SPP2265
Random Geometric Systems

The vertex-reinforced jump process and a non-linear supersymmetric hyperbolic sigma model with long-range interactions

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Joint work with Margherita Disertori and Franz Merkl

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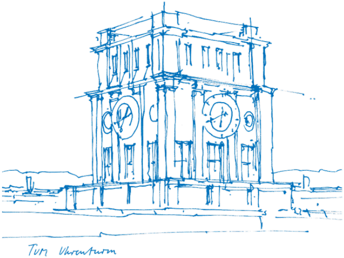

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A marginal of the $H^{2|2}$-model

- Let $G = (\Lambda, E)$ be an undirected finite graph endowed with edge weights $W_e > 0$, $e \in E$.
- Fix a pinning point $\rho \in \Lambda$.
- Let $\mathcal{S}$ be the set of spanning trees of the graph G.

Consider the following probability measure on $\{u \in \mathbb{R}^\Lambda : u_\rho = 0\}$:

$$\mu_{W,\rho}(du) =$$

$$= \sqrt{\sum_{S \in \mathcal{S}} \prod_{\{i,j\} \in S} W_{ij} e^{u_i + u_j}} \prod_{\{i,j\} \subseteq \Lambda} e^{-W_{ij}(\cosh(u_i - u_j) - 1)} \prod_{i \in \Lambda \setminus \{\rho\}} \frac{e^{-u_i}}{\sqrt{2\pi}} du_i,$$

$\mu_{W,\rho}$ is the u-marginal of the $H^{2|2}$ model (non-linear supersymmetric hyperbolic sigma model) in horospherical coordinates.
It was introduced by Zirnbauer in 1991.

Long-range interactions

Consider the complete graph with vertex set $\mathbb{Z}^d$ and longe-range weights

$$W_{ij} = w(\|i-j\|_\infty), i,j \in \mathbb{Z}^d, i \neq j,$$

with a decreasing weight function $w : [1,\infty) \to (0,\infty)$. Assume that

$$\sum_{i\in\mathbb{Z}^d\setminus\{0\}} w(\|i\|_\infty) < \infty \quad \text{and}$$

$$w(x) \geq \overline{W}\frac{(\log_2 x)^\alpha}{x^{2d}} \text{ for all } x \geq 1$$

for some $\alpha > 1$ and $\overline{W} > 0$.

Wired boundary conditions

For $N \in \mathbb{N}$, we consider the boxes

$$\Lambda_N := \{0, 1, \ldots, 2^N - 1\}^d \subseteq \mathbb{Z}^d$$

with wired boundary conditions given by

$$W_{i\rho} = \sum_{j \in \mathbb{Z}^d \setminus \Lambda_N} W_{ij}, \quad i \in \Lambda_N,$$

with wiring point $\rho \notin \Lambda_N$.

A first bound

Assumptions:

- box $\Lambda_N := \{0, 1, \ldots, 2^N - 1\}^d$ with wired boundary conditions
- long-range weights $W_{ij} = w(\|i - j\|_\infty)$ as above
- for some $\alpha > 1$ and $\overline{W} > 0$,

$$w(x) \geq \overline{W} \frac{(\log_2 x)^\alpha}{x^{2d}} \quad \text{for all } x \geq 1.$$

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Theorem. (Disertori, Merkl & R. AOP 2025+) For the corresponding $H^{2|2}$ model $\mu_W^{\Lambda_N}$, for all $m \geq 1$ and $\sigma \in \{\pm 1\}$, one has

$$\mathbb{E}_W^{\Lambda_N}[e^{\sigma m u_i}] \leq c = c(\overline{W}, d, \alpha, m)$$

with a constant $c = \exp\left(\frac{d^\alpha 2^{2d}(2m+1)^2}{\overline{W}} \sum_{l=2}^\infty l^{-\alpha}\right) > 1$ uniformly in $i \in \Lambda_N \cup \{\rho\}$ and N.

A refined version of the result for large weights

Assumptions: as above with the stronger lower bound

$$w(x) \geq \overline{W} \frac{(\log_2 x)^\alpha}{x^{2d}} \text{ for all } x \geq 1.$$

for some $\alpha > 3$ and $\overline{W} > 0$.

Theorem. (Disertori, Merkl & R. 2023) Let $\kappa \in (0, 1]$. There exist $c_1, c_2 > 0$ such that for all $\overline{W} \geq -c_1 \kappa^{-1} \log \kappa$ and $m \in [0, c_2 \kappa \overline{W}]$, one has

$$\begin{aligned} \mathbb{E}_W^{\Lambda_N} [(\cosh u_i)^m] &\leq 1 + \kappa \\ \mathbb{E}_W^{\Lambda_N} [(\cosh(u_i - u_j))^{m/2}] &\leq 2^{m/2}(1 + \kappa). \end{aligned}$$

uniformly in $i, j \in \Lambda_N \cup \{\rho\}$ and N.

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The vertex-reinforced jump process (vrjp)

Let $G = (\Lambda, E)$ be an undirected finite or infinite graph endowed with edge weights $W_e > 0$, $e \in E$.

The vertex-reinforced jump process $Y = (Y_t)_{t \geq 0}$ is a process in continuous time, where given $(Y_s)_{s \leq t}$, the particle jumps from site i to site $j \sim i$ with rate $W_{ij} L_j(t)$, where

$$L_j(t) = 1 + \int_0^t 1_{\{Y_s = j\}} \, ds = 1 + \text{local time at } j.$$

Denote the distribution by P_W.

- The process was conceived by Wendelin Werner around 2000.
- There is a huge literature.
- Small W corresponds to strong reinforcement, large W to weak reinforcement.

Connection with the $H^{2|2}$ model

Let $G = (\Lambda, E)$ be a finite graph with weights $W = (W_e)_{e \in E}$. For $t \geq 0$, consider

$$D(t) = \sum_{i \in \Lambda} (L_i(t)^2 - 1).$$

Theorem. (Sabot-Tarrès 2015)
Let $(Y_t)_{t \geq 0}$ be vrjp on G. The time-changed process $(Z_\sigma := Y_{D^{-1}(\sigma)})_{\sigma \geq 0}$ is a mixture of Markov jump processes:

$$P_W(Z \in A) = \int J_u(A)\, \mu_W(du) \quad \text{for any event } A.$$

- μ_W is the u-marginal of the $H^{2|2}$ model in horospherical coordinates
- J_u is the law of the Markov jump process jumping from i to j at rate $W_{ij} e^{u_j - u_i}$.

$(Z_\sigma)_{\sigma \geq 0}$ is called vrjp in exchangeable time scale.

The embedded discrete-time process

Let $(X_n)_{n\in\mathbb{N}_0}$ be the discrete-time process obtained from the vrjp by taking only the values right before the jumps.

Corollary. (Sabot-Tarrès 2015)
$(X_n)_{n\in\mathbb{N}_0}$ is a mixture of reversible Markov chains:

$$P_W(X \in A) = \int Q_u(A)\, \mu_W(du) \qquad \text{for any event } A.$$

- As before μ_W is the u-marginal of the $H^{2|2}$ model in horospherical coordinates.
- Q_u is the law of a Markovian random walk jumping from i to j with probability proportional to $W_{ij}e^{u_i+u_j}$.

$$\text{Note that} \quad \frac{W_{ij}e^{u_j-u_i}}{\sum_k W_{ik}e^{u_k-u_i}} = \frac{W_{ij}e^{u_j+u_i}}{\sum_k W_{ik}e^{u_k+u_i}}.$$

Transience on $\mathbb{Z}^d$ with long range interactions

Theorem. (Disertori, Merkl & R. AOP 2025+) Fix a dimension $d \in \mathbb{N}$. Consider the vertex-reinforced jump process on the complete graph with vertex set $\mathbb{Z}^d$ and edge weights

$$W_{ij} = w(\|i-j\|_\infty), i, j \in \mathbb{Z}^d, i \neq j,$$

with a decreasing weight function $w : [1,\infty) \to (0,\infty)$. If

$$\sum_{i \in \mathbb{Z}^d \setminus \{0\}} w(\|i\|_\infty) < \infty \quad \text{and}$$

$$w(x) \geq \overline{W} \frac{(\log_2 x)^\alpha}{x^{2d}} \text{ for all } x \geq 1$$

for some $\alpha > 1$ and $\overline{W} > 0$. Then the expected number of visits to any given vertex is finite. In particular, the vrjp is a.s. transient.

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For $d \geq 3$, there is $\overline{W}_1 > 0$ such that for all $\overline{W} \geq \overline{W}_1$, the lower bound $W_{ij} \geq \overline{W} 1_{\{\|i-j\|_2 = 1\}}$ implies transience.

Relation with Markovian random walks on $\mathbb{Z}^d$

Poudevigne 2022: If the Markovian random walk on a weighted graph (G, W) is recurrent, then the vertex-reinforced jump process on G with parameters W is recurrent.

Caputo-Faggionato-Gaudillière 2009, Bäumler 2022:
Consider $\mathbb{Z}^d$ with long-range weights. The random walk on this weighted graph is

- recurrent if $W_{ij} \leq \overline{W} \|i - j\|^{-2d}$ for all $i, j \in \mathbb{Z}^d$, $d \in \{1, 2\}$;
- transient if $W_{ij} \geq \overline{W} \|i - j\|^{-s}$ for some $s < 2d$ and all $i, j \in \mathbb{Z}^d$

Bäumler's argument can be extended to give transience if

$$W_{ij} \geq \overline{W} \frac{(\log_2 \|i - j\|)^\alpha}{\|i - j\|^{2d}}$$

for some $\alpha > 1$.

Transience on $\mathbb{Z}^d$, $d \geq 3$, for large weights

Theorem. (Sabot-Tarrès 2015)
For any $d \geq 3$, there exists $W_2 < \infty$ such that vrjp on $\mathbb{Z}^d$ with nearest-neighbor jumps and constant initial weights $W \geq W_2$ is transient.

Vrjp on hierarchical lattices

Wang and Zeng 2025 analyse recurrence and transience for vrjp with long-range jumps on hierarchical lattices.

More about this in Xiaolin's talk.

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The world is an Ising model – you just have to look at it in the right way.

Saying of unknown origin.

Ancestor of sigma-models: Ising model

Ising model on a finite graph $G = (\Lambda, E)$ (Lenz 1920, Ising 1925)

- State space $\Omega = \{\pm 1\}^\Lambda \ni (\sigma_i)_{i\in\Lambda}$
- Coupling constants $W_{ij} > 0$, $\{i,j\} \in E$,
- Energy function, “Hamiltonian”

$$H(\sigma) = -\sum_{\{i,j\}\in E} W_{ij}\sigma_i \cdot \sigma_j$$

- Probability function $\mu(\{\sigma\}) = \dfrac{e^{-H(\sigma)}}{Z}$

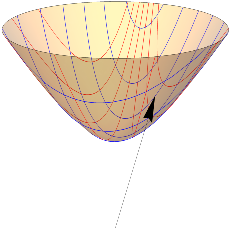
  Normalizing constant $Z = \sum_{\sigma\in\Omega} e^{-H(\sigma)}$

The $H^{2|2}$ model

It is like an Ising model with a more complicated state space.

The model involves for every vertex i

- three commuting variables x_i, y_i, z_i and
- two Grassmann, anticommuting variables ξ_i, η_i
  E.g. $\xi_i \eta_j = -\eta_j \xi_i$ and $\xi_i^2 = 0$.
- "spin variables" $\sigma_i = (x_i, y_i, z_i, \xi_i, \eta_i)$
- supersymmetric hyperbolic inner product for $i, j \in \Lambda$:
  $\langle \sigma_i, \sigma_j \rangle = x_i x_j + y_i y_j - z_i z_j + \xi_i \eta_j - \eta_i \xi_j$



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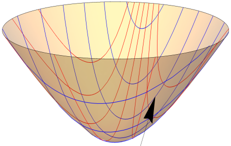

Constraints:

- supersymmetric hyperboloid $H^{2|2}$:
  $\langle \sigma_i, \sigma_i \rangle = x_i^2 + y_i^2 - z_i^2 + 2\xi_i \eta_i = -1$
- positive shell: body$(z_i) > 0$
- **pinning**: Fix $\rho \in \Lambda$ and set $\sigma_\rho = o := (0, 0, 1, 0, 0)$.

supersymmetric Hamiltonian: $H(\sigma) = -\sum_{\{i,j\} \in E} W_{ij} \langle \sigma_i, \sigma_j \rangle$

The $H^{2|2}$ model

The model with pinning at ρ is described by the following superintegration form acting on test superfunctions f:

$$\langle f \rangle_{W,\rho} := \int_{(H^{2|2})^{\Lambda \setminus \{\rho\}}} \frac{\mathcal{D}\sigma}{(2\pi)^{|\Lambda|-1}} \exp\left\{ \sum_{\{i,j\} \in E} W_{ij}(1 + \langle \sigma_i, \sigma_j \rangle) \right\} f(\sigma)$$

with the canonical supermeasure $\mathcal{D}\sigma$.

[Disertori-Spencer-Zirnbauer 2010]:
This supermeasure is normalized.

$$\text{Supersymmetry invariance} \quad \Longrightarrow \quad \langle 1 \rangle_{W,\rho} = 1$$

Change of coordinates

Recall the constraints $x_i^2 + y_i^2 - z_i^2 + 2\xi_i\eta_i = -1$ and $\mathrm{body}(z_i) > 0$ defining $H^{2|2}$. Transform cartesian coordinates $\sigma_i = (x_i, y_i, z_i, \xi_i, \eta_i)$ in $H^{2|2}$ to horospherical coordinates

$$(\underbrace{u_i, s_i}_{\text{even}}, \underbrace{\overline{\psi}_i, \psi_i}_{\text{odd}})$$

by

$$x_i = \sinh u_i - \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}, \qquad y_i = s_i e^{u_i},$$

$$z_i = \cosh u_i + \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}$$

$$\xi_i = \overline{\psi}_i e^{u_i}, \qquad \eta_i = \psi_i e^{u_i}$$

After this change of coordinates, we integrate over all Grassmann variables $\overline{\psi}_i$, ψ_i and over s_i. This yields the probability measure $\mu_{W,\rho}(du)$, which describes the random environment for vrjp.

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Estimates for the $H^{2|2}$ model

If we have two sets of weights $W_e^- \le W_e^+$, $e \in E$, for $m \ge 1$, a monotonicity result of Poudevigne 2022 implies

$$\mathbb{E}_{W^+}\left[e^{\pm m u_i}\right] \le \mathbb{E}_{W^-}\left[e^{\pm m u_i}\right].$$

This is used to decuce the result for long-range weights

$$W_{ij}^+ := W_{ij} \ge \overline{W} 1_{\{\|i-j\|_2 = 1\}} =: W_{ij}^-$$

with $\overline{W}$ large on $\mathbb{Z}^d$, $d \ge 3$, from known bounds of Disertori-Spencer-Zirnbauer 2010.

Estimates for the $H^{2|2}$ model with longe-range weights

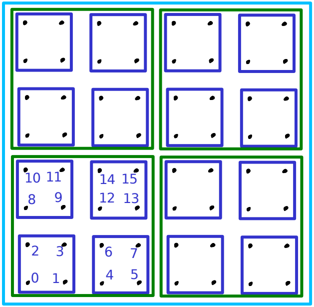
- For the general long-range result, we compare the $H^{2|2}$ model with long-range weights with an $H^{2|2}$ model with hierarchical weights.
- The distribution of u_i in the hierarchical $H^{2|2}$ model agrees with the distribution in an effective $H^{2|2}$ model.
- Studying the effective $H^{2|2}$ model can be reduced to studying a one-dimensional model with inhomogeneous weights, where the vertices represent length scales.

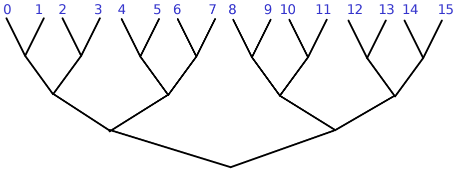
The $H^{2|2}$ model with hierarchical interactions

- Interpret $\{0,1\}^{Nd}$ as the set of leaves of a binary tree.
- For $k,l \in \{0,1\}^{Nd}$, let $d_H(k,l)$ be their hierarchical distance, i.e. the distance to the least common ancestor in the binary tree.

We can identify $\Lambda_N = \{0,1,\ldots,2^N-1\}^d \subseteq \mathbb{Z}^d$ and $\{0,1\}^{Nd}$ via a bijection $\phi : \Lambda_N \to \{0,1\}^{Nd}$ such that

$$2^{\lceil d_H(\phi(i),\phi(j))/d \rceil} > \|i-j\|_\infty \quad \forall i,j \in \mathbb{Z}^d.$$





Comparison with a hierarchical model

- We compare our long-range $H^{2|2}$ model with weights $W_{ij} = w(\|i-j\|_\infty)$ using the embedding $\phi : \Lambda_N \to \{0,1\}^{Nd}$.
- Since w is decreasing and $2^{\lceil d_H(\phi(i),\phi(j))/d \rceil} > \|i-j\|_\infty$, one has

$$W_{ij} = w(\|i-j\|_\infty) \geq w(2^{\lceil d_H(\phi(i),\phi(j))/d \rceil}) = W_{ij}^{-}$$

By a monotonicity result of Poudevigne 2022, we obtain

$$\mathbb{E}_W^{\Lambda_N}\left[(\cosh u_i)^m\right] \leq \mathbb{E}_{W^-}^{\Lambda_N}\left[(\cosh u_i)^m\right]$$

W_{ij}^{-} depends only on the hierarchical distance of i and j.

The $H^{2|2}$ model with hierarchical interactions

Theorem. (Disertori, Merkl & R. AOP 2025+) Consider the $H^{2|2}$ model on the complete graph with vertex set $\Lambda_N = \{0,1\}^N \cup \{\rho\}$ and edge weights

$$W_{ij}^H = w^H(d_H(i,j)), \quad W_{i\rho}^H = h^H > 0, \quad i,j \in \Lambda_N.$$

Assume that for some $\alpha > 1$ and $\overline{W} > 0$, one has

$$w^H(l) \geq \overline{W} 2^{-2l} l^\alpha \text{ for } l \in \mathbb{N}, \quad h^H \geq 4\overline{W} 2^{-N}(N+1)^\alpha.$$

Then for all $m \geq 1$ and $\sigma \in \{\pm 1\}$, one has

$$\mathbb{E}_W^{\Lambda_N}[e^{\sigma m u_i}] \leq c$$

with a constant $c = \exp\left(\frac{(2m+1)^2}{\overline{W}} \sum_{l=2}^\infty l^{-\alpha}\right)$ uniformly in $i \in \Lambda_N$ and N.

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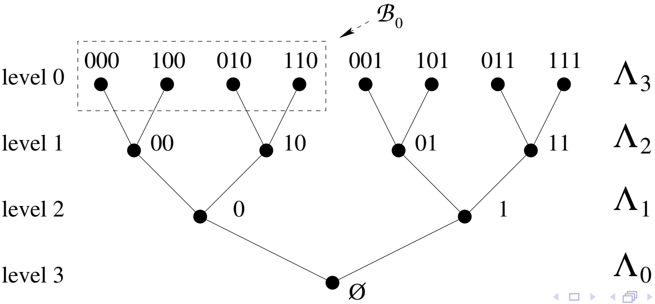
Analysing the hierarchical $H^{2|2}$ model

Extend the hierarchical model from the set of leaves $\Lambda_N = \{0,1\}^N$ to subsets $A \subseteq \mathcal{T}^N$ of the whole binary tree $\mathcal{T}^N = \cup_{n=0}^N \{0,1\}^n$:

- The level $\ell(i)$ of i is the distance from the leaves.
- $i \wedge j$ is the least common ancestor of i and j.
- $\mathcal{B}_i$ denotes the set of leaves above i.

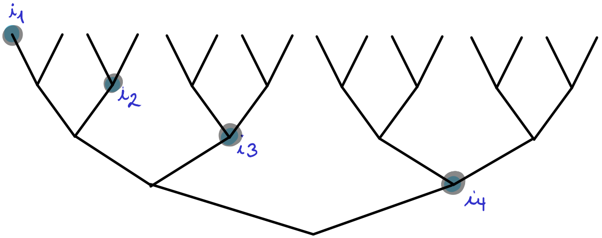
We set for $i, j \in A$

$$W_{ij}^H := 2^{\ell(i)+\ell(j)} w^H(\ell(i \wedge j)), \quad W_{i\rho}^H := |\mathcal{B}_i| h^H.$$



Antichains

- $i \preceq j$ means that the vertex $i \in \mathcal{T}^N$ is below or at j.
- A set $A \subseteq \mathcal{T}^N$ is called an antichain if $i \npreceq j$ holds for all $i, j \in A$, $i \neq j$.



$A = \{i_1, \ldots, i_4\}$ is an antichain.

Effective $H^{2|2}$ model

- We identify $\Lambda_N = \{0, 1, \ldots, 2^N - 1\}^d \subseteq \mathbb{Z}^d$ with the set of leaves of the binary tree $\mathcal{T}^N$.
- Let A be a maximal antichain.
- For $j \in A$, let $\mathcal{B}_j$ be the set of leaves above j.

Theorem. (Disertori, Merkl & R. ALEA 2022) The joint distribution of the block spin variables

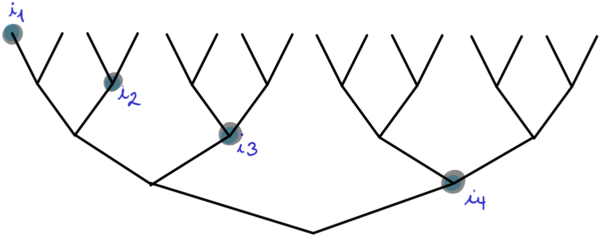
$$\left(|\mathcal{B}_j|^{-1} \sum_{i \in \mathcal{B}_j} e^{u_i} \right)_{j \in A}$$

with respect to the $H^{2|2}$ model on Λ_N with hierarchical weights agrees with the joint distribution of $(e^{u_j})_{j \in A}$ with respect to the $H^{2|2}$ model on the antichain.
There is a version involving also the s-variables.

Effective $H^{2|2}$ model in our application

Goal: Estimate $\mathbb{E}_W^{\Lambda_N}[(\cosh u_i)^m]$, $i \in \Lambda_N$, in the $H^{2|2}$ model on $\Lambda_N = \{0,1\}^N$ with hierarchical interactions.

- By symmetry, the expectation is independent of i.
- The distribution of $\cosh u_0$ in the $H^{2|2}$ model on Λ_N equals the distribution of $\cosh u_0$ in the $H^{2|2}$ model on the following antichain A. Note that $|\Lambda_N| = 2^N$, $|A| = N$.



$$W^H_{i_1,i_2} = 2w^H(2), W^H_{i_2,i_3} = 8w^H(3), W^H_{i_3,i_4} = 32w^H(4), h^H = 8h^H$$

Effective $H^{2|2}$ model in our application

Consider the antichain $A = \{i_1, \dots, i_N\} \subseteq \mathcal{T}^N$, where

$$
\begin{aligned}
i_1 =& 0^N = (0, \dots, 0), \\
i_l =& (1, 0^{N-l}) = (1, 0, \dots, 0) \in \{0, 1\}^{N-l+1}, 2 \le l \le N.
\end{aligned}
$$

For $2 \le l \le N$, one has

$$
\begin{aligned}
W^H_{i_{l-1}, i_l} =& 2^{2l-3} w^H(l), \\
W^H_{i_l \rho} =& 2^{l-1} h^H, \quad W_{i_1 \rho} = h^H.
\end{aligned}
$$

- In the effective $H^{2|2}$ model, all vertices interact with each other.
- However, using monotonicity from Poudevigne or in a determinant in our argument, we only keep the above interactions. This yields an inhomogeneous one-dimensional model.

Estimates for an inhomogeneous one-dimensional model

Rough bound for $\mathbb{E}_W^{\Lambda_N}[e^{\sigma m u_i}]$

- The proof uses that in one dimension, $u_{j+1} - u_j$ are independent.
- The rough bound is sufficient to prove transience of the vertex-reinforced jump process.

Refined bound for $\mathbb{E}_W^{\Lambda_N}[(\cosh u_i)^m]$

- The proof requires a careful analysis using supersymmetry similar to Disertori-Spencer-Zirnbauer 2010.