rangle_{cycle} \equiv y,$$

  $$\Omega_{DR} = m\omega_c\, dx \wedge dy, \quad H_{DR} = e\phi(x,y).$$

- Guiding-center drift (slow variables x, y).

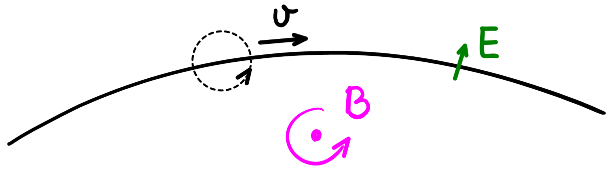
  Recall $dH = \Omega(\cdot, X_H)$ (Hamiltonian vector field X_H)

  Equations of motion: $|B|\,\dot{x} = -\frac{\partial \phi}{\partial y}, \quad |B|\,\dot{y} = +\frac{\partial \phi}{\partial x}$.

  $\Longrightarrow \phi(x(t), y(t)) = \text{const}$,

  guiding-center drift with speed $|v| = |E|/|B|$ (actually, $E = \iota(v)B$).

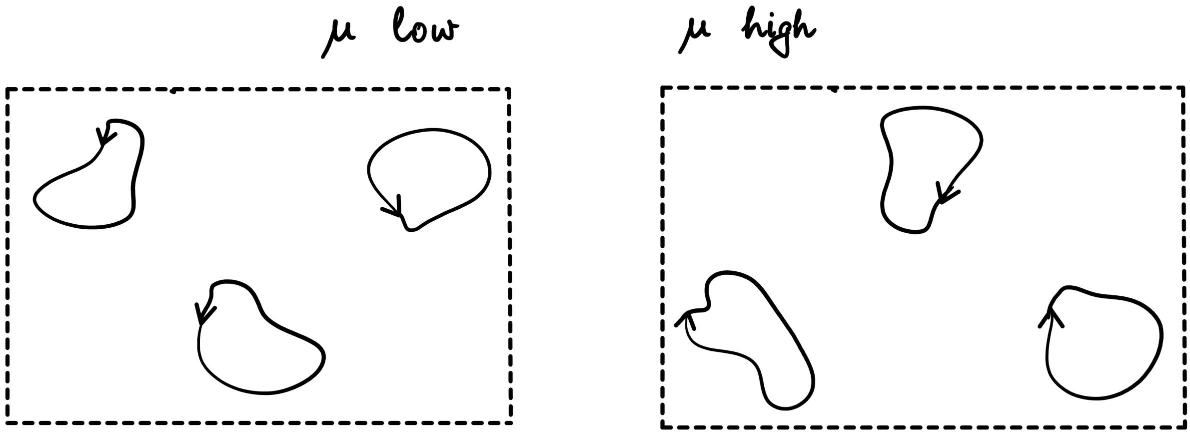
Illustration.



$\phi = \text{const}$

1.3 Quantum Percolation Scenario

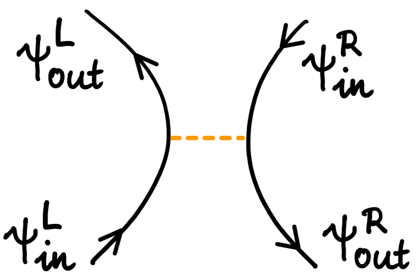
Recall: guiding center drift along equipotentials $\phi(\cdot) = \mu = \text{const}$ (chemical potential μ).



Critical point $\mu = \mu_c$: phase transition of percolation type

<u>Observation</u>. Must take into account quantum tunneling across saddle points, as the phase transition in the classical limit ($\rightarrow$ percolating equipotential) falls into a different universality class.

quantum tunneling:

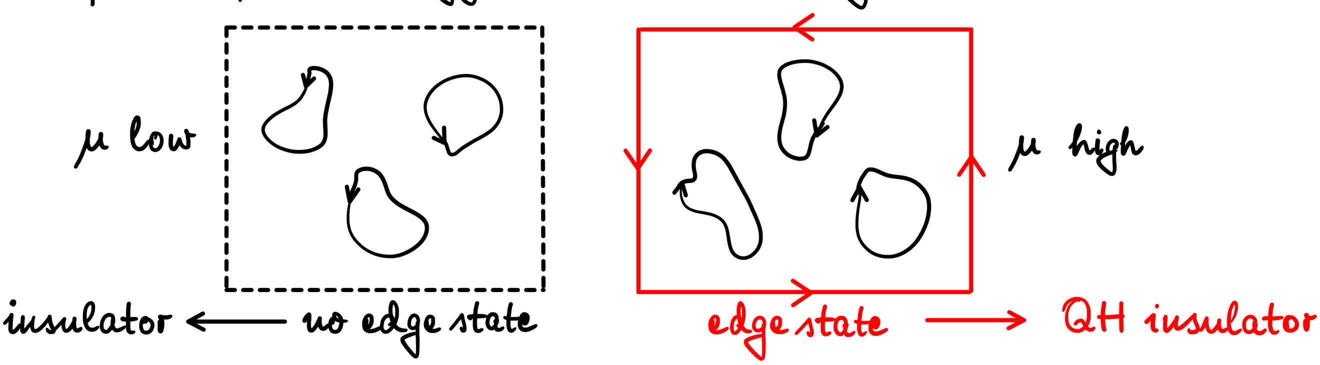


current conservation $\Longrightarrow$

$$|\psi^L_{out}|^2 - |\psi^L_{in}|^2 = |\psi^R_{in}|^2 - |\psi^R_{out}|^2 .$$

<u>Challenge</u>. Compute the universal properties of this quantum Hall percolation transition.

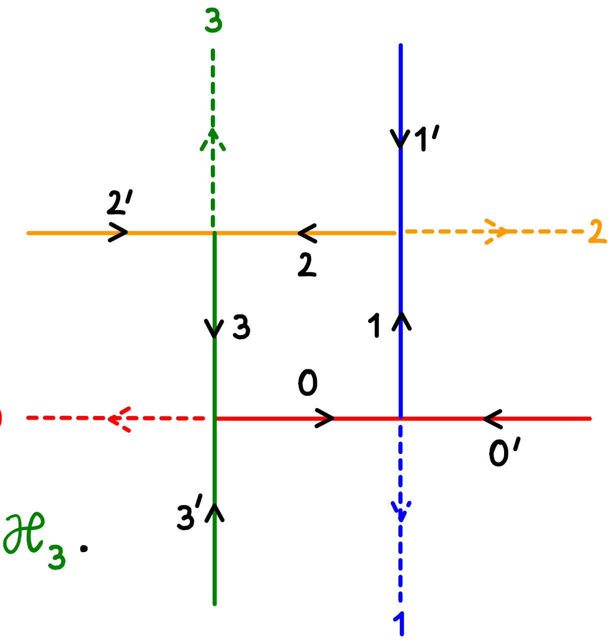
- Perspective from topology (bulk $\longleftrightarrow$ boundary).



Lecture 2 : Network Model

2.1 Chalker-Coddington model ($N_c = 1$).

— Square lattice $\Lambda \subset \mathbb{Z}^2$
with directed links. Unit cell :



Hilbert space :

$$\mathcal{H} = \bigoplus_{\text{links } \ell} \mathbb{C}(\ell) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 .$$

— Unitary quantum dynamics (discrete time): $\psi_{t+1} = \mathcal{U}\psi_t$, where
$\mathcal{U} : \mathcal{H}_\nu \longrightarrow \mathcal{H}_{\nu+1}$ $(\nu = 0, 1, 2, 3, 4 \equiv 0)$ is of Floquet type,
$\mathcal{U} = \mathcal{U}_r \mathcal{U}_s$, with $\mathcal{U}_r$ link-diagonal, $\mathcal{U}_r(\ell) = e^{i\theta(\ell)}$,

i.i.d. random variables,
Haar distributed on $U(1)$,

and $\mathcal{U}_s$ deterministic scattering at the nodes,
with amplitude a_L (a_R) for left (right) turn, and (unitarity →)

$$|a_L|^2 + |a\ |^2 = 1, \quad \arg(a_L) = -\arg(a_R) = \frac{\pi}{4} \quad \text{[Kac-Ward]}.$$

<u>Note</u>. The model is critical (localization length $= \infty$) for $|a_L|^2 = |a_R|^2 = \frac{1}{2}$.

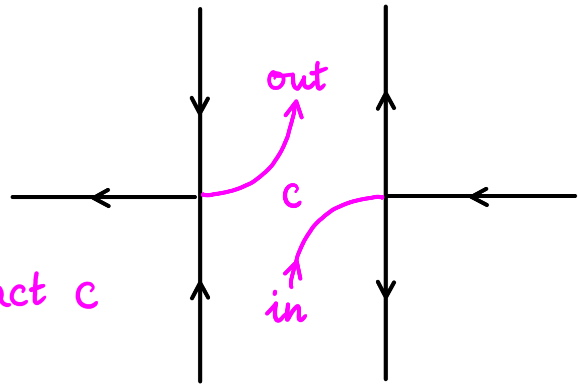
<u>Remark</u>.

$$\mathcal{U}_s\, e_\nu(\square) = e_{\nu+1}(\square)\, a_L + e'_{\nu+1}(\square + t_\nu)\, a_R ,$$

$$\mathcal{U}_s\, e'_\nu(\square) = e_{\nu+1}(\square)\, a_R + e'_{\nu+1}(\square + t_\nu)\, a_L ,$$

$$t_0 = -\delta_y, \quad t_1 = +\delta_x, \quad t_2 = +\delta_y, \quad t_3 = -\delta_x .$$

2.2 Point-contact conductance.



point contact c

- Consider the case of two point contacts, located at links c_A, c_B.

  Let $Q := \mathrm{Id}_{\mathcal{H}} - P_A - P_B$ (projection operators).

— Conductance: $G_{AB} = \left| \mathcal{U}(1-T)^{-1}(c_A, c_B) \right|^2, \quad T = Q\,\mathcal{U}.$

  Mean conductance: $\mathbb{E}\, G_{AB} = \sum\limits_n \mathbb{E} \left| (\mathcal{U}T^n)(c_A, c_B) \right|^2.$

The path sum $(\mathcal{U}T^n)(c_A, c_B) = \sum\limits_{\ell_1, \ldots, \ell_n \neq c_A, c_B} \mathcal{U}(c_A, \ell_n)\, T(\ell_n, \ell_{n-1}) \cdots T(\ell_1, c_B)$

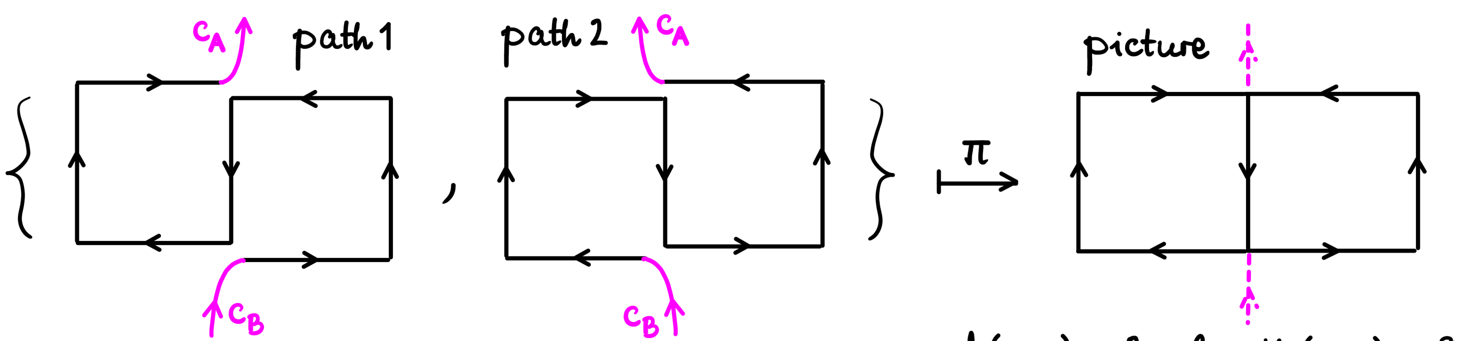
doubles for $\left| (\mathcal{U}T^n)(c_A, c_B) \right|^2 \longrightarrow$ "retarded" and "advanced" paths.

<u>Remark</u>. Valid paths start at c_B, never return to c_B, and terminate upon first arrival at c_A.

- Def (Bettelheim, Gruzberg & Ludwig, 2012). [BGL12]

  Organize all equal-length paths that visit the <u>same</u> set of links the <u>same</u> number of times, into an equivalence class called a "picture" (pic).

  Very simple example:

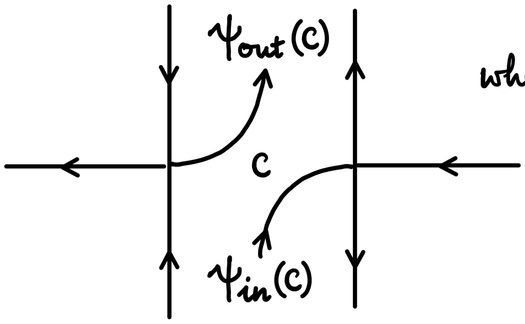


card(pic) $= 2$, length(pic) $= 8$.

<u>Fact</u>. The path double sum for $\mathbb{E}\, G_{AB}$ reduces to a positive sum over pictures. At criticality: $\mathbb{E}\, G_{AB} = \sum\limits_{\text{pic } B \to A} \mathrm{card}^2(\text{pic})\, 2^{-\mathrm{length}(\text{pic})}.$

2.3 Point-contact stationary states. [BWZ17]

Consider the simplified situation of a <u>single</u> point contact, c:



where $\psi = \mathcal{U}\psi$ stationary scattering state,

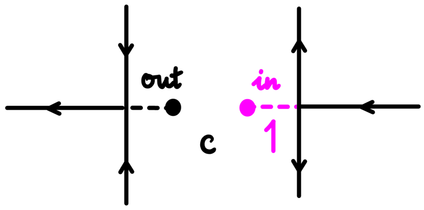
and $\psi_{out}(c) = S\,\psi_{in}(c)$,

with S scattering "matrix" $S \in U(1)$.

- Precise formulation: impose boundary conditions. P projector for contact link c

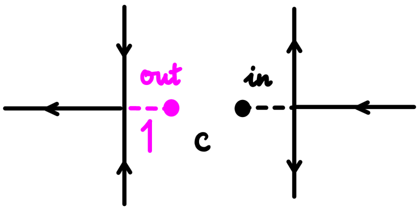
— incoming-wave b.c. ($\psi \equiv \psi^+$, $Q = 1 - P$)

$$\psi^+ = Q\,\mathcal{U}\,\psi^+ + 1 \cdot e_c$$



— outgoing-wave b.c. ($\psi \equiv \psi^-$)

$$\psi^- = Q\,\mathcal{U}^{-1}\psi^- + 1 \cdot e_c$$



<u>Remark</u>. $\psi^-(\ell) = S\,\psi^+(\ell)$ (for all ℓ), $|S| = 1$.

- Random variable of interest:

$$|\psi_c(\ell)|^2 \equiv |\psi^\pm(\ell)|^2 = \left|(1-T)^{-1}(\ell,c)\right|^2, \quad T = Q\,\mathcal{U}.$$

<u>Lemma</u>. $|\psi_c(\ell)|^2 \overset{\ell\neq c}{=} G(\ell,\ell)$, $\quad G = T(1-T)^{-1} + (1-T^\dagger)^{-1}$.

Proof. $G := \psi_c(\psi_c, \cdot)$

$$= \psi^-(\psi^-, \cdot) \overset{\text{owbc}}{=} Q(1-T^\dagger)^{-1}\mathcal{U}^{-1}P\,\mathcal{U}(1-T)^{-1}Q.$$

Use $\mathcal{U}^{-1}P\,\mathcal{U} = 1 - T^\dagger T = (1-T^\dagger)(1-T) + (1-T^\dagger)T + T^\dagger(1-T)$,

hence $G = Q + T(1-T)^{-1}Q + Q(1-T^\dagger)^{-1}T^\dagger$

$$= T(1-T)^{-1}Q + Q(1-T^\dagger)^{-1} \quad \blacksquare$$

<u>Corollary</u>. $\mathbb{E}\,|\psi_c(\ell)|^2 = 1$ for all ℓ.

Lecture 3: Methods of Analysis

3.1 Wegner–Efetov SUSY method (first step).

Express $\mathbb{E}\,\left|(E \overset{R}{\underset{A}{\pm}} i\varepsilon - H)^{-1}(x,y)\right|^2$ by

$$A^{-1}(x,y) = \frac{\mathrm{Cof}_{y,x}(A)}{\mathrm{Det}(A)}$$

fermionic $\prod \frac{\partial^2}{\partial\xi\,\partial\eta}\, e^{-(\xi, A\eta)}\, \eta(x)\,\xi(y)$,

bosonic $\int e^{-(\bar\varphi, A\varphi)}$

and take disorder average.

Then, Hubbard–Stratonovich transformation, etc.

3.2 Variant: "color–flavor transformation"

$$\mathbb{E}\,\left|(1 - z\mathcal{U})^{-1}(x,y)\right|^2 \overset{|z|<1}{=} \int_B \int_F \eta_R(x)\,\xi_R(y)\,\eta_A(x)\,\xi_A(y)$$

$$\times\ \mathbb{E}\exp\Big(-\big(\xi_R, (1 - z\mathcal{U})\eta_R\big) - \big(\xi_A, (1 - \bar z\mathcal{U}^\dagger)\eta_A\big) - \text{"bosons"}\Big).$$

Now for $\mathcal{U} = \mathcal{U}_r \mathcal{U}_s$, $\mathcal{U}_r \in U(N_c) \times \cdots \times U(N_c)$, taking averages w.r.t. Haar measure on $U(N_c)$ leads to an unwieldy expression (modified Bessel functions). What to do?

$\rightarrow$ CFT (here in schematic form): [Zi96]

$$\mathbb{E}_{\mathcal{U}} \exp\left(\bar\psi_R \mathcal{U} \psi_R + \bar\psi_A \mathcal{U}^\dagger \psi_A\right) = \mathbb{E}_Z \exp\left(\bar\psi_R Z \psi_A + \bar\psi_A \tilde{Z} \psi_R\right)$$

[replaces the HS-transformation of Wegner–Efetov].

<u>Warning</u>: complications for $N_c = 1$!

3.3 Read's method ("second quantization"). [BWZ17]

$$H \in \mathrm{End}(V) \begin{array}{l} \nearrow \; \hat{H}_F \in \mathrm{End}(\Lambda(V)) \quad \text{"fermionic" SQ,} \\ \searrow \; \hat{H}_B \in \mathrm{End}(S(V)) \quad \text{"bosonic" SQ.} \end{array}$$

$$\hat{H}_F = f_i^\dagger H^i{}_j \otimes f^j$$

$$H = e_i H^i{}_j \otimes e^j$$

$f_i^\dagger = \varepsilon(e_i)$ | $f^j = \iota(e^j)$

particle creation | annihilation

$b_i^\dagger = \mu(e_i)$ | $b^j = \delta(e^j)$

$$\hat{H}_B = b_i^\dagger H^i{}_j \otimes b^j .$$

- Character formulas.

— $\mathrm{STr}_{\Lambda(V)}\, e^{\hat{H}_F} = \mathrm{Det}(1 \pm e^H)$, $\quad \mathrm{STr}_{\Lambda(V)} \equiv \mathrm{Tr}_{\Lambda^{ev}(V)} - \mathrm{Tr}_{\Lambda^{odd}(V)}$.

— $\mathrm{Tr}_{S(V)}\, e^{\hat{H}_B} = \mathrm{Det}^{-1}(1 - e^H)$, $\quad$ if $\mathrm{Re}\, H \equiv \frac{1}{2}(H + H^\dagger) < 0$.

— $Z \equiv \mathrm{STr}_{\mathcal{F}}\, e^{\hat{H}} = 1$, $\quad \mathcal{F} = S(V) \otimes \Lambda(V)$, $\quad \hat{H} = \hat{H}_B + \hat{H}_F$.

<u>Note</u>. The Lie algebra representations $H \mapsto \hat{H}_X$ $(X = B, F)$ exponentiate to (semi-)group representations, i.e. we may pass to $U \mapsto \rho_X(U)$, $\rho_X(e^H) \equiv e^{\hat{H}_X}$ $(X = B, F)$.

- <u>Key relations.</u> Let $\rho(U) \equiv \rho_B(U)\,\rho_F(U)$.

— $(1 - U)^{-1}(x, y) = \mathrm{STr}_{\mathcal{F}}\, \rho(U)\, f(x)\, f^\dagger(y)$,

— $\rho(U_r U_s) = \rho(U_r)\, \rho(U_s)$,

— $\mathbb{E}\, \rho(U_r) = \prod_{\text{links}} P(\ell)$, where $P(\ell)$ projects on $\ker(\hat{n}_R - \hat{n}_A)(\ell)$;

for $N_c = 1$: $\; P = \int \frac{d\theta}{2\pi}\, e^{i\theta(\hat{n}_R - \hat{n}_A)}$, $\; \hat{n}_Y = b_Y^\dagger b_Y + f_Y^\dagger f_Y$ $(Y = R, A)$.

Remark. Leads to SUSY vertex model repn of the network model.

<u>Corollary</u> (from SUSY vertex model).

$$\mathbb{E}\,|\psi_c(r)|^{2q} = \mathbb{E}\,|\psi_c(r)|^{2(1-q)} \quad (q \in \tfrac{1}{2} + i\mathbb{R}).$$

Question. The marginal field e^t of the $H^{2|2}$-model corresponds to (the classical version of) $B^\dagger B > 0$, $B = b_R + b_A^\dagger$. Can one find the marginal distribution of the latter?

Generating function. $\mathbb{E}\,(1 + t\,|\psi_c(r)|^2)^{-1} = \mathrm{STr}_{\mathcal{V}}\, \pi(c)\, \varrho(\hat{\mathcal{U}}_s)\, \varrho(e^{-tY(r)})$, after projection $\mathcal{F} \to \mathcal{V}$ by $\mathbb{E}\,\varrho(\mathcal{U}_r) = \prod_{\text{links}} P(\ell)$, and with $Y = B^\dagger B$.

3.4 Cauchy transform.

Let $A_h = \dfrac{1+h}{1-h}$. Then if all of $1-g$, $1-h$, and $1-gh$ are invertible one has the identity

$$\begin{aligned}(1-gh)^{-1} &= (1-h)^{-1}\left(\tfrac{1}{2}(A_g + A_h)\right)^{-1}(1-g)^{-1} \\ &= (1-g)^{-1} - g(1-g)^{-1}\left(\tfrac{1}{2}(A_g + A_h)\right)^{-1}(1-g)^{-1}.\end{aligned}$$

Apply this to the $N_c = 1$ network model, setting $g = \mathcal{U}_r$, $h = \mathcal{U}_s$. Then for $x \neq y$,

$$|(1 - \mathcal{U}_r \mathcal{U}_s)^{-1}(x,y)|^2 = q(x)\,|(T+V)^{-1}(x,y)|^2\, q(y),$$

$$T = \frac{1+\mathcal{U}_s}{1-\mathcal{U}_s}, \quad V = \frac{1+\mathcal{U}_r}{1-\mathcal{U}_r}, \quad q(\ell) = \frac{2}{|1-e^{i\Theta(\ell)}|^2}.$$

<u>Remark</u>.

T nonlocal, deterministic, translation-invariant;

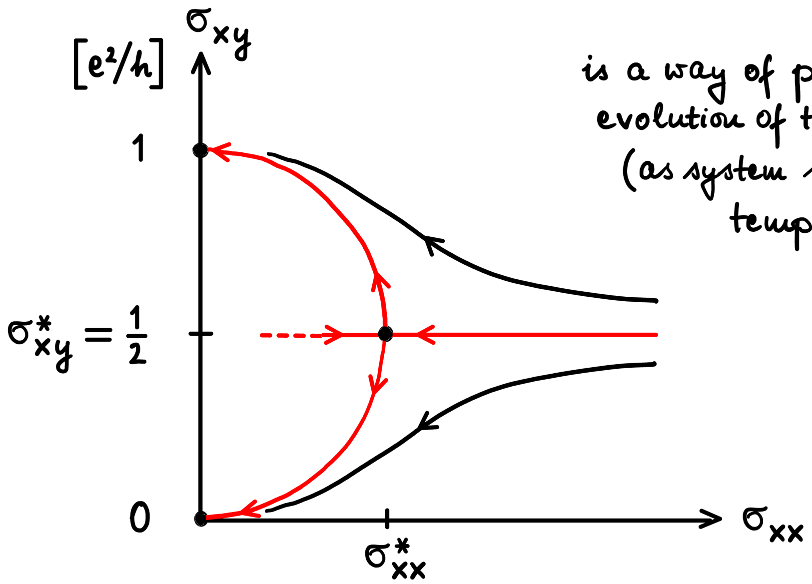
V local, random with Cauchy-distributed matrix element

$$V(\ell) = \frac{1+e^{i\Theta(\ell)}}{1-e^{i\Theta(\ell)}}.$$

$\longrightarrow$ methods of Wegner–Efetov available here!

Lecture 4: Conjectures

- Pruisken–Khmelnitskii scaling / flow diagram:



is a way of plotting the evolution of the experimental data (as system size $L \to \infty$, temperature $T \to 0$).

Question (for Theorists): is this the renormalization group flow diagram for a nonlinear σ–model (Pruisken, 1983) with metric coupling σ_{xx}, topological coupling $\sigma_{xy} = \theta/2\pi$ (angle θ), and RG–beta vector field

$$\beta = \frac{1}{\nu}\,(\sigma_{xy} - \sigma^*_{xy})\,\frac{\partial}{\partial \sigma_{xy}} - y\,(\sigma_{xx} - \sigma^*_{xx})\,\frac{\partial}{\partial \sigma_{xx}} \quad ?$$

Remark. By standard RG reasoning, the positive number ν is the critical exponent for the localization length, $\xi \sim |\sigma_{xy} - \sigma^*_{xy}|^{-\nu}$, while y is the exponent for the leading corrections to the scaling limit.

Note. There exists no consensus on the values of the critical exponents ν, y from numerical simulations. Values reported by different groups vary in the range of $0 \leqslant y < 1$, $2.3 < \nu < 3.9$.

[Proposed explanation [Zi21]: the zero of the RG–β vector field is actually of higher order than what is assumed in the nonlinear σ–model picture of Pruisken–Khmelnitskii scaling; i.e. $\frac{1}{\nu} = 0 = y$. The finite-size scaling analysis of the numerical data has to be carried out with a modified scaling ansatz.]

Non-controversial prediction from Pruisken–Khmelnitskii: the metallic phase is absent, i.e., all states off of the critical line $\sigma_{xy} = \sigma^*_{xy} = \frac{1}{2}$ are localized.

- **Gaussian Free Field Hypothesis** [BWZ17].

In the scaling limit at criticality, the law of the random variable $\ln|\psi_c(r)|^2$ is expected to be that of a Gaussian Free Field φ ($|\psi_c(r)|^2 \sim e^{\varphi}$) with "background charge" $Q = 1$. This means that

$$\mathbb{E}\, |\psi_c(r)|^2 \sim |r - c|^{-2\Delta_q},$$

and the spectrum of multifractal scaling exponents is parabolic:

$$\Delta_q = \frac{1}{n} q (Q - q), \quad Q = 1.$$

- **Conformal Field Theory** [Zi19].

The renormalization-group fixed point for the critical model is expected to be a conformal field theory, and has been argued to be a Wess–Zumino–Witten model $GL_{n,\gamma}(1|1)$ with current-algebra of level $n = 4$ and Abelian current–current deformation $\gamma = 1$:

$$S_* = \frac{n}{8\pi} \int_\Sigma d^2x\, \mathrm{STr}\,(M^{-1}\nabla M)^2 + \frac{n}{12\pi} \int_\Sigma \omega$$

$$- \frac{\gamma}{8\pi} \int_\Sigma d^2x\, (\mathrm{STr}\, M^{-1}\nabla M)^2, \quad \text{where } \omega \text{ anomalous 2-form}$$

with exterior derivative $d\omega = \mathrm{STr}\,(M^{-1}dM \wedge M^{-1}dM \wedge M^{-1}dM)$.

In the parametrization $M = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ this becomes

$$S_* = \frac{in}{4\pi} \int_\Sigma \left(\partial\varphi \wedge \bar\partial\varphi + \partial\theta \wedge \bar\partial\theta + 2 e^{\varphi - i\theta} \partial\xi \wedge \bar\partial\eta \right)$$

$$- \frac{i\gamma}{4\pi} \int_\Sigma (\partial\varphi - i\partial\theta) \wedge (\bar\partial\varphi - i\bar\partial\theta), \quad \text{with}$$

$$d = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} = dz \frac{\partial}{\partial z} + d\bar z \frac{\partial}{\partial \bar z} \equiv \partial + \bar\partial,$$

$$z = x + iy, \quad \bar z = x - iy.$$

The $GL(1|1)$–invariant Berezin integration form is

$$DM = \frac{1}{2\pi} d\varphi\, d\theta\, e^{-\varphi + i\theta} \frac{\partial^2}{\partial\xi\,\partial\eta}, \qquad \int DM\, e^{-\beta\, \mathrm{STr}(M + M^{-1})} = 1.$$

[$\rightsquigarrow$ functional integration measure $\mathcal{D}M = \prod_{\bullet} DM(\cdot)$]

CFT predictions (small selection).

- The fixed-point dissipative conductivity is
$$\sigma^*_{xx} = \frac{n}{2\pi} = 0.6366\ldots \quad (n=4).$$

- From the current algebra and the stress-energy tensor of the CFT one infers that the WZW field M has scaling dimension $\Delta_M = \frac{1-\gamma}{n^2} = 0$ (as required by phenomenology). The scaling dimension of the vertex field $(M^0{}_0)^q = e^{q\varphi}$ is $\Delta_q = \frac{1}{4} q(1-q)$, in keeping with the GFF hypothesis.

- The point-contact conductance at criticality decays as
$\mathbb{E} G^*_{AB} \sim |c_A - c_B|^{-2\Delta_{\frac{1}{2}}}$ (times a logarithmic correction).

<u>Open question</u>: the point-contact conductance (among other correlation functions) is expected to exhibit exponential decay, $\mathbb{E} G_{AB} \sim e^{-|c_A - c_B|/\xi}$, away from the critical point, but how does ξ diverge on approaching criticality (faster than any power? Kosterlitz-Thouless type?).

- **Singular continuous spectral measure.**

Define a spectral measure for the case of $\Sigma = \mathbb{Z}^2$ as usual by $(|\lambda| > 1)$
$$(\lambda \cdot 1 - \mathcal{U})^{-1}(e,e) = \oint_{S^1} \frac{d\mu_{\mathcal{U},e}(\theta)}{\lambda - e^{i\theta}}.$$

<u>Conjecture</u>. At criticality, the integrated local density of states,
$$\mathcal{I}(\theta) = \int_0^\theta d\mu_{\mathcal{U},e},$$ is singular continuous as
$$\lim_{\theta \to \theta'} |\ln(\theta - \theta')| \, |\mathcal{I}(\theta) - \mathcal{I}(\theta')| < \infty.$$

Research Talk: WZW-Anomaly of the Chalker–Coddington Model

- Conserved current in 2D-CFT.

— current conservation (stationary situation).

current density as a vector field: $\operatorname{div}\vec{j} = \partial_\mu j^\mu = 0$;

—"— as a twisted 2-form $j = \varepsilon_{\mu\nu} j^\mu dx^\nu$: $dj = 0$.

— CFT-principle: $dj = 0 \xrightarrow[2D]{CFT} \mathbb{E}(\bullet\ d{*j}) = 0$.

In words: conformal symmetry requires that a divergence-free current flow be also irrotational, $\operatorname{curl}\vec{j} = 0$, in expectation.

[Note. For the conserved current of our model of interest (CC) this property will be intuitively clear.]

Using the complex structure (or equivalently, the Hodge star operator $*$) of Σ, decompose the current into its (1,0) and (0,1) tensor components:

$j = j^{1,0} + j^{0,1}$, $j^{1,0} = \frac{1}{2}(j + i{*j})$, $j^{0,1} = \frac{1}{2}(j - i{*j})$, where

$i = \sqrt{-1}$, ${*dz} = *(dx + i dy) = dy - i dx = -i dz$, ${*d\bar{z}} = i d\bar{z}$.

Also, let $d = \partial + \bar\partial$ and ${*d} = -i\partial + i\bar\partial$. Then

$$dj = 0 = d{*j} \iff \bar\partial j^{1,0} = 0 = \partial j^{0,1}.$$

So, $j^{1,0}$ is holomorphic and $j^{0,1}$ antiholomorphic (in expectation).

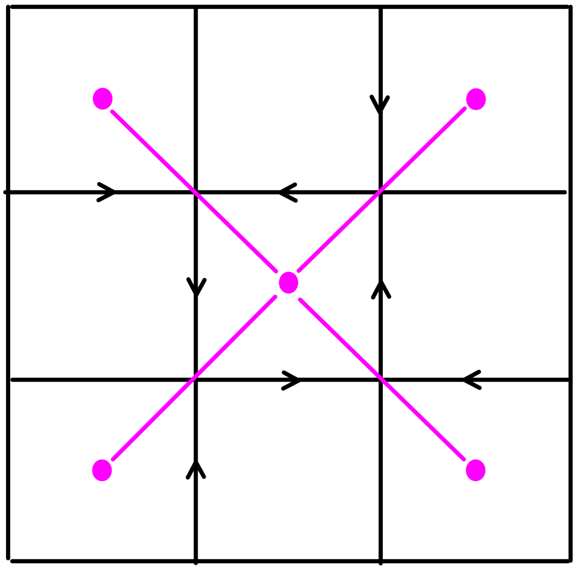
Example. GFF φ: $j^{1,0} = \partial\varphi$, $j^{0,1} = \bar\partial\varphi$.

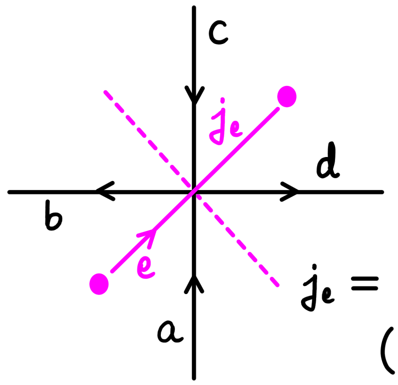
$\bar\partial j^{1,0} = \bar\partial\partial\varphi = 0$, and $\partial j^{0,1} = \partial\bar\partial\varphi = -\bar\partial\partial\varphi = -\bar\partial j^{1,0} = 0$.

by eqn of motion, valid in $\mathbb{E}(\bullet)$.

Network model current.

"coarse graining": the 0-cells of the medial lattice $\mathcal{M}$ (square) are in bijection with the 2-cells of counterclockwise orientation on the original square lattice.





$$j_e = |a|^2 - |b|^2$$
$$(= |d|^2 - |c|^2)$$

By Kirchhoff's law, the current 1-cochain $j = \sum_e j_e e^* \in C^1(\mathcal{M})$ satisfies the conservation law $\boxed{dj = 0}$ (coboundary operator d).

<u>Note</u>. In the continuum limit at criticality, we expect an emergent conservation law $\boxed{\mathbb{E}(\bullet\, d{*}j) = 0}$ (cf. CFT-principle). Heuristically, this is clear from left-right symmetry ($|a_L| = |a_R|$): critical CC random paths do <u>not</u> rotate/circulate in expectation.

Field theory implementation.

simplify the notation for continuum currents at criticality: $j^{1,0} \equiv j$, $j^{0,1} \equiv \bar{j}$.

By SUSY technology & principles, we expect the field theory to have a quadruplet ($\alpha, \beta = 0,1$ or B,F) of conserved currents $\bar\partial j^\alpha{}_\beta = 0 = \partial \bar{j}^\alpha{}_\beta$. In field theory, such conservation laws arise as Ward identities. How can this work in the non-Abelian setting with a matrix of conserved currents?

→ Make an inspired guess by generalization from the GFF example:

Replace $\left\{ \begin{array}{l} \bar{j} = \bar\partial\varphi = e^{-\varphi}\bar\partial e^{\varphi} \\ j = \partial\varphi = (\partial e^{\varphi})e^{-\varphi} \end{array} \right\}$ by $\left\{ \begin{array}{l} \bar{J} = g^{-1}\bar\partial g \\ J = (\partial g)\, g^{-1} \end{array} \right\}$,

where g takes values in a non-Abelian Lie group G, in our case the Lie supergroup $G = GL(1|1)$.

Notice that $\bar\partial J = (\bar\partial\partial g)\, g^{-1} + (\partial g)\, g^{-1}(\bar\partial g)\, g^{-1}$

$= (-\partial\bar\partial g)\, g^{-1} - g\,(\partial g^{-1})(\bar\partial g)\, g^{-1} = g\left(-\partial\,(g^{-1}\bar\partial g)\right) g^{-1} = g\,(-\partial\bar{J})\, g^{-1}.$

Thus one has mutual consistency: $\bar\partial J = 0 \iff \partial\bar{J} = 0$.

- Wess–Zumino–(Novikov–) Witten functional.

WANTED: a field theory action functional, S, that yields $\bar\partial J = 0 = \partial\bar{J}$ as equations of motion (a.k.a. Ward identities). More precisely, we seek the solution to the following problem: find S such that (change notation $g \to M$)

$$\boxtimes \quad \frac{d}{dt}\Big|_{t=0} S[e^{-tY}M] = \frac{n}{2\pi i}\int \mathrm{STr}\, J \wedge \bar\partial Y,$$
$$\text{and} \quad \frac{d}{dt}\Big|_{t=0} S[Me^{+tY}] = \frac{n}{2\pi i}\int \mathrm{STr}\, \partial Y \wedge \bar{J}.$$

Remark. The substitution $M(\cdot) \longrightarrow e^{-tY(\cdot)} M(\cdot)$ is a change of variable that leaves the functional integration measure invariant. Hence, $\frac{d}{dt}\Big|_{t=0} S[e^{-tY}M] = 0$ holds in expectation (i.e. under the functional integral sign). Since the variation field $Y(\cdot)$ is arbitrary, it follows by the Fundamental Lemma of Variational Calculus that $\bar\partial J = 0$. The case of $\partial\bar{J} = 0$ is similar.

Solution to problem posed (Wess & Zumino 1971, Witten 1984, Polyakov & Wiegmann 1984):

$$S[M] = \frac{in}{4\pi}\int \mathrm{STr}\left(M^{-1}\partial M \wedge M^{-1}\bar\partial M + d^{-1}(M^{-1}dM)^{\wedge 3}\right).$$

anomalous term

Remark. Due to the presence of $d^{-1}\mathrm{STr}\,(M^{-1}dM)^{\wedge 3}$ (the anomaly) [meaning any 2-form potential for the closed 3-form $\mathrm{STr}\,(M^{-1}dM)^{\wedge 3}$], which makes $S[M]$ ambiguous by amounts in $2\pi i n \mathbb{Z}$, the given solution $S[M]$ (known as the WZNW-functional) is acceptable only for $n \in \mathbb{Z}$. The integer n is called the "level" of the WZNW field theory (or its current algebra).

Proof of $\boxtimes$ is best done by way of the Polyakov–Wiegmann relation:

$$S[g h^{-1}] = S[g] + S[h^{-1}] + \frac{in}{2\pi}\int \mathrm{STr}\,(\partial h)\, h^{-1} \wedge g^{-1}\bar{\partial} g\,.$$

- Fermion determinant.

How could a WZNW functional arise in the field-theory reformulation of the Chalker–Coddington network model at criticality?

→ Compute the determinant of a massless Dirac operator in the presence of a background gauge field A_μ (Orlando Alvarez, 1984):

$$\ln \mathrm{Det}\,(\gamma^\mu(\partial_\mu + A_\mu)) \overset{D=2}{=} \ln \mathrm{Det}\begin{pmatrix} 0 & \partial_z + A_z \\ \partial_{\bar z} + A_{\bar z} & 0 \end{pmatrix}$$

$$= \ln \mathrm{Det}\left(\begin{pmatrix} g_R^{-1} & 0 \\ 0 & g_L^{-1} \end{pmatrix}\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar z} & 0 \end{pmatrix}\begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}\right) = S_{n=1}[M], \quad M = g_L g_R^{-1}.$$

$$A_z = \tfrac{1}{2}(A_x - i A_y) = g_R^{-1}\partial_z g_R\,, \quad A_{\bar z} = g_L^{-1}\partial_{\bar z} g_L$$

Remark. This is adapted to our case by taking A_μ to be supermatrix-valued, replacing Det by SDet, etc.

- Network model: Dirac spinor structure.

We claim that the (field theory of the) Chalker–Coddington network model has a WZNW anomaly of level $n=4$ (@ criticality, i.e. $|a_L|^2 = |a_R|^2 = \frac{1}{2}$).
To support that claim, we exhibit four massless Dirac modes for the deterministic factor $\mathcal{U}_s$ in $\mathcal{U} = \mathcal{U}_r \mathcal{U}_s$.

Start-up computations.

$$a_L + a_R = \frac{1}{\sqrt{2}}\left(e^{i\pi/4} + e^{-i\pi/4}\right) = \sqrt{2}\cos(\pi/4) = 1,$$

$$a_L - a_R = \frac{1}{\sqrt{2}}\left(e^{i\pi/4} - e^{-i\pi/4}\right) = \sqrt{2}\, i \sin(\pi/4) = i.$$

$$\begin{pmatrix} a_L & 0 \\ 0 & a_R \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 1, \quad \begin{pmatrix} a_L & 0 \\ 0 & a_R \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot i.$$

In words: the single-vertex scattering matrix is diagonalized by passing to the "spinor basis" $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with eigenvalues that are fourth roots of unity.

Now define momentum k by Fourier-Bloch transform with respect to the unit cell of Lecture 2.1. Then $\mathcal{U}_s(k=0) = \mathrm{Id}_{\mathcal{H}}$, and recalling $\mathcal{U} : \mathcal{H}_\nu \to \mathcal{H}_{\nu+1}$ $(\nu = 0, 1, 2, 3, 4 \equiv 0)$, $\mathcal{U}^4 : \mathcal{H}_\nu \to \mathcal{H}_\nu$, one has the

Consequence.

$$\mathrm{Id}_{\mathcal{H}} - \mathcal{U}_s^4(k) = 0 + \text{"linear in } k\text{"} \quad (= \text{Dirac operator})$$

$\longrightarrow$ four $(\nu = 0, \ldots, 3)$ massless Dirac modes.

Here the real work begins: we show that certain "spin-replica locked" Dirac modes (with "replica" meaning the duplicity of retarded & advanced) remain massless when the strong disorder due to the factor $\mathcal{U}_r$ is turned on.

First step (starting from Cauchy transform):

Use $\dfrac{1+x}{1-x} = \dfrac{1+x^4}{1-x^4} + 2\,\dfrac{x+x^2+x^3}{1-x^4}$ with $x \equiv e^{i\pi/4}\,\mathcal{U}_s$.

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[Zi21] = arXiv:2106.01291

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