Reinforcement, supersymmetry and isomorphism theorems

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Outline I

Edge-Reinforced Random Walk (ERRW): introduction

Definition

Simulations

Partial exchangeability (Diaconis-Freedman, 1980)

Statistical view

How to find the limit measure?

"Magic formula" for ERRW

$\star$-Edge Reinforced Random Walk ($\star$-ERRW): introduction

Yaglom reversible Markov Chains

Definition

Partial exchangeability

Results

The Vertex-Reinforced Jump Process (VRJP)

ERRW $\longleftrightarrow$ VRJP (Vertex Reinforced Jump Process)

VRJP: three timescales

VRJP $\longleftrightarrow$ random Schrödinger operator

Applications : recurrence/transience

Outline II

The $\star$-Vertex-Reinforced Jump Process ($\star$-VRJP)

- Correspondence $\star$-ERRW $\longleftrightarrow$ $\star$-VRJP
- Limiting manifold
- Partial exchangeability after randomization of initial local time
- Limiting behavior of non-randomized process
- *-VRJP: Random Schrödinger version

Dynkin and Ray-Knight isomorphisms

- $\mathbb{H}^{2|2}$ and the VRJP
- $\mathbb{H}_\star^{5|4}$ and the $\star$-VRJP

Edge-Reinforced Random Walk (Coppersmith and Diaconis, 1986)

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- $a_e > 0$, $e \in E$, initial weights

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$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\, k \leqslant n) = \mathbb{1}_{\{j \sim i\}} \frac{Z_{\{i,j\}}(n)}{\sum_{k \sim X_n} Z_{\{ik\}}(n)}$$

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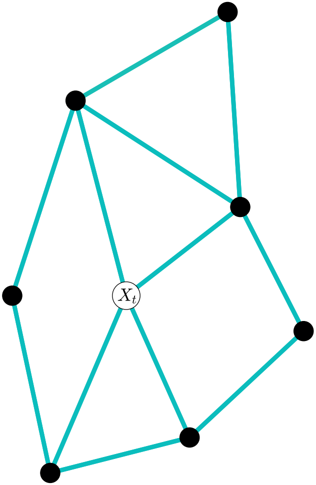
where

$$Z_{\{i,j\}}(n) = a_{i,j} + \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1}, X_k\} = \{i,j\}}.$$

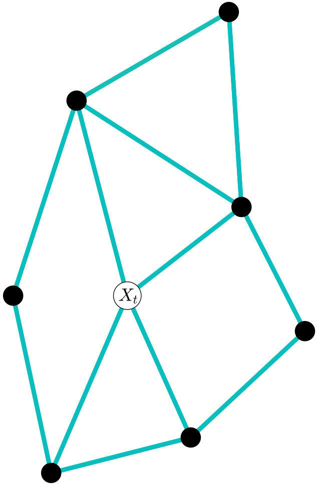
- a_e small: strong reinforcement
- a_e large: small reinforcement

The ERRW



A simulation due to Andrew Swan.

The Mixing Measure of ERRW



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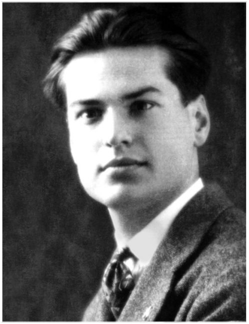

(a) de Finetti Bruno: 1906-1985

Sul significato soggettivo della probabilità.

Memoria di

Bruno de Finetti (Roma).

Sunto.

Si spiega come si possa con tutto rigore introdurre il concetto di probabilità e dimostrarne le proprietà fondamentali ben note attenendosi esclusivamente al punto di vista soggettivo. Dopo aver indicato un modo di procedere di natura quantitativa, che particolarmente si presta alla trattazione analitica, se ne analizzano criticamente i principi dimostrando che sono di natura puramente qualitativa e elementare.

(b) On the subjective meaning of probability, 1931

The notion of exchangeability (de Finetti)

Definition

Let $(X_i)_{i \geqslant 1}$ random process taking values in $\{0, 1\}$. Then X is called exchangeable if, for all $n \in \mathbb{N}$ and $\sigma \in S_n$,

$$\mathcal{L}\left((X_{\sigma(i)})_{1 \leqslant i \leqslant n}\right) = \mathcal{L}\left((X_i)_{1 \leqslant i \leqslant n}\right).$$

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Theorem (de Finetti)

If $(X_i)_{i\geqslant 1}$ *is exchangeable, then there exists a random variable* $\alpha \in [0,1]$ *such that*

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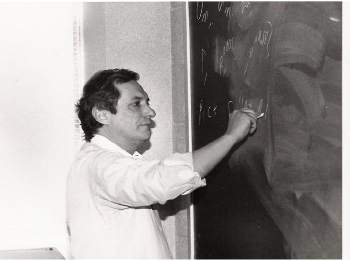
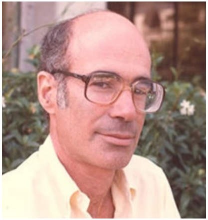
Conditionally on α, $(X_i)_{i \geqslant 1}$ is an i.i.d. sequence of Bernoulli random variables with success probability α, which we call P^{α}.



(a) Diaconis Persi: 1945-



(b) Freedman David: 1938-2008

The Annals of Probability
1980, Vol. 8, No. 1, 115–130

DE FINETTI'S THEOREM FOR MARKOV CHAINS

BY P. DIACONIS[1] AND D. FREEDMAN[2]

Bell Laboratories, Murray Hill, New Jersey and University of California at Berkeley

Let $Z = (Z_0, Z_1, \cdots)$ be a sequence of random variables taking values in a countable state space I. We use a generalization of exchangeability called partial exchangeability. Z is partially exchangeable if for two sequences $\sigma, \tau \in I^{n+1}$ which have the same starting state and the same transition counts, $P(Z_0 = \sigma_0, Z_1 = \sigma_1, \cdots, Z_n = \sigma_n) = P(Z_0 = \tau_0, Z_1 = \tau_1, \cdots, Z_n = \tau_n)$. The main result is that for recurrent processes, Z is a mixture of Markov chains if and only if Z is partially exchangeable.

Partial exchangeability (Diaconis and Freedman, 1980) (I)

Definition

Let $(Y_n)_{n \geqslant 0}$ a random process on a graph $G = (V, E)$, E oriented (resp. non-oriented) edges. It is called partially exchangeable (resp. reversibly partially exchangeable) if, for any nearest-neighbor path $\gamma = (\gamma_0, \ldots, \gamma_n)$ on V,

$$\mathbb{P}[(Y_0, \ldots, Y_n) = (\gamma_0, \ldots, \gamma_n)]$$

only depends on its starting point and on the number of crossings of directed (resp. undirected) edges by γ.

Partial exchangeability (Diaconis and Freedman, 1980) (II)

Theorem (Diaconis and Freedman, 1980)

Assume $(Y_n)_{n \geqslant 0}$ a.s. recurrent (i.e. $Y_n = Y_0$ infinitely often) and partially exchangeable (resp. reversibly partially exchangeable, and each edge is traversed is traversed in both directions a.s.).
Then it is a mixture of Markov chains (resp. reversible Markov chains), i.e.

$$\mathcal{L}(Y) = \int P^{\omega}(.)\, d\mu(\omega).$$

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$$\mathcal{L}(Y) = \int P^{\omega}(.)\, d\mu(\omega).$$

Here P^{ω} denotes the Markov Chain with transition probability $\omega(i,j)$ from i to j. If P^{ω} is reversible, then there exists $x = (x_e) \in (0,\infty)^E$ such that

$$\omega(i,j) = \omega^x(i,j) = \frac{x_{ij}}{x_i},\ x_i = \sum_{j\sim i} x_{ij}. \quad \textit{Let } P^x = P^{\omega^x}.$$

Edge-Reinforced random walk (ERRW): partial exchangeability

Let $\mathbb{P}^{a,i_0}$ law of ERRW with initial weights $a = (a_e)_{e \in E}$ and starting from i_0.

Lemma

The ERRW is reversibly partially exchangeable: more precisely,

$$\mathbb{P}^{i_0,a}(X_0 = i_0, \dots, X_n = i_n = j_0) = \frac{\gamma(j_0, \alpha)}{\gamma(i_0, a)},$$

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where n_e (resp. v_i) is the number of crossings (resp. visits) of edge e (resp. site i) by the path $(i_0, \ldots, i_n)$, $\alpha = (a_e + n_e)_{e \in E}$, and

$$\gamma(i_0, a) = \frac{\prod_{e \in E} \Gamma(a_e)}{\prod_{i \in V} \Gamma\left(\frac{1}{2}(a_i + 1 - \delta_{i_0}(i))\right) 2^{\frac{1}{2}(a_i - \delta_{i_0}(i))}}.$$

Edge-Reinforced random walk (ERRW): partial exchangeability

Proof.

Note that $\mathbb{P}^{i_0,a}(X_0 = i_0, \ldots, X_n = i_n = j_0)$ equals

$$\frac{\prod_{e\in E}(a_e, n_e)}{\prod_{i\in V} 2^{v_i-\delta_{j_0}(i)}\left(\frac{a_i+1-\delta_{i_0}(i)}{2}, v_i - \delta_{j_0}(i)\right)},$$

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and that

$$n_i := \sum_{j\sim i} n_{ij} = 2v_i - \delta_{i_0}(i) - \delta_{j_0}(i)$$

so that

$$v_i - \delta_{j_0}(i) = \frac{(a_i + n_i) - \delta_{j_0}(i)}{2} - \frac{a_i - \delta_{i_0}(i)}{2}.$$

□

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- Hence, the posterior distribution after n first steps is given by $\mathcal{L}(Z(n), X_n)$.
- Thus prior and posterior are conjuguate priors.
- (Diaconis and Rolles, 2006) $Z(n)$ is a minimal sufficient statistic for the model, also provide method of simulation of the posterior.

Edge Reinforced Random Walks (ERRW): how to find the limit measure?

Bayesian statistics again: assume $\mathcal{L}(i_0, a)$ has integrable smooth density $\varphi^{i_0,\alpha}$ w.r.t $d\alpha = \prod_{e\in E\setminus\{e_0\}} d\alpha_e$, $e_0 \in E$, on the simplex $\mathcal{L}_1 = \{\sum \alpha_e = 1,\, \alpha_e > 0\}$.

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Bayes' formula:

$$\mathbb{P}^{i_0,a}(x \in [y, y+dy] | X_0 = i_0, \ldots, X_n = i_n = j_0) = \mathbb{P}^{j_0,\alpha}(x \in [y, y+dy])$$

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$$= \frac{\varphi^{i_0,a}(y)\, dy}{\mathbb{P}^{i_0,a}(X_0 = i_0, \ldots, X_n = i_n = j_0)} \frac{\prod_{e\in E} y_e^{n_e}}{\prod_{i\in V} y_i^{v_i - \delta_{j_0}(i)}}.$$

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Therefore

$$\varphi^{j_0,y}(y) = \varphi^{i_0,a}(y) \frac{\prod_{e\in E} y_e^{n_e}}{\prod_{i\in V} y_i^{v_i - \delta_{j_0}(i)}} \frac{\gamma(i_0, a)}{\gamma(j_0, \alpha)}.$$

Edge Reinforced Random Walks (ERRW): Limit measure (Diaconis and Coppersmith, 1986, Keane and Rolles, 2000)

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- x *has the following density w.r.t to measure* $\prod_{e\in E\setminus\{e_0\}} dy_e$ *on the simplex* $\{\forall e \in E,\ y_e > 0\ \sum_{e\in E} y_e = 1\}$

$$C\gamma(i_0, a)^{-1}\sqrt{y_{i_0}}\frac{\prod_{e\in E} y_e^{a_e-1}}{\prod_{i\in V} y_i^{\frac{1}{2}a_i}}\sqrt{D(y)}.$$

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Recall that

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and

$$C = \frac{2^{3/2-|V|}}{\sqrt{\pi^{|V|-1}}}, \; D(y) = \sum_{T \in \mathcal{T}} \prod_{e \in T} y_e,$$

where $\mathcal{T}$ is the set of (non-oriented) spanning trees of G.

Early results on recurrence of Edge-Reinforced random walk ('86-'09)

- Pemantle '88: recurrence/transience phase transition on trees:
  - Root the tree at i_0 for simplicity.
  - Between two visits to each vertex, once an edge is crossed the walk comes back through it.
  - Hence, independently at each vertex, Pólya urn with initial number of balls $((a_{ij} + \delta_{\{j \text{ father of } i_0\}})/2)_{j\sim i}$.
  - Hence the environment is independent Dirichlet at each vertex i: Random Walk in (independent) Random Environment (RWRE)
- Merkl Rolles '09: recurrence on a $2d$ graph (but not $\mathbb{Z}^2$)

Yaglom reversible Markov chains

- Markov Chain on discrete locally finite directed graph $G = (V, E)$, with involution $\star$ on V s.t.

$$(i, j) \in E \Rightarrow (j^\star, i^\star) \in E$$

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- $\Longrightarrow$ π invariant measure for MC.

Initial motivation: continuous time and space setting
$\star : (x, \dot{x}) \mapsto (x, -\dot{x})$.

Example of Yaglom reversibility: Reversible k-dependent Markov chains

- (Y_i) k-dependent Markov chain on S finite (i.e. law of Y_{n+1} depends only on $(Y_{n-k+1}, \dots, Y_n)$).

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- Induces Markov chain (X_n) on the (directed) de Bruijn graph $G = (V = S^k, E)$ with

$$\omega = (i_1, \ldots, i_k) \to \tilde{\omega} = (i_2, \ldots, i_{k+1})$$

with transition rate $p(\omega, \tilde{\omega})$, and invariant measure $\pi(\omega)$.

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- Reversibility $\iff$ Yaglom reversibility on de Bruijn graph with involution $*$:

$$\omega = (i_1, \ldots, i_k) \mapsto \omega^* = (i_k, \ldots, i_1) \text{ flipped } k\text{-string}.$$

Other examples of Yaglom reversibility of higher-order Markov chains (Bacallado, 2006)

- Variable-order MC with context set $\mathcal{C} \subseteq S \cup S^2 \cup \cdots \cup S^k$ on de Bruijn graph: $\forall (i_1, \ldots, i_\ell) \in \mathcal{C}$, transition probabilities out of x and y are the same whenever x and y both end in $(i_1, \ldots, i_\ell)$.

Other examples of Yaglom reversibility of higher-order Markov chains (Bacallado, 2006)

- Variable-order MC with context set $\mathcal{C} \subseteq S \cup S^2 \cup \cdots \cup S^k$ on de Bruijn graph: $\forall (i_1, \dots, i_\ell) \in \mathcal{C}$, transition probabilities out of x and y are the same whenever x and y both end in $(i_1, \dots, i_\ell)$.
- Random walk with amnesia: RW on $G = (V, E)$ defined by $V = S \cup S^2 \cup \dots S^k$ with two types of edges: "forgetting" ones $(i_1, \dots, i_m) \to (i_2, \dots, i_m)$, if $m > 1$, "appending" ones $(i_1, \dots, i_m) \to ((i_1, \dots, i_m, j)$, for each $j \in V$, if $m < k$. Generalization of the above.

$\star$-Edge-Reinforced random walk: Definition

- $G = (V, E)$ directed graph with involution $\star$ on V s.t.

$$(i,j) \in E \Rightarrow (j^\star, i^\star) \in E$$

- $\alpha_{i,j} > 0$, $(i,j) \in E$ such that $\alpha_{i,j} = \alpha_{j^\star,i^\star}$.

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We call $\star$-ERRW with initial weights (α_e), the discrete time process (X_n) defined by

$$\mathbb{P}(X_{n+1}=j\,|\,X_k,\ k\leqslant n)=\mathbb{1}_{\{X_n\to j\}}\frac{Z_n((X_n,j))}{\sum_{l,X_n\to l}Z_n((X_n,l))}$$

where

$$Z_n((i,j))=\alpha_{i,j}+N_{i,j}(n)+N_{j^\star,i^\star}(n)$$

$$N_{i,j}(n)=\sum_{k=1}^{n}\mathbb{1}_{\{(X_{k-1},X_k)=(i,j)\}}.$$

Define $V_0 = \{i \in V : i = i^\star\}$, and write $V = V_0 \sqcup V_1 \sqcup V_1^\star$ disjoint.

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$$\mathrm{div}(z)(i) = \sum_{j, i \to j} z_{i,j} - \sum_{j, j \to i} z_{j,i}.$$

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Given $a = (a_e)_{e \in E}$ s.t. $a_{ij} = a_{j^\star i^\star}$, let

$$a_i = \sum_{j, i \to j} a_{ij},$$

and let $\gamma(i_0, a)$ be

$$\frac{\prod_{e \in E} \Gamma(a_e)}{\prod_{i \in V} \Gamma(\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0})) 2^{\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0})} \prod_{i \in V_1} \Gamma(\min(a_i, a_{i^\star}))}.$$

$\star$-ERRW : particular case, see ST (2025), Perrel-Sabot (2025)

- Given $G = (V, E)$ directed graph, let $\check{G} = (\check{V} \simeq V, \check{E})$ obtained by reversing each edge of E.

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- Given $G = (V, E)$ directed graph, let $\check{G} = (\check{V} \simeq V, \check{E})$ obtained by reversing each edge of E.
- Glue G and $\check{G}$ at $i_0 \in V$ into $\mathcal{G}$, and let $\star$ be the involution mapping V to its copy in $\check{V}$. In particular, $i_0 = i_0^\star$.

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- $\star$-ERRW on $\mathcal{G}$ starting at i_0 with initial weights a.

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- Given $G = (V, E)$ directed graph, let $\check{G} = (\check{V} \simeq V, \check{E})$ obtained by reversing each edge of E.
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- $\star$-ERRW on $\mathcal{G}$ starting at i_0 with initial weights a.
- Map all excursions in $\check{G}$ to reversed excursions in G: resulting path has distribution of annealed law of the directed ERRW, since $\mathrm{div}(a) = 0$, by the time-reversal property of Sabot and Tournier (2011).

$\star$-Edge-Reinforced random walk: partial exchangeability

Proposition (Bacallado '11, Baccalado, Sabot and T. '21)

Let $i_0 \in V$. If $div(\alpha) = \delta_{i_0^\star} - \delta_{i_0}$, then the $\star$-ERRW starting from i_0 is partially exchangeable. Given path $\sigma = (\sigma_0 = i_0, \sigma_1, \ldots, \sigma_n)$, let n_e be its number of crossings of edge $e \in E$, and let $a = \alpha + n$. Then

$$\mathbb{P}^{\star - ERRW}(X \text{ follows } \sigma) = \frac{\gamma(\sigma_n, a)}{\gamma(i_0, \alpha)}.$$

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$$\mathbb{P}^{\star-ERRW}(X \text{ follows } \sigma) = \frac{\gamma(\sigma_n, a)}{\gamma(i_0, \alpha)}.$$

Proof.

- $\text{div}(Z(k)) = \delta_{X_k^\star} - \delta_{X_k}$ for all k;
- $Z(i)$ increases by 2 at each visit to $i \in V_0$;
- $\min(Z(i), Z(i^\star))$ increases by 1 at each visit to $\{i, i^\star\}$, for all $i \in V_1$.

□

-Edge Reinforced Random Walks (-ERRW): results

Theorem (Bacallado, Sabot and T., 2021)

- $(Z_n(e)/n)_{n\in\mathbb{N}}$ *converges a.s. to a random vector* $x = (x_e)_{e\in E}$ *in*

$$\mathcal{L}_1 = \left\{ (y_e) \in (0,\infty)^E : \, y_{i,j} = y_{j^*,i^*}, \, div(y) = 0, \, \sum_{e\in E} y_e = 1 \right\}.$$

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- *Conditionally on* x, **-ERRW is a Markov chain* P_x *with jump probability* x_{ij}/x_i *from* i *to* j, $x_i = \sum_{i\to k} x_{ik}$.

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- *Conditionally on* x, **-ERRW is a Markov chain* P_x *with jump probability* x_{ij}/x_i *from* i *to* j, $x_i = \sum_{i\to k} x_{ik}$.
- *The random variable* x *has the following density on* $\mathcal{L}_1$, *w.r.t pullback of Lebesgue measure on* $\mathbb{R}^B$ *by the projection* $(y_e) \in \mathcal{L}_0 \mapsto (y_e)_{e\in B}$, B *basis of* $\mathcal{L}_1$:

$$C\gamma(i_0,\alpha)^{-1}\sqrt{y_{i_0}}\left(\frac{\prod_{(i,j)\in\tilde{E}} y_{i,j}^{\alpha_{i,j}-1}}{\prod_{i\in V} y_i^{\frac{1}{2}\alpha_i}}\right)\frac{1}{\prod_{i\in V_0}\sqrt{y_i}}\sqrt{D(y)}\,dy_{\mathcal{L}_1},$$

-Edge Reinforced Random Walks (-ERRW): results

We let $\tilde{E}$ be the set of edges quotiented by the relation $(i,j) \sim (j^*, i^*)$,

$$C = \frac{2}{\sqrt{2\pi}^{|V_0|-1}\sqrt{2}^{|V_0|+|V_1|}},$$

and

$$D(y) = \sum_T \prod_{(i,j)\in T} y_{i,j}.$$

The last sum runs on spanning trees directed towards a root $j_0 \in V$ (value does not depend on the choice of the root j_0).

ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (I)

Let $(W_e)_{e\in E}$ be conductances on edges, $W_e > 0$.
VRJP $(Y_s)_{s\geqslant 0}$ is a continuous-time process defined by $Y_0 = i_0$ and, if $Y_s = i$, then, conditionally to the past,

$$Y \text{ jumps to } j \sim i \text{ at rate } W_{i,j} L_j(s),$$

with

$$L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

Proposed by Werner and first studied **on trees** by Davis, Volkov ('02,'04).

ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (II)
Random conductances $(W_e)_{e \in E}$

Theorem (T. '11, Sabot, T. '15)

ERRW $(X_n)_{n\in\mathbb{N}}$ *with weights* $(\alpha_e)_{e\in E}$

"law" $=$ *VRJP* $(Y_t)_{t\geqslant 0}$ *with conductances* $W_e \sim \Gamma(\alpha_e)$ *indep.*

(at jump times)

- Similar equivalence applies to any linearly reinforced RW on its continuous time version (initially proved for VRRW, T'. 11)

Proof of ERRW $\longleftrightarrow$ VRJP (I)
Rubin construction : continuous equivalent of ERRW

Similar to continuous-time version of discrete-time Markov chain

Clocks at each edge:

- $(\zeta_i^e)_{e \in E, i \in \mathbb{N}}$ collection of i.i.d variables, $Exp(1)$ distributed.
- Alarms at each edge $e \in E$, at times

$$V_k^e := \sum_{i=0}^{k} \frac{\zeta_i^e}{\alpha_e + i}, \; k \in \mathbb{N} \cup \{\infty\}.$$

Process $(\tilde{X}_t)_{t \geqslant 0}$ starting from $i_0 \in V$:

- Clock e only runs when $(\tilde{X}_t)_{t \geqslant 0}$ adjacent to e.
- Alarm e rings $\Longrightarrow$ $\tilde{X}_t$ traverses it.

Then $(\tilde{X}_t)_{t \in \mathbb{R}_+}$ (at jump times) $\overset{\text{"law"}}{=}$ $(X_n)_{n \geqslant 0}$.

Proof of ERRW $\longleftrightarrow$ VRJP (II)
Yule process: a result of D. Kendall ('66)

For all $e \in E$, $t \geqslant 0$, let

$$N_t^e := \text{nb. of alarms at time } t \text{ for } e.$$

Then $\exists W_e \sim \mathrm{Gamma}(\alpha_e)$ s.t., conditionally to W_e,

$$N_\cdot^e \text{ increases between } t \text{ and } t+dt \text{ with prob. } W_e e^t\, dt.$$

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Consequences on Rubin construction:

- Let $T_x(t)$ time spent in $x \in V$ at time t
- Then, conditionally to W_e, $e \in E$, and to the past $\leqslant t$, if $\tilde{X}_t = x$, $\tilde{X}$ jumps to $y \sim x$ between t and $t+dt$ with prob. $W_{xy} e^{T_x(t)+T_y(t)} d(T_x(t)) = W_{xy} L_y(t) d(L_x(t))$, where

$$L_z(t) := e^{T_z(t)}.$$

VRJP: three timescales (I)

Jump rates from i to j

- Initial timescale process Y, with local time L :

$$W_{ij}L_j(t), \text{with} \quad L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u=j\}} du.$$

- Reversible timescale process Z, with local time T:

$$W_{ij}e^{T_i(t)+T_j(t)}, \text{with} \quad T_j(s) = \int_0^s \mathbb{1}_{\{Z_u=j\}} du.$$

- Exchangeable timescale process X:

$$\frac{1}{2}W_{ij}\sqrt{\frac{1+\ell_j}{1+\ell_i}}, \text{with} \quad \ell_j(s) = \int_0^s \mathbb{1}_{\{X_u=j\}} du.$$

VRJP: three timescales (II)

Proof: Change "clocks" at all sites:

- Z: $T_j = \log L_j$, or $L_j = e^{T_j}$ (already appears in the proof of ERRW $\longleftrightarrow$ VRJP)
- X: $\ell_j = L_j^2 - 1$, or $L_j = \sqrt{1+\ell_j}$.

Then

$$W_{ij} L_j dL_i = \frac{1}{2} W_{ij} \sqrt{\frac{1+\ell_j}{1+\ell_i}} d\ell_i = e^{T_i + T_j} dT_i.$$

Partial exchangeability of VRJP

Theorem

The VRJP is partially exchangeable in the sense of Diaconis and Freedman.
VRJP(i_0, W) X is a mixture of Markov Jump Processes (MJP) P^u with jump rate from i to j

$$\frac{1}{2} W_{ij} e^{u_j - u_i},$$

where u has measure $\mu^{i_0,W}(du)$ described next slide.

VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (I) Fixed conductances $(W_e)_{e\in E}$, G finite (Sabot-T.'15)

The measure $\mu^{i_0,W}(du)$ has density on $\mathcal{L}_0 = \{(u_i), \sum u_i = 0\}$

$$\frac{N}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)},$$

where

$$H(W,u) = 2 \sum_{\{i,j\}\in E} W_{i,j} \sinh^2((u_i - u_j)/2).$$

and

$$D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{\{i,j\}} e^{u_i+u_j}.$$

VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (II)
Fixed conductances $(W_e)_{e\in E}$, G finite (Merkl-Rolles-T.'19)

- $\mathcal{Q}^{i_0,W}(du)$ marginal of Gibbs "measure" on supermanifold extension $\mathbb{H}^{2|2}$ of hyperbolic plane with action $A_W(v,v)=\sum_{i,j}W_{ij}(v_i-v_j,v_i-v_j)$, taken in horospherical coordinates after integration over fermionic variables.

- Merkl-Rolles-T.'19: Other variables in extension of SuSy model arise on two different time scales as limits of
  - local times on logarithmic scale
  - rescaled fluctuations of local times
  - rescaled crossing numbers
  - last exit trees of the walk (tree version of fermionic variables)
- Bauerschmidt-Helmuth-Swan '19 (AP and AIHP): very nice interpretation of in terms of Brydges-Fröhlich-Spencer-Dynkin isomorphism for the supersymmetric field.

VRJP $\longleftrightarrow$ random Schrödinger (Sabot-T.-Zeng '15) (I)

Let, for all $i \in V$,

$$\beta_i = \frac{1}{2} \sum_{j \sim i} W_{ij} e^{u_j - u_i} + \delta_{i_0}(i)\gamma,$$

$\gamma \sim \Gamma(1/2)$ indep. of u.

- $\forall i \neq i_0$, $\beta_i =$ jump rate from i
- β field 1-dependent: $\beta_{|V_1}$ and $\beta_{|V_2}$ are independent if $dist_{\mathcal{G}}(V_1, V_2) \geqslant 2$.
- On $\mathbb{Z}^d$ with $W_{ij} = W$ constant, $(\beta_i)_{i \in V}$ translation-invariant
- The marginals β_i are such that $(2\beta_i)^{-1}$ have inverse Gaussian law.

VRJP $\longleftrightarrow$ random Schrödinger: Range and law of β (II)

- V finite
- $\Delta = (\Delta_{i,j})_{i,j\in V}$ discrete Laplacian, letting $W_i := \sum_{j\sim i} W_{i,j}$,

$$\Delta_{i,j} := \begin{cases} W_{i,j}, & \text{if } i \sim j,\ i \neq j \\ -W_i, & \text{if } i = j \end{cases}$$

- $H_\beta := -\Delta + 2\beta$, W diagonal with coefficients $(W_i)_{i\in V}$.
- $H_\beta > 0$ (positive definite): $\Longrightarrow$ $(H_\beta)^{-1}$ has positive entries.
- $\beta = (\beta_i)_{i\in V}$ has distribution

$$\nu^W(d\beta) = \sqrt{\frac{2}{\pi}}^{|V|} \mathbb{1}_{\{H_\beta>0\}} \frac{e^{\sum_{i\in V}(W_i/2-\beta_i)}}{\sqrt{|H_\beta|}} \prod_{i\in V} d\beta_i.$$

VRJP $\longleftrightarrow$ random Schrödinger: Retrieve u from β (III)

- Set $G = (H_\beta)^{-1}$.
- Then

$$\beta_i = \frac{1}{2}\sum_{j\sim i} W_{ij} e^{u_j - u_i}, \;\; i \neq i_0$$
$$\iff H_\beta(e^{u_\cdot})(i) = (-\Delta + 2\beta)(e^{u_\cdot})(i) = 0, \;\; i \neq i_0$$
$$\iff e^{u_i} = \frac{G(i_0, i)}{G(i_0, i_0)}, \;\; i \in V$$

  where $(u_i)_{i\in V}$ defined above and follows the law $\mathcal{Q}_{i_0}^{W}(du)$.

- Hence, time-changed VRJP starting from i_0 mixture of Markov jump processes with jump rate

$$\frac{1}{2} W_{i,j} e^{u_j - u_i} = \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

ERRW/VRJP and statistical physics: implications

Using link with QFT and localisation/delocalisation results of Disertori, Spencer, Zirnbauer '10 :

Theorem (ST'15, Angel-Crawford-Kozma'14, G bded degree)

ERRW (resp.VRJP) is positive recurrent at strong reinforcement, i.e. for a_e (resp. W_e) uniformly small in $e \in E$.

Theorem (ST'15, Disertori-ST'15, $G = \mathbb{Z}^d$, $d \geqslant 3$)

ERRW (resp. VRJP) is transient at weak reinforcement, i.e. for a_e (resp. W_e) uniformly large in $e \in E$.

Using link with Random Schrödinger operator:

Theorem (Sabot-Zeng '19, Sabot -19, Merkl-Rolles '09)

ERRW with constant weights $a_e = a$ (resp. $W_e = W$) is recurrent in dimension 2.

Theorem (Poudevigne'22)

Increasing initial weights of ERRW and VRJP makes them more transient (unique phase transition).

Correspondence $\star$-ERRW $\longleftrightarrow$ $\star$-VRJP (I)

Let $(W_e)_{e \in E}$ be conductances on edges, $W_{ij} = W_{j^\star i^\star} > 0$.
The $\star$-Vertex-Reinforced Jump Process ($\star$-VRJP) $(Y_s)_{s \geqslant 0}$ is a continuous-time process defined by $Y_0 = i_0$ and, if $Y_s = i$, then, conditionally to the past,

$$Y \text{ jumps to } j \sim i \text{ at rate } W_{i,j} L_{j^\star}(s),$$

with

$$L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_r = j\}} dr.$$

Correspondence $\star$-ERRW $\longleftrightarrow$ $\star$-VRJP (II)
Random conductances $(W_e)_{e\in E}$

Theorem (Bacallado-Sabot-T. '21)

$\star$-*ERRW* $(X_n)_{n\in\mathbb{N}}$ *with weights* $(\alpha_e)_{e\in E}, \alpha_{ij}=\alpha_{j^\star i^\star}$

$\overset{\text{"law"}}{=}$ $\star -$ *VRJP* $(Y_t)_{t\geqslant 0}$ *with conductances* $W_e \sim \Gamma(\alpha_e), e\in \tilde{E}$ *indep.*

(at jump times)

Proof.
Similar to [T.'11, Sabot-T.'15], as for any linearly reinforced RW on its continuous time version. □

$\star$-VRJP: again three timescales

Jump rates from i to j

- Initial timescale process Y, with local time L :

$$W_{ij} L_j^\star(t), \text{with} \quad L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

- Reversible timescale process Z, with local time T:

$$W_{ij} e^{T_i(t) + T_j^\star(t)}, \text{with} \quad T_j(s) = \int_0^s \mathbb{1}_{\{Z_u = j\}} du.$$

- Exchangeable timescale process X:

$$\frac{1}{2} W_{ij} \sqrt{\frac{1 + \ell_j^\star}{1 + \ell_i}}, \text{with} \ \ \ell_j(s) = \int_0^s \mathbb{1}_{\{X_u = j\}} du.$$

The limiting manifold

Set $\mathcal{L}_0^W = \{(u_i)_{i\in V},\ \mathrm{div}(W^u) = 0,\ \sum_{i\in V} u_i = 0\}$.

Proposition

The following limit

$$\tilde{U}_i = \lim_{t\to\infty} T_i(t) - t/|V|$$

exists a.s. and $\tilde{U} \in \mathcal{L}_0^W$.

Proof of $\tilde{U} \in \mathcal{L}_0^W$.

If X is at i, it jumps to j with probability $W_{ij}d(e^{T_i(t)+T_{j^*}(t)})$ on infinitesimal time interval. Hence

$$W_{ij}e^{T_i(t)+T_{j^*}(t)}/Z_t(ij) \to_{t\to\infty} 1.$$

On the other hand, by Kirchoff's law,

$$|\sum_{j:\, i\to j} Z_t(ij) - \sum_{k:\, k\to i} Z_t(ki)| \leqslant 1.$$

⋆-VRJP : Randomize initial local time

- Also appears in the context of self-repelling motion: T., Tóth and Valkó'12, Horváth, Tóth and Vetö '12.

$\star$-VRJP : Randomize initial local time

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- For $i_0 \in V$, consider the probability measure given by

$$\nu^{i_0,W}(da) = \frac{1}{F(W,i_0)} e^{a_{i_0^\star}} e^{-\frac{1}{2}\sum_{i\to j} W_{i,j} e^{a_{j^\star}-a_{i^\star}}}(da),$$

on

$$\mathcal{A} = \{(a_i) \in \mathbb{R}^V, a_{i^\star} = -a_i\}.$$

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on

$$\mathcal{A} = \{(a_i) \in \mathbb{R}^V, a_{i^\star} = -a_i\}.$$

Let

$$\overline{\mathbb{P}}^{i_0,W}(\cdot) = \mathbb{E}^{A\sim\nu^{i_0,W}}(\mathbb{P}^{i_0,W^A}(\cdot)), \quad (W^A)_{i,j} = W_{i,j}e^{A_i+A_{j^\star}}$$

law of the $\star$-VRJP after expectation w.r.t. $A \sim \nu_{i_0}^W$.

$\star$-VRJP : Randomize initial local time

Lemma (Sabot-T. 2024)

Under $\overline{\mathbb{P}}_{i_0}^{W}$, conditionally on $\sigma\{X_s,\ s \leqslant t\}$, (A_i) is distributed according to $\frac{1}{F(X_t, W^{T(t)})}\nu^{X_t, W^{T(t)}}$.

Proposition

Let (α_e) be positive weights with $div(a) = \delta_{i_0^*} - \delta_{i_0}$, and $W_e \sim Gamma(\alpha_e)$ indep. Then $W^A \overset{law}{=} W$.

$\star$-VRJP : partial exchangeability

Let

$$C(t) = \frac{1}{2}\sum_{i\in V}(e^{T_i(t)+T_{i^\star}(t)} - 1),$$

and $Z_s = X_{C^{-1}(s)}$.

Proposition (Sabot-T. 2024)

Under $\overline{\mathbb{P}}_{i_0}^{W}(\cdot)$, Z has jump rate

$$W_{i,j}^{T(t)}\frac{F(W^{T(t)},j)}{F(W^{T(t)},i)}$$

and is partially exchangeable.

$\star$-VRJP : mixing measure

Theorem (Sabot-T. 2024)

i) Under $\overline{\mathbb{P}}_{i_0}^{W}$*, the following limit exists*

$$U_i := \lim_{t\to\infty} A_i + T_t(i) - t/N,$$

and

$$U \in \mathcal{L}_0^W = \{(u_i)_{i\in V},\ div(W^u) = 0,\ \sum_{i\in V} u_i = 0\}.$$

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ii) Under $\overline{\mathbb{P}}_{i_0}^W$, conditionally on U, the $\star$-VRJP in partially exchangeable time scale, $(Z_t)_{t\geqslant 0}$, is a Markov jump process with jump rate from i to j equal to

$$W_{i,j} e^{U_{j^\star} - U_{i^\star}}.$$

$\star$-VRJP : mixing measure

Theorem (Sabot-T. 2024)

Under $\overline{\mathbb{P}}^{W}_{i_0}$, U *has distribution* $\mu^{i_0,W}(du)/F^{W}_{i_0}$, *where* $\mu^{i_0,W}(du)$ *has density on* $\mathcal{L}^{W}_{0}$

$$C^G e^{u_{i_0^\star}} e^{-\frac{1}{2}\sum_{i\to j} W_{i,j} e^{u_{j^\star}-u_{i^\star}}} e^{-\sum_{i\in V_0} u_i} \frac{\sqrt{D(W^u)}}{\det_{\mathcal{A}}(-K^u)},$$

with

$$D(W^u) = \sum_{T} \prod_{\{i,j\}\in T} W^u_{i,j} :$$

sum on all **rooted** *spanning trees of the graph,* K^u *generator of MJP at rate* $W^u_{i,j} = W_{ij} e^{u_i + u_{j^\star}}$.

Given Z Markov process $P_{i_0}^{W,u}$, there exists a random variable

$$B^{\theta}(s) = \frac{1}{2}\int_0^s \frac{\mathbb{1}_{Z_u=i} - \mathbb{1}_{Z_u=i^*}}{\theta + \ell_i(u) + \ell_{i^*}(u)} du, \quad \forall i \in V,\ s \geqslant 0.$$

$B^{\theta}(\infty)$ has density on $\mathcal{A}$, which we denote by $f_{i_0}^{W,u,\theta}$. Let $f_{i_0}^{W,u} = f_{i_0}^{W,u,1}$ for simplicity.

Theorem (Non-randomized $\star$-VRJP)

(i) $B^1(t) \to \frac{1}{2}(u - u^*)$ as $t \to \infty$.
(ii) The law of U for the non-randomized $\star$-VRJP is

$$f_{i_0}^{W,u}\left(\frac{1}{2}(u - u^*)\right)\mu_{i_0}^{W}(du).$$

(iii) At time $t \geqslant 0$, the jump rate of the non-randomized $\star$-VRJP Z conditioned on $U = u$ from $Z_t = i$ to j is

$$\frac{f_j^{W,u,1+\ell(t)+\ell^*(t)}\left(\frac{1}{2}(u - u^*) - B^1(t)\right)}{f_i^{W,u,1+\ell(t)+\ell^*(t)}\left(\frac{1}{2}(u - u^*) - B^1(t)\right)} W_{ij} e^{u_j^* - u_i^*}.$$

*-VRJP: Random Schrödinger version

Theorem

For all $\theta \in (0,\infty)^V$, $\eta \in (\mathbb{R}_+)^V$, we have

$$\prod_{i\in V_0} \theta_i \int_{\mathcal{S}} \frac{\mathbb{1}_{H_\beta>0}}{\sqrt{2\pi}^{|\mathcal{S}|}} \exp\left(-\frac{1}{2}\langle\theta, H_\beta\theta\rangle - \frac{1}{2}\langle\eta, G_\beta\eta\rangle\right)\frac{d\beta}{\sqrt{|H_\beta|}}$$
$$= \int_{\mathcal{A}} \frac{1}{\sqrt{2\pi}^{|\mathcal{A}|}} \exp\left(-\frac{1}{2}\langle e^a\theta, We^a\theta\rangle + \frac{1}{2}\langle\theta, W\theta\rangle - \langle\eta, e^a\theta\rangle\right) da.$$

When $X_0 = i_0$, the measure on β is associated to a differentiation with respect to η_{i_0} of a combination of the two measures above at $\eta = 0$, $\theta = 1$ on $\{i_0, i_0^*\}^c$.