Restrictions of vertex-reinforced jump processes to subgraphs

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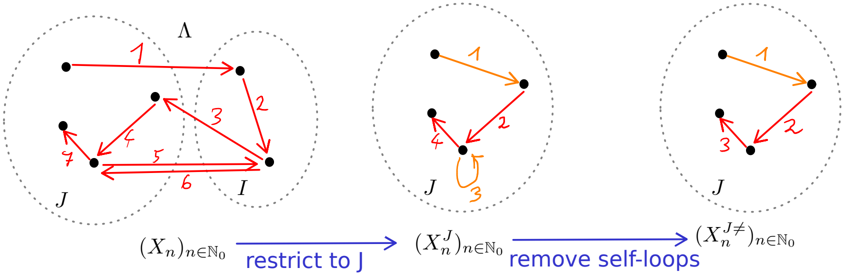
Vertex-reinforced jump processes (vrjp)

- $G = (\Lambda, E)$: undirected locally finite connected graph without self-loops
- vertex set Λ, edge set E
- edge weights $W_e = W_{ij} > 0$, $e = \{i, j\} \in E$
- vrjp: Λ-valued non-Markovian continuous-time jump process $(Y_t)_{t \geq 0}$
- $L_j(t) := 1 + \int_0^t 1_{\{Y_s = j\}}\, ds$ local time at $j \in \Lambda$ with an offset of 1.
- Conditioned on $(Y_s)_{s \leq t}$ and $Y_t = i \in \Lambda$, the rate to jump to $j \neq i$ equals $W_{ij} L_j(t)$.
- Time change introduced by Sabot and Tarrès 2012: $(Z_t := Y_{D^{-1}(t)})_{t \geq 0}$ with $D(t) = \sum_{i \in \Lambda} (L_i(t)^2 - 1)$.
- In this time scale,
  Vrjp is a mixture of reversible Markov jump processes.
  (Sabot and Tarrès 2012, Sabot and Zeng 2015) We use only this time scale below.

Encoding of a jump process in two sequences

- $(X_n)_{n\in\mathbb{N}_0}$: random sequence of vertices $X_n \in \Lambda$ visited
- $(T_n)_{n\in\mathbb{N}_0}$: random sequence of waiting times
- $T_n > 0$: waiting time for the jump from X_n to X_{n+1}
- Self-loops not a priori forbidden: $X_{n+1} = X_n$ may occur.

Restriction to a subset J and removal of self-loops



Visualization:

- Editing a film of the continuous time representation of the jump process
- Cut out all parts of the film where the jumping particle is not in J. Cut locations remain visible as self-loops.
- Removal of self-loops in the edited film:
  Cut locations become invisible.

The β-field and its law ν_Λ^W

Sabot, Tarrès, and Zeng (2017) (for finite Λ) and Sabot and Zeng (2019) (for infinite Λ) have described the mixing measure for vrjp in terms of a random field $\beta = (\beta_i)_{i\in\Lambda}$.

Definition for finite Λ:

- Symmetric matrix of weights: $W \in [0,\infty)^{\Lambda\times\Lambda}$,
- Random Schrödinger operator: $H_\beta := 2\,\mathrm{diag}(\beta) - W$
- $\langle 1, H_\beta 1\rangle$: Sum of all entries of H_β
- Law of β:

$$\nu_\Lambda^W(d\beta) := \left(\frac{2}{\pi}\right)^{\frac{|\Lambda|}{2}} 1_{\{H_\beta \text{positive definite}\}} \frac{e^{-\frac{1}{2}\langle 1, H_\beta 1\rangle}}{\sqrt{\det H_\beta}}\, d\beta$$

This law ν_Λ^W appears in this talk in an additional role.

Restriction of vrjp as a mixture of vrjps

Assumptions:

- G finite
- partition $\Lambda = I \cup J$ with $|J| \geq 2$
- (X, T): vrjp on Λ with weights W
- starting point $\rho \in J$
- β_I: restriction to I of the random field $\beta \sim \nu_\Lambda^W$

Theorem (DMR)

The restrictions (X^J, T^J) and $(X^{J\neq}, T^{J\neq})$ to J without or with self-loops removed are mixtures of vrjps on J with random weights

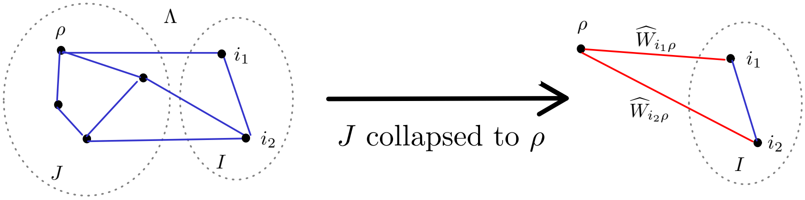
$$W^J(\beta_I) := W_{JJ} + W_{JI}([H_\beta]_{II})^{-1} W_{IJ},$$
$$W^{J\neq}(\beta_I) := \left(W^J_{ij}(\beta_I) 1_{\{i\neq j\}}\right)_{i,j\in J},$$

respectively.

Alternative description of β_I

Following Sabot, Tarrès, and Zeng, $\beta_{I\cup\{\rho\}} \in \mathbb{R}^{I\cup\{\rho\}}$ is distributed according to $\nu^{\widehat{W}}_{I\cup\{\rho\}}$ with $\widehat{W} \in \mathbb{R}^{(I\cup\{\rho\})\times(I\cup\{\rho\})}$ obtained by restricting the parameters W to I and wiring all points in J at ρ:

$$\widehat{W}_{ij} = \widehat{W}_{ji} = \begin{cases} W_{ij} & \text{for } i,j \in I, \\ \sum_{k\in J} W_{ik} & \text{for } i \in I, j = \rho, \\ 0 & \text{for } i = j = \rho. \end{cases}$$



Restriction of vrjp as a mixture of vrjps in formulas

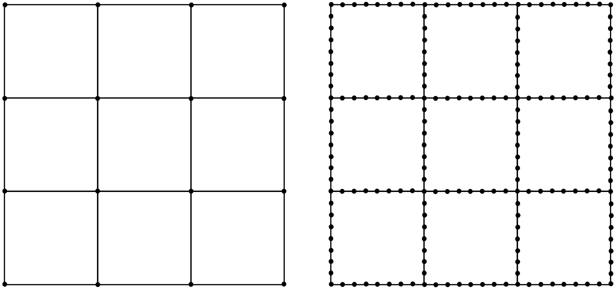
Theorem (DMR): Let $P_\rho^{W,\Lambda}$ denote the probability measure underlying the vrjp (X,T) with data Λ, W, ρ.
Then for any event A,

$$
\begin{aligned}
&P_\rho^{W,\Lambda}((X^{J\neq}, T^{J\neq}) \in A) \\
&= \int_{\mathbb{R}^{I\cup\{\rho\}}} P_\rho^{W^{J\neq}(\beta_I),J}((X,T)\in A)\, \nu_{I\cup\{\rho\}}^{\widehat{W}}(d\beta_{I\cup\{\rho\}}) \\
&= \int_{\mathbb{R}^\Lambda} P_\rho^{W^{J\neq}(\beta_I),J}((X,T)\in A)\, \nu_\Lambda^W(d\beta_\Lambda).
\end{aligned}
$$

The analogous statement holds for (X^J, T^J).

Subdivisions

- 2^r-subdivision $G_r = (\Lambda_r, E_r)$ of the undirected graph $G = (\Lambda, E)$ obtained by replacing every edge in G by a series of 2^r edges.
- $\Lambda_0 = \Lambda$ and $\Lambda_l \subseteq \Lambda_r$ for $l \leq r$.



A part of $\mathbb{Z}^2$ on the left and its 8-subdivided version on the right

vrjp on subdivided graphs

Setup

- $G = (\Lambda, E)$: connected undirected graph without self-loops
- vertex degree $\leq d$.
- $l \leq r$
- i.i.d random weights $W_e > 0$, $e \in E_r$
- (X, T): vrjp on G_r started in $\rho \in \Lambda$

Theorem (DMR)

- *$(X^{\Lambda_l \neq}, T^{\Lambda_l \neq})$ is again a mixture of vrjps on G_l with i.i.d. random weights denoted by $W^{(l)} = (W_e^{(l)})_{e \in E_l}$.*
- *Assume that $\mathbb{E}[W_e^{\alpha}] \leq c_1 2^{\alpha(r-l)}$ for some $\alpha \in (0, \frac{1}{4}]$, with an appropriate constant $c_1(d, \alpha)$. Then, $X^{\Lambda_l \neq}$ is a mixture of positive recurrent reversible Markov chains. In particular, X visits a.s. ρ infinitely often.*

Induction over scales: initial case

The proof of the theorem is by induction over scales. For the initial case, the following fact is used:

Assumptions:

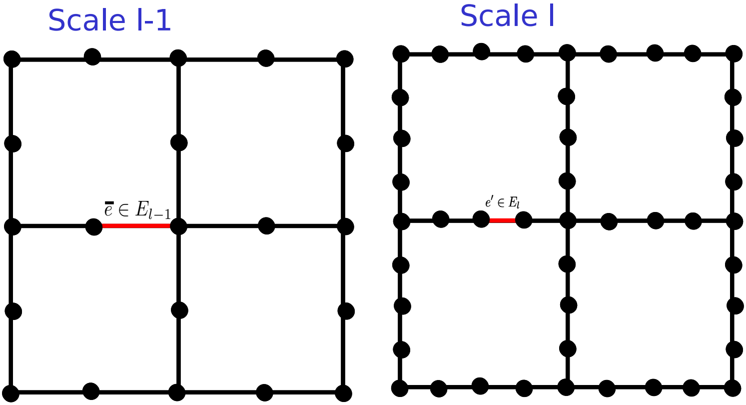
- $0 < \alpha \leq \frac{1}{4}$
- $G = (\Lambda, E)$ connected undirected graph
- vertex degree $\leq d$
- starting point ρ
- $W = (W_e)_{e \in E}$ independent random weights with $\mathbb{E}[W_e^\alpha] \leq c_1(d, \alpha)$ for all e
- X: discrete-time process associated to vrjp in random environment with these (random) parameters

Fact (Sabot and Tarrès (2015); variant proven by Angel, Crawford, and Kozma (2014))

Under these assumptions,

X is a mixture of positive recurrent Markov chains.

Induction over scales: visualisation



Induction over scales: induction step

Lemma (DMR)

Consider the setup of the theorem. For $l \in \{1, \ldots, r\}$, $\bar{e} \in E_{l-1}$ (coarser scale), and $e' \in E_l$ (finer scale), we have the following with some explicitly known constants C_α and c_2:

$$\mathbb{E}[(W_{\bar{e}}^{(l-1)})^\alpha] \leq 2^{-\alpha}\mathbb{E}[(W_{e'}^{(l)})^\alpha] \quad (1)$$

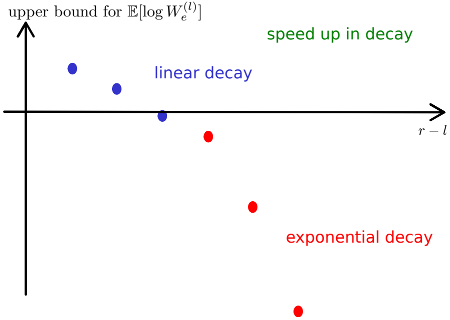
$$\mathbb{E}[(W_{\bar{e}}^{(l-1)})^\alpha] \leq C_\alpha \mathbb{E}[(W_{e'}^{(l)})^\alpha]^2 \quad (2)$$

$$\mathbb{E}[\log W_{\bar{e}}^{(l-1)}] \leq \mathbb{E}[\log W_{e'}^{(l)}] - \log 2 \quad (3)$$

$$\mathbb{E}[\log W_{\bar{e}}^{(l-1)}] \leq 2\mathbb{E}[\log W_{e'}^{(l)}] + c_2 \quad (4)$$

- $[(1) \Rightarrow (2)] \iff \mathbb{E}[(W_{e'}^{(l)})^\alpha] \geq 2^{-\alpha} C_\alpha^{-1}$
- $[(3) \Rightarrow (4)] \iff \mathbb{E}[\log W_{e'}^{(l)}] \geq -\log 2 - c_2$

Change of regimes: visualization of speed up



The flow along scales drives the effective weights towards the positive recurrent regime.

- For large effective weights:
  Linear decay of logarithms corresponds to exponential decay of moments.
- For small effective weights:
  Exponential decay of logarithms corresponds to doubly exponential decay of moments.

Linearly edge-reinforced random walk

Definition Linearly edge-reinforced random walk (errw) is a discrete-time non-Markovian random walk $(X_n)_{n\in\mathbb{N}_0}$ on G:

- Let $X_0 = \rho$.
- Initial edge weights: $w_0(e) := a$, $e \in E$.
- In each time step, the random walker jumps to a neighboring vertex with probability proportional to the weight of the traversed edge.
- Each time an edge is traversed, its weight is increased by 1.

Sabot and Tarrès (2012) have shown that
errw is a mixture of the discrete-time process associated to vrjp
with i.i.d. Gamma(a,1)-distributed weights W_e, $e \in E$.

Consequences for errw on subdivides graphs

Assumptions:

- graph $G = (\Lambda, E)$ as in the theorem
- vertex degree $\leq d$
- $l \leq r$
- G_r: subdivided graph
- $\rho \in \Lambda$: starting point
- constant initial weights $a > 0$
- X: errw on Λ_r
- $0 < \alpha \leq \frac{1}{4}$
- $\Gamma(a+\alpha)/\Gamma(a) \leq c_1(d,\alpha)2^{\alpha(r-l)}$

Corollary (DMR)

Under the above assumptions,
$X^{\Lambda_l \neq}$ is a mixture of positive recurrent Markov chains.
Consequently, X visits a.s. ρ infinitely often.

Setup for the non-linear hyperbolic supersymmetric sigma model ($H^{2|2}$-model)

- Λ: Finite set of vertices
- $\rho \in \Lambda$: pinning point
- interactions $W = (W_{ij} = W_{ji})_{i,j\in\Lambda}$, such that the graph (Λ, E_+) with edge set $E_+ := \{\{i,j\} \subseteq \Lambda : W_{ij} > 0\}$ is connected.
- Grassmann-algebra-valued spin variable $\sigma_i = (x_i, y_i, z_i, \xi_i, \eta_i)$ for every vertex i, $\sigma_\Lambda = (\sigma_i)_{i\in\Lambda}$
  - x_i, y_i, z_i even ($=$ commuting)
  - ξ_i, η_i odd ($=$ anticommuting)
- hyperbolic constraint $x_i^2 + y_i^2 - z_i^2 + 2\xi_i\eta_i = -1$, body$(z_i) > 0$
  body(z) is the unique real number such that $z -$ body(z) is nilpotent.
- hyperbolic inner product $\langle\sigma, \sigma'\rangle := xx' + yy' - zz' + \xi\eta' - \eta\xi'$
- pinning $\sigma_\rho = o := (0, 0, 1, 0, 0)$, δ_o: Dirac measure in o

The $H^{2|2}$-model

- reference superintegration form

$$f \mapsto \int \mathcal{D}\sigma f(\sigma) := \frac{1}{2\pi} \int_{\mathbb{R}^2} dx\, dy\, \partial_\xi \partial_\eta \left(\frac{1}{z} f(x, y, z, \xi, \eta) \right)$$

- $\mathcal{D}\sigma_{\Lambda \setminus \{\rho\}} := \prod_{i \in \Lambda \setminus \{\rho\}} \mathcal{D}\sigma_i$
- Definition of the $H^{2|2}$-model for spin-variables $\sigma_\Lambda = (\sigma_i)_{i \in \Lambda}$:

$$\mu^W_{\Lambda,\rho}(\sigma_\Lambda) := \delta_o(d\sigma_\rho) \mathcal{D}\sigma_{\Lambda \setminus \{\rho\}} \exp\left(\frac{1}{2} \sum_{i,j \in \Lambda} W_{ij} (1 + \langle \sigma_i, \sigma_j \rangle) \right)$$

Reminder to some notation

- partition $\Lambda = I \cup J$ with $|J| \geq 2$
- effective weights for restriction:

$$W^J(\beta_I) := W_{JJ} + W_{JI}([H_\beta]_{II})^{-1} W_{IJ}$$

- wiring weights:

$$\widehat{W}_{ij} = \widehat{W}_{ji} = \begin{cases} W_{ij} & \text{for } i, j \in I, \\ \sum_{k \in J} W_{ik} & \text{for } i \in I, j = \rho, \\ 0 & \text{for } i = j = \rho. \end{cases}$$

- $\nu_\Lambda^W(d\beta)$: Law of the random field β over Λ with weights W

Restriction of the $H^{2|2}$-model as a mixture of $H^{2|2}$-models

Theorem (DMR)

Given the assumptions, we have

$$\int_{(H^{2|2})^\Lambda} \mu_\Lambda^W(\sigma_\Lambda) f(\sigma_J) = \int_{\mathbb{R}^\Lambda} \nu_\Lambda^W(d\beta) \int_{(H^{2|2})^J} \mu_J^{W^J(\beta_I)}(\sigma_J) f(\sigma_J)$$

$$= \int_{\mathbb{R}^{I\cup\{\rho\}}} \nu_{I\cup\{\rho\}}^{\widehat{W}}(d\beta) \int_{(H^{2|2})^J} \mu_J^{W^J(\beta_I)}(\sigma_J) f(\sigma_J)$$

for any superfunction f on $(H^{2|2})^J$ which is compactly supported or decays at least sufficiently fast.