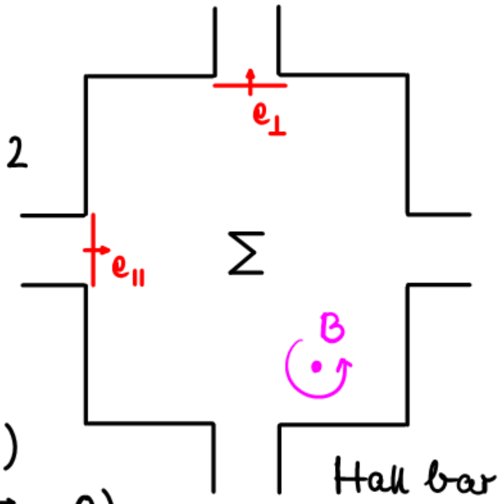
Lecture 1: Quantum Hall Basics

1.1 Conductance

(Hall and dissipative)

$D = 2$



[Assume d.c. limit]

a) The current-density $(D-1)$-form j (twisted) is closed: $dj = 0$ ("Kirchhoff's rule", $\operatorname{div} \vec{j} = 0$).
Current $I \equiv [j] \in H^{D-1}(\Sigma, \mathcal{L})$.

b) The electric-field 1-form E is exact: $E = -d\phi$.
Voltage $V \equiv [E] \in H^1_c(\Sigma)$.

c) "Power" pairing: (non-degenerate)

$$H^{D-1}(\Sigma,\mathcal{L}) \otimes H^1_c(\Sigma) \longrightarrow \mathbb{R},$$
$$[j] \otimes [E] \longmapsto \int_\Sigma j \wedge E .$$

d) Conductance G:

$$H^1_c(\Sigma) \longrightarrow H^{D-1}(\Sigma,\mathcal{L}) \cong H^1_c(\Sigma)^*,$$
$$V \longmapsto I = GV .$$

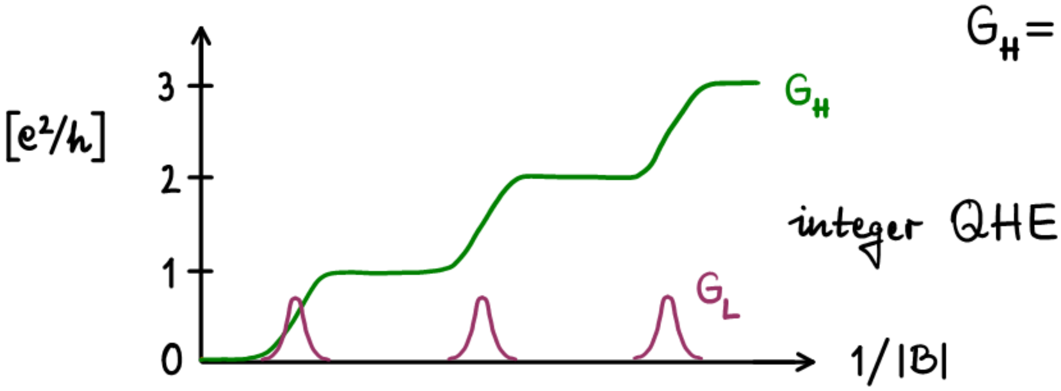
- Decompose conductance as $G = G^{sym} + G^{skew}$ (dissipative + Hall).

  [Onsager relation: $G^{sym}(-B) = +G^{sym}(B)$, $G^{skew}(-B) = -G^{skew}(B)$.]

  Quantum Hall Effect: $G^{sym} = 0$, G^{skew} quantized (in some range of B).

  [Warning/disclaimer: experiments measure the resistance, $R = G^{-1}$.]

  Given cohomology generators $e_\parallel, e_\perp \in H^1_c(\Sigma)$, let $G_L = e_\parallel(G e_\parallel)$,
  $G_H = e_\parallel(G e_\perp)$.



1.2 High-Field (Semiclassical) Limit

<u>Warning</u>: Assume the single-electron approximation!

- Charged particle in a (static) field E, B as a Hamiltonian system (M, Ω, H):
  - phase space $M = T^*\Sigma$,
  - symplectic form $\Omega = dp_i \wedge dq^i + \frac{1}{2} e B_{ij}\, dq^i \wedge dq^j$,
  - Hamiltonian $(D=2)$: $H = \frac{p_1^2 + p_2^2}{2m} + e\phi(q^1, q^2)$.

- High-field limit: cyclotron frequency $\omega_c = \frac{|eB|}{m}$ largest scale. Can take average over one cycle of the fast variables (p_1, p_2); cf. Arnol'd.

$\longrightarrow$ Dimensional reduction (DR):

$$M_{DR} = \Sigma, \quad \langle q^1 \rangle_{\text{cycle}} \equiv x, \quad \langle q^2 \rangle_{\text{cycle}} \equiv y,$$

$$\Omega_{DR} = m\omega_c\, dx \wedge dy, \quad H_{DR} = e\phi(x,y).$$

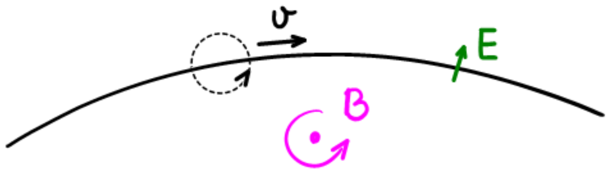
- Guiding-center drift (slow variables x, y).

Recall $dH = \Omega(\cdot, X_H)$ (Hamiltonian vector field X_H)

Equations of motion: $|B|\,\dot{x} = -\frac{\partial\phi}{\partial y}, \quad |B|\,\dot{y} = +\frac{\partial\phi}{\partial x}$.

$\Longrightarrow \phi(x(t), y(t)) = \text{const}$,

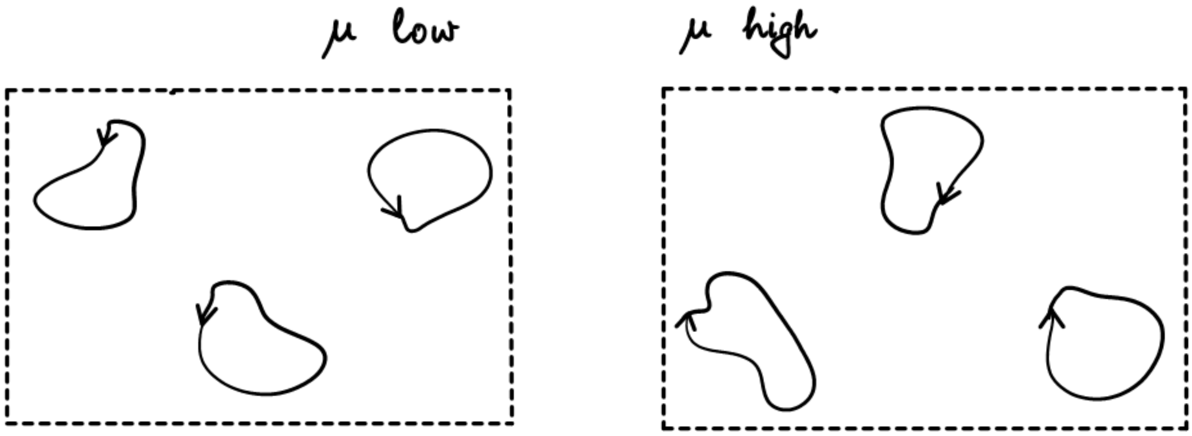
guiding-center drift with speed $|v| = |E|/|B|$ (actually, $E = \iota(v)B$).

Illustration. 

$\phi = \text{const}$

1.3 Quantum Percolation Scenario

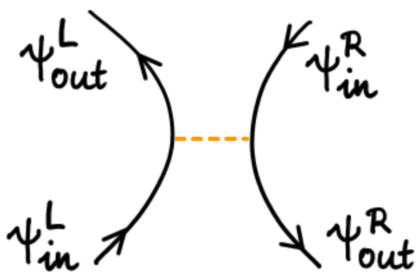
Recall: guiding center drift along equipotentials $\phi(\cdot) = \mu = \text{const}$ (chemical potential μ).



Critical point $\mu = \mu_c$: phase transition of percolation type

Observation. Must take into account quantum tunneling across saddle points, as the phase transition in the classical limit ($\rightarrow$ percolating equipotential) falls into a different universality class.

quantum tunneling:

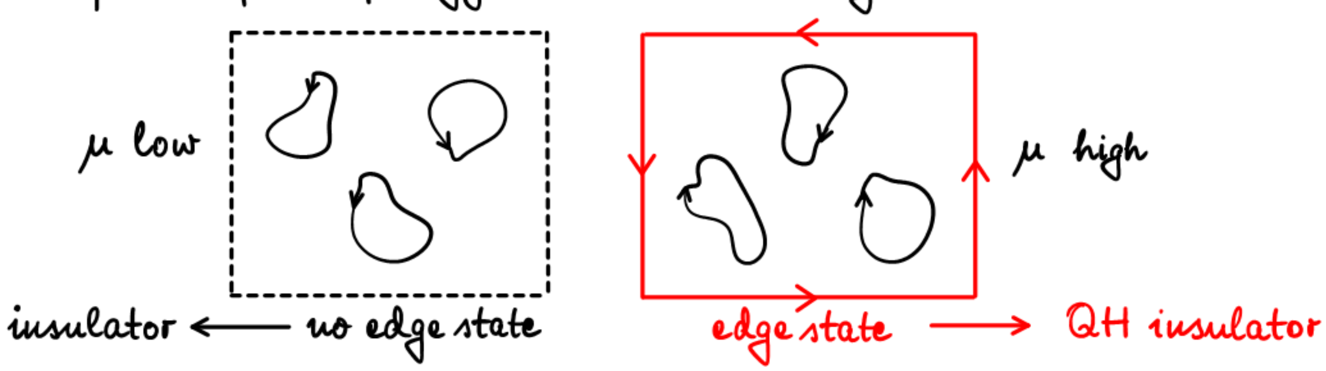


current conservation $\Longrightarrow$

$$|\psi^L_{out}|^2 - |\psi^L_{in}|^2 = |\psi^R_{in}|^2 - |\psi^R_{out}|^2 .$$

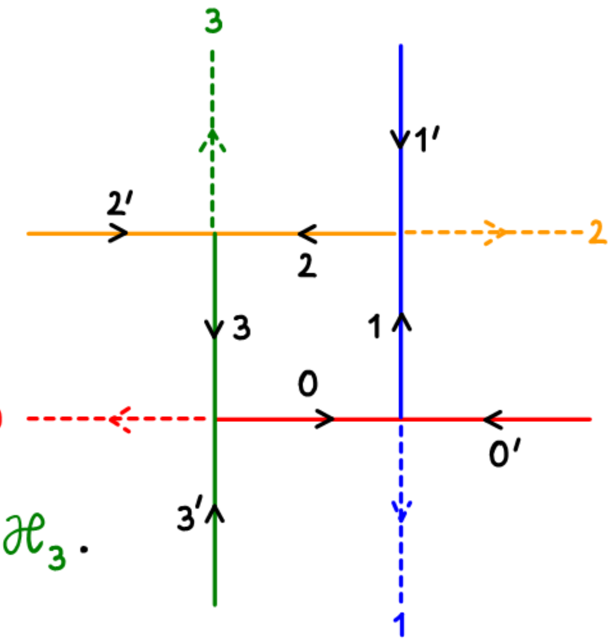
Challenge. Compute the universal properties of this quantum Hall percolation transition.

- Perspective from topology (bulk $\longleftrightarrow$ boundary).



Lecture 2 : Network Model

2.1 Chalker-Coddington model ($N_c = 1$).



— Square lattice $\Lambda \subset \mathbb{Z}^2$
with directed links. Unit cell :

Hilbert space :

$$\mathcal{H} = \bigoplus_{\text{links } \ell} \mathbb{C}(\ell) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 .$$

— Unitary quantum dynamics (discrete time): $\psi_{t+1} = \mathcal{U}\psi_t$, where
$\mathcal{U} : \mathcal{H}_\nu \longrightarrow \mathcal{H}_{\nu+1}$ $(\nu = 0, 1, 2, 3, 4 \equiv 0)$ is of Floquet type,
$\mathcal{U} = \mathcal{U}_r \mathcal{U}_s$, with $\mathcal{U}_r$ link-diagonal, $\mathcal{U}_r(\ell) = e^{i\Theta(\ell)}$,
i.i.d. random variables,
Haar distributed on $U(1)$,
and $\mathcal{U}_s$ deterministic scattering at the nodes,
with amplitude a_L (a_R) for left (right) turn, and (unitarity →)

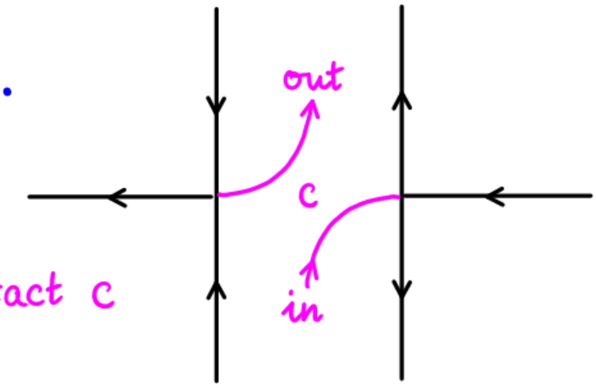
$$|a_L|^2 + |a_{\ }|^2 = 1, \quad \arg(a_L) = -\arg(a_R) = \frac{\pi}{4} \quad \text{[Kac-Ward]}.$$

<u>Note</u>. The model is critical (localization length $= \infty$) for $|a_L|^2 = |a_R|^2 = \frac{1}{2}$.

<u>Remark</u>.
$$\mathcal{U}_s\, e_\nu(\square) = e_{\nu+1}(\square)\, a_L + e'_{\nu+1}(\square + t_\nu)\, a_R ,$$
$$\mathcal{U}_s\, e'_\nu(\square) = e_{\nu+1}(\square)\, a_R + e'_{\nu+1}(\square + t_\nu)\, a_L ,$$
$$t_0 = -\delta_y, \quad t_1 = +\delta_x, \quad t_2 = +\delta_y, \quad t_3 = -\delta_x .$$

2.2 Point-contact conductance.

point contact c



- Consider the case of two point contacts, located at links c_A, c_B.

  Let $Q := \mathrm{Id}_{\mathcal{H}} - P_A - P_B$ (projection operators).

  — Conductance: $G_{AB} = \left|(1-T)^{-1}(c_A, c_B)\right|^2$, $T = QU$.

  Mean conductance: $\mathbb{E}\, G_{AB} = \sum_n \mathbb{E} \left| T^n(c_A, c_B)\right|^2$.

  The path sum $T^n(c_A, c_B) = \sum_{\ell_1, \ldots, \ell_n \neq c_A, c_B} T(c_A, \ell_n) \cdots T(\ell_1, c_B)$

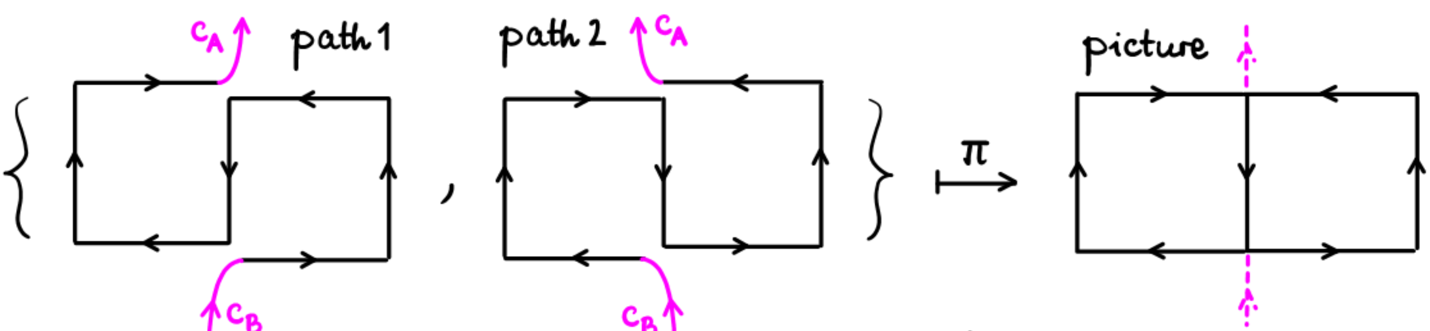
  doubles for $\left|T^n(c_A,c_B)\right|^2 \longrightarrow$ "retarded" and "advanced" paths.

  <u>Remark</u>. Valid paths start at c_B, never return to c_B, and terminate upon first arrival at c_A.

- Def (Bettelheim, Gruzberg & Ludwig, 2012).

  Organize all equal-length paths that visit the <u>same</u> set of links the <u>same</u> number of times, into an equivalence class called a "picture" (pic).

  Very simple example:

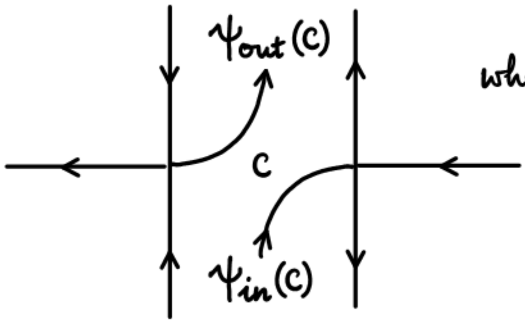


card(pic) = 2, length(pic) = 8.

<u>Fact</u>. The path double sum for $\mathbb{E}G_{AB}$ reduces to a positive sum over pictures. At criticality: $\mathbb{E}G_{AB} = \sum_{\text{pic } B \to A} \mathrm{card}^2(\text{pic})\, 2^{-\mathrm{length}(\text{pic})}$.

2.3 Point-contact stationary states.

Consider the simplified situation of a <u>single</u> point contact, c :



where $\psi = \mathcal{U}\psi$ stationary scattering state,

and $\psi_{out}(c) = S\,\psi_{in}(c)$,

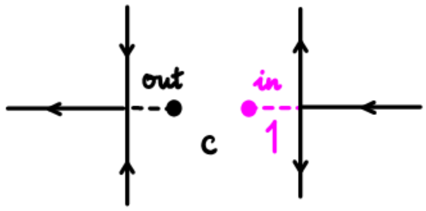
with S scattering "matrix" $S \in U(1)$.

- Precise formulation: impose boundary conditions.

P projector for contact link c

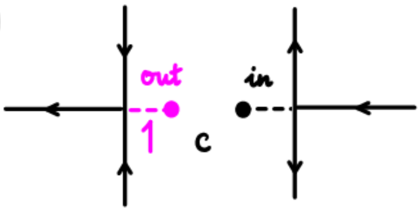
— incoming-wave b.c. ($\psi \equiv \psi^+$, $Q = 1 - P$)

$$\psi^+ = Q\,\mathcal{U}\,\psi^+ + 1 \cdot e_c$$



— outgoing-wave b.c. ($\psi \equiv \psi^-$)

$$\psi^- = Q\,\mathcal{U}^{-1}\psi^- + 1 \cdot e_c$$



<u>Remark</u>. $\psi^-(\ell) = S\,\psi^+(\ell)$ (for all ℓ), $|S| = 1$.

- Random variable of interest:

$$|\psi_c(\ell)|^2 \equiv |\psi^\pm(\ell)|^2 = \left|(1-T)^{-1}(\ell,c)\right|^2, \quad T = Q\,\mathcal{U}.$$

<u>Lemma</u>. $|\psi_c(\ell)|^2 \overset{\ell \neq c}{=} G(\ell,\ell)$, $\quad G = T(1-T)^{-1} + (1-T^\dagger)^{-1}$.

Proof. $G := \psi_c\,(\psi_c\,,\cdot)$

$$= \psi^-(\psi^-,\cdot) \overset{\text{owbc}}{=} Q(1-T^\dagger)^{-1}\mathcal{U}^{-1}P\,\mathcal{U}(1-T)^{-1}Q.$$

Use $\mathcal{U}^{-1}P\,\mathcal{U} = 1 - T^\dagger T = (1-T^\dagger)(1-T) + (1-T^\dagger)T + T^\dagger(1-T)$,

hence $G = Q + T(1-T)^{-1}Q + Q(1-T^\dagger)^{-1}T^\dagger$

$$= T(1-T)^{-1}Q + Q(1-T^\dagger)^{-1} \quad \blacksquare$$

<u>Corollary</u>. $\mathbb{E}\,|\psi_c(\ell)|^2 = 1$ for all ℓ.

Lecture 3: Methods of Analysis

3.1 Wegner–Efetov SUSY method (first step).

Express $\mathbb{E}\,\left|(E \underset{A}{\overset{R}{\pm}} i\varepsilon - H)^{-1}(x,y)\right|^2$ by

$$A^{-1}(x,y) = \frac{\mathrm{Cof}_{y,x}(A)}{\mathrm{Det}(A)}$$

fermionic $\displaystyle \prod \frac{\partial^2}{\partial\xi\,\partial\eta}\, e^{-(\xi, A\eta)}\, \eta(x)\,\xi(y)$,

bosonic $\displaystyle \int e^{-(\bar\varphi, A\varphi)}$

and take disorder average.

Then, Hubbard–Stratonovich transformation, etc.

3.2 Variant: "color–flavor transformation"

$$\mathbb{E}\,\left|(1 - z\mathcal{U})^{-1}(x,y)\right|^2 \overset{|z|<1}{=} \int_B \int_F \eta_R(x)\,\xi_R(y)\,\eta_A(x)\,\xi_A(y)$$

$$\times\ \mathbb{E}\exp\Big(-(\xi_R, (1 - z\mathcal{U})\eta_R) - (\xi_A, (1 - \bar z\mathcal{U}^\dagger)\eta_A) - \text{"bosons"}\Big).$$

Now for $\mathcal{U} = \mathcal{U}_r \mathcal{U}_s$, $\mathcal{U}_r \in U(N_c) \times \cdots \times U(N_c)$, taking averages w.r.t. Haar measure on $U(N_c)$ leads to an unwieldy expression (modified Bessel functions). What to do?

$\longrightarrow$ CFT (here in schematic form):

$$\mathbb{E}_U \exp\left(\bar\psi_R U \psi_R + \bar\psi_A U^\dagger \psi_A\right) = \mathbb{E}_Z \exp\left(\bar\psi_R Z \psi_A + \bar\psi_A \tilde Z \psi_R\right)$$

[replaces the HS-transformation of Wegner – Efetov].

<u>Warning</u>: complications for $N_c = 1$!

3.3 Read's method ("second quantization").

$$H \in \mathrm{End}(V) \quad \begin{cases} \nearrow \ \hat{H}_F \in \mathrm{End}(\Lambda(V)) & \text{"fermionic" SQ,} \\ \searrow \ \hat{H}_B \in \mathrm{End}(S(V)) & \text{"bosonic" SQ.} \end{cases}$$

$$H = e_i H^i{}_j \times e^j \quad \begin{cases} \nearrow \ \hat{H}_F = f^\dagger_i H^i{}_j \otimes f^j \\ \searrow \ \hat{H}_B = b^\dagger_i H^i{}_j \otimes b^j . \end{cases}$$

particle creation	annihilation
$f^\dagger_i = \varepsilon(e_i)$	$f^j = \iota(e^j)$
$b^\dagger_i = \mu(e_i)$	$b^j = \delta(e^j)$

- **Character formulas.**

— $\mathrm{STr}_{\Lambda(V)}\, e^{\hat{H}_F} = \mathrm{Det}(1 \pm e^H)$, $\quad \mathrm{STr}_{\Lambda(V)} \equiv \mathrm{Tr}_{\Lambda^{ev}(V)} - \mathrm{Tr}_{\Lambda^{odd}(V)}$.

— $\mathrm{Tr}_{S(V)}\, e^{\hat{H}_B} = \mathrm{Det}^{-1}(1 - e^H)$, $\quad$ if $\mathrm{Re}\, H \equiv \frac{1}{2}(H + H^\dagger) < 0$.

— $Z \equiv \mathrm{STr}_{\mathcal{F}}\, e^{\hat{H}} = 1$, $\quad \mathcal{F} = S(V) \otimes \Lambda(V)$, $\quad \hat{H} = \hat{H}_B + \hat{H}_F$.

Note. The Lie algebra representations $H \mapsto \hat{H}_X$ $(X = B, F)$ exponentiate to (semi-)group representations, i.e. we may pass to $U \mapsto \rho_X(U)$, $\rho_X(e^H) \equiv e^{\hat{H}_X}$ $(X = B, F)$.

- **Key relations.** Let $\rho(U) \equiv \rho_B(U)\, \rho_F(U)$.

— $(1 - U)^{-1}(x, y) = \mathrm{STr}_{\mathcal{F}}\, \rho(U)\, f(x)\, f^\dagger(y)$,

— $\rho(U_r U_s) = \rho(U_r)\, \rho(U_s)$,

— $\mathbb{E}\, \rho(U_r) = \prod_{\text{links}} P(\ell)$, where $P(\ell)$ projects on $\ker(\hat{n}_R - \hat{n}_A)(\ell)$;

for $N_c = 1$: $\quad P = \int \frac{d\theta}{2\pi}\, e^{i\theta(\hat{n}_R - \hat{n}_A)}$, $\quad \hat{n}_Y = b^\dagger_Y b_Y + f^\dagger_Y f_Y$ $(Y = R, A)$.

Remark. Leads to SUSY vertex model repn of the network model.

Corollary (from SUSY vertex model).

$$\mathbb{E}\,|\psi_c(r)|^{2q} = \mathbb{E}\,|\psi_c(r)|^{2(1-q)} \quad (q \in \tfrac{1}{2} + i\mathbb{R}).$$

Question. The marginal field e^t of the $H^{2|2}$-model corresponds to (the classical version of) $B^\dagger B > 0$, $B = b_R + b_A^\dagger$. Can one find the marginal distribution of the latter?

Generating function. $\mathbb{E}\,(1 + t|\psi_c(r)|^2)^{-1} = \mathrm{STr}_{\mathcal{V}}\, \pi(c)\, \rho(\hat{\mathcal{U}}_s)\, \rho(e^{-tY(r)})$,

after projection $\mathcal{F} \to \mathcal{V}$ by $\mathbb{E}\,\rho(\mathcal{U}_r) = \prod_{\text{links}} P(\ell)$, and with $Y = B^\dagger B$.

3.4 Cauchy transform.

Let $A_h = \dfrac{1+h}{1-h}$. Then if all of $1-g$, $1-h$, and $1-gh$ are invertible one has the identity

$$\begin{aligned}(1-gh)^{-1} &= (1-h)^{-1}\left(\tfrac{1}{2}(A_g + A_h)\right)^{-1}(1-g)^{-1} \\ &= (1-g)^{-1} - g(1-g)^{-1}\left(\tfrac{1}{2}(A_g+A_h)\right)^{-1}(1-g)^{-1}.\end{aligned}$$

Apply this to the $N_c = 1$ network model, setting $g = \mathcal{U}_r$, $h = \mathcal{U}_s$.

Then for $x \neq y$,

$$\left|(1-\mathcal{U}_r\mathcal{U}_s)^{-1}(x,y)\right|^2 = q(x)\,\left|(T+V)^{-1}(x,y)\right|^2 q(y),$$

$$T = \frac{1+\mathcal{U}_s}{1-\mathcal{U}_s}, \quad V = \frac{1+\mathcal{U}_r}{1-\mathcal{U}_r}, \quad q(\ell) = \frac{2}{|1-e^{i\Theta(\ell)}|^2}.$$

Note.

T nonlocal, deterministic, translation-invariant;

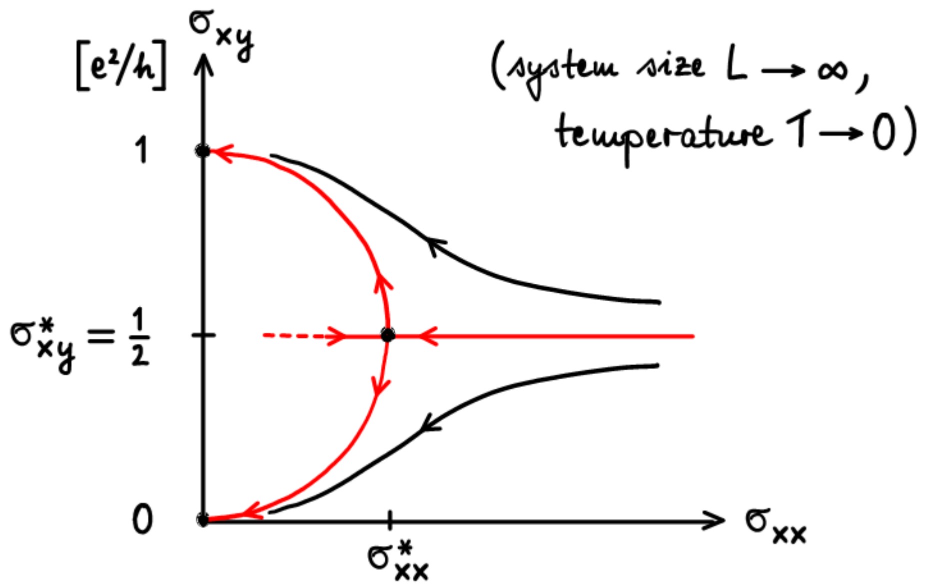
V local, random with Cauchy-distributed matrix element

$$V(\ell) = \frac{1+e^{i\Theta(\ell)}}{1-e^{i\Theta(\ell)}}.$$

$\longrightarrow$ methods of Wegner–Efetov available here!

Lecture 4: Conjectures

- Pruisken–Khmelnitskii scaling / flow diagram:



Is this the renormalization group flow diagram for a nonlinear σ–model (Pruisken, 1983) with RG–beta vector field

$$\beta = \frac{1}{\nu}\,(\sigma_{xy} - \sigma^*_{xy})\,\frac{\partial}{\partial \sigma_{xy}} - \eta\,(\sigma_{xx} - \sigma^*_{xx})\,\frac{\partial}{\partial \sigma_{xx}} \quad ?$$

<u>Note</u>. There exists no consensus on the values of the scaling exponents ν, η from numerical simulations.

Consequence. The metallic phase is absent, i.e., all states off of the critical line $\sigma_{xy} = \sigma^*_{xy} = \frac{1}{2}$ (or $|a_L|^2 - |a_R|^2 \neq 0$) are localized.

In particular, the point–contact conductance has exponential decay,

$$\mathbb{E} G_{AB} \sim e^{-|c_A - c_B|/\xi},$$

with $\xi \to \infty$ as $|a_L|^2 - |a_R|^2 \to 0$.

The CFT prediction for the decay at criticality is $\mathbb{E} G^*_{AB} \sim |c_A - c_B|^{-\frac{1}{8}}$.

<u>Q</u>: What is the scaling behavior of ξ ? $\quad \xi \sim \big|\,|a_L|^2 - |a_R|^2\big|^{-\nu}$??

- **Gaussian Free Field Hypothesis.**

In the scaling limit at criticality, the law of the random variable $\ln|\psi_c(r)|^2$ is expected to be that of a Gaussian Free Field φ ($|\psi_c(r)|^2 \sim e^{\varphi}$) with "background charge" $Q = 1$. This means that

$$\mathbb{E}\,|\psi_c(r)|^2 \sim |r-c|^{-2\Delta_q},$$

and the spectrum of multifractal scaling exponents is parabolic:

$$\Delta_q = \frac{1}{n}\,q\,(1-q).$$

- **Conformal Field Theory.**

The renormalization-group fixed point for the critical model has been argued to be a Wess–Zumino–Witten model $GL_{n,\gamma}(1|1)$ with current-algebra level $n = 4$ and Abelian current–current deformation $\gamma = 1$. The fixed-point dissipative conductivity following from this CFT is

$$\sigma^*_{xx} = \frac{n}{2\pi} = 0.6366\ldots$$

- **Singular continuous spectral measure.**

Define a spectral measure for the case of $\Sigma = \mathbb{Z}^2$ as usual by $(|\lambda| > 1)$

$$(\lambda\cdot 1 - \mathcal{U})^{-1}(e,e) = \oint_{S^1} \frac{d\mu_{\mathcal{U},e}(\theta)}{\lambda - e^{i\theta}}.$$

<u>Conjecture</u>. At criticality, the integrated local density of states,

$$\mathcal{I}(\theta) = \int_0^\theta d\mu_{\mathcal{U},e}\,,$$ is singular continuous as

$$\lim_{\theta\to\theta'} |\ln(\theta-\theta')|\;|\mathcal{I}(\theta)-\mathcal{I}(\theta')| < \infty.$$